THE QUANTUM TORUS AS AN $\mathbb{E}_{\mathrm{M}}$-CATEGORY

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Abstract. Given an oriented 2-manifold M, a locally constant sheaf of lattices Λ over M, and a pointed morphism $q : \mathsf{B}^2\Lambda \to \mathsf{B}^4\mathbf{C}^\times$, we define an $\mathbb{E}_{\mathrm{M}}$-category $\mathrm{Rep}_q(\check{\mathrm{T}})$ which we call the "quantum torus" at level q. We explain why this terminology is deserved and calculate the factorization homology of $\mathrm{Rep}_q(\check{\mathrm{T}})$. When M arises from a global complex curve, we confirm (a version of) a conjecture of Ben-Zvi and Nadler for tori.

Contents

Introduction

Given a reductive group G and a "level" $q \in \mathbf{C}^\times$, one obtains the category $\mathrm{Rep}_q(\check{\mathrm{G}})$ of representations of the quantum group. It can be regarded as a braided monoidal deformation of the category $\mathrm{Rep}(\check{\mathrm{G}})$ of representations of the Langlands dual group $\check{\mathrm{G}}$.

Morever, the category $\mathrm{Rep}_q(\check{\mathrm{G}})$ admits a ribbon structure, which allows one to "spread" $\mathrm{Rep}_q(\check{\mathrm{G}})$ onto an oriented 2-manifold M, thus defining an $\mathbb{E}_{\mathrm{M}}$-category. It is then possible to extract a global invariant

$$\int_{\mathrm{M}} \mathrm{Rep}_q(\check{\mathrm{G}}),$$

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called its "factorization homology", which plays a prominent role in the Betti quantum geometric Langlands program. We refer the reader to [BZBJ18, BZN18] for details.

The consideration above makes us suspect that $\mathrm{Rep}_q(\check{\mathrm{G}})$ starts life naturally as an $\mathbb{E}_{\mathrm{M}}$-category. Indeed, in the context of the quantum geometric Langlands program, the level q is not expected to be a complex number in general, but a (suitably categorified) degree-4 reduced cohomology class of $\mathsf{B}\mathrm{G}$. Unless G is simply connected, a level of this kind can vary along M, so we do not expect $\mathrm{Rep}_q(\check{\mathrm{G}})$ to come from a single ribbon category.

In this note, we confirm this suspicion for quantum tori: We give a direct definition of $\mathrm{Rep}_q(\check{\mathrm{T}})$ as an $\mathbb{E}_{\mathrm{M}}$-category, which accommodates the general notion of levels as well as nonsplit tori. From a classical perspective, one can say that this note explains how quantum tori behave in family.

Contents of this note. Our definition of the "quantum torus" $\mathrm{Rep}_q(\check{\mathrm{T}})$ takes as input a triple (M, Λ, q), where

(1) M is an oriented 2-manifold (with underlying ∞-groupoid $\mathrm{Sing}\,\mathrm{M}$);
(2) Λ is a functor from $\mathrm{Sing}\,\mathrm{M}$ to the category of finite free $\mathbf{Z}$-modules;
(3) $q : \mathsf{B}^2\Lambda \to \mathsf{B}^4\mathbf{C}^\times$ is a morphism in the ∞-category $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathsf{Spc}_*)$, where Spc_* denotes the ∞-category of pointed ∞-groupoids.

Given the triple (M, Λ, q), we shall define $\mathrm{Rep}_q(\check{\mathrm{T}})$ as an $\mathbb{E}_{\mathrm{M}}$-algebra in the ∞-category DGCat of DG categories in §2.2.

Our definition is conceptually simple, but not quite explicit. To argue that we have given a reasonable definition, we shall show that $\mathrm{Rep}_q(\check{\mathrm{T}})$ has the expected behavior of a quantum torus: It is completely determined by its heart $\mathrm{Rep}_q(\check{\mathrm{T}})^\heartsuit$, which is a family of "twisted" braided monoidal categories over M whose local invariants can be expressed explicitly in terms of those of q (*cf.* Proposition 2.4.5, Proposition 2.4.9).

As for global invariants, we shall compute the factorization homology of $\mathrm{Rep}_q(\check{\mathrm{T}})$. In Theorem 3.1.3, we shall construct a canonical equivalence in DGCat:

$$\int_{\mathrm{M}} \mathrm{Rep}_q(\check{\mathrm{T}}) \simeq \mathrm{LS}_q(\Gamma_c(\mathrm{M}, \mathsf{B}^2\Lambda)), \tag{0.1}$$

where the right-hand-side denotes the DG category of "q-twisted" local systems over the ∞-groupoid $\Gamma_c(\mathrm{M}, \mathsf{B}^2\Lambda)$ of compactly supported sections.

In fact, our definition of $\mathrm{Rep}_q(\check{\mathrm{T}})$ turns (0.1) into an immediate corollary of nonabelian Poincaré duality, due to Salvatore, Segal, and Lurie (*cf.* [Sal01, Seg10, Lur17]).

When M is the underlying oriented 2-manifold of a global complex curve X and Λ is defined by the (locally constant) sheaf of cocharacters of an X-torus T, the equivalence (0.1) implies a version of [BZN18, Conjecture 4.27], the quantum Betti geometric Langlands conjecture for tori (*cf.* Corollary 3.2.8).

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1. Preparation

In this section, we recall the notions of $\mathbb{E}_{\mathrm{M}}$-algebras and factorization homology, and gather all of their properties that we shall use later. These properties are comprehensively established in [Lur17, §5.5] and, from an alternative point of view, in [AF15]. We also recall the formalism of local systems of [GKRV22].

Needless to say, this section contains no originality.

1.1. Factorization homology.

1.1.1. Fix an integer $n \geq 0$.

Denote by Mfd_n the topological category of n-dimensional topological manifolds admitting finite "good covers" (*cf.* [AF15, Definition 2.1]).

We shall suppress the operation of taking homotopy coherent nerves from our notation, and regard Mfd_n as an ∞-category.

Thus, the mapping space between $\mathrm{M}_1, \mathrm{M}_2 \in \mathsf{Mfd}_n$ is the ∞-groupoid $\mathrm{Sing}\,\mathrm{Emb}(\mathrm{M}_1, \mathrm{M}_2)$, where $\mathrm{Emb}(\mathrm{M}_1, \mathrm{M}_2)$ is the set of embeddings $\mathrm{M}_1 \to \mathrm{M}_2$, endowed with the compact-open topology, and Sing denotes the functor of singular chains.

1.1.2. Denote by $\mathsf{BTop}(n)$ the full subcategory of Mfd_n consisting of objects homeomorphic to the Euclidean space $\mathbf{R}^n$.

The Kister–Mazur theorem shows that $\mathsf{Top}(n) \coloneqq \mathrm{Sing}\,\mathrm{Emb}(\mathbf{R}^n, \mathbf{R}^n)$ is a *grouplike* monoid. In particular, $\mathsf{BTop}(n)$ may be identified with the classifying space of $\mathsf{Top}(n)$.

Given $\mathrm{M} \in \mathsf{Mfd}_n$, we have a forgetful functor

$$\mathsf{BTop}(n)_{/\mathrm{M}} \to \mathsf{BTop}(n). \tag{1.1}$$

Remark 1.1.3. The slice ∞-category $\mathsf{BTop}(n)_{/\mathrm{M}}$ is a Kan complex equivalent to $\mathrm{Sing}\,\mathrm{M}$ (*cf.* [Lur17, Remark 5.4.5.2]), so (1.1) determines a morphism of ∞-groupoids

$$\tau_{\mathrm{M}} : \mathrm{Sing}\,\mathrm{M} \to \mathsf{BTop}(n).$$

By [AF15, Corollary 2.13], this morphism classifies the tangent microbundle of M.

1.1.4 *The ∞-operad* $\mathbb{E}_{\mathrm{M}}^{\otimes}$**.** Recall that $\mathsf{BTop}(n)$ is the underlying ∞-category of an ∞-operad $\mathsf{BTop}(n)^{\otimes}$ (*cf.* [Lur17, Definition 5.4.2.1]).

Moreover, each ∞-category $\mathcal{C}$ functorially determines an ∞-operad $\mathcal{C}^{\sqcup}$ (*cf.* [Lur17, §2.4.3]) and we have a natural morphism $\mathsf{BTop}(n)^{\otimes} \to \mathsf{BTop}(n)^{\sqcup}$ (*cf.* [Lur17, Remark 5.4.2.7]).

For each $\mathrm{M} \in \mathsf{Mfd}_n$, the ∞-operad $\mathbb{E}_{\mathrm{M}}^{\otimes}$ is defined as the fiber product

$$\mathbb{E}_{\mathrm{M}}^{\otimes} \coloneqq \mathsf{BTop}(n)^{\otimes} \times_{\mathsf{BTop}(n)^{\sqcup}} (\mathsf{BTop}(n)_{/\mathrm{M}})^{\sqcup}.$$

Remark 1.1.5. Let us give an informal description of $\mathbb{E}_{\mathrm{M}}^{\otimes}$.

By definition, the underlying ∞-category of $\mathbb{E}_{\mathrm{M}}^{\otimes}$ is $\mathsf{BTop}(n)_{/\mathrm{M}}$. Given objects $x : \mathrm{U} \to \mathrm{M}$ and $x_j : \mathrm{U}_j \to \mathrm{M}$ ($j = 1, \cdots, m$) of $\mathsf{BTop}(n)_{/\mathrm{M}}$, an m-ary operation from $\{x_j\}_{j=1,\cdots,m}$ to $\{x\}$ in $\mathbb{E}_{\mathrm{M}}^{\otimes}$ consists of an embedding $\mathrm{U}_1 \sqcup \cdots \sqcup \mathrm{U}_m \to \mathrm{U}$ together with an identification of the composite

$$\mathrm{U}_j \to \mathrm{U}_1 \sqcup \cdots \sqcup \mathrm{U}_m \to \mathrm{U} \xrightarrow{x} \mathrm{M}$$

with x_j as morphisms in Mfd_n, for each $j = 1, \cdots, m$.

1.1.6. We fix $\mathrm{M} \in \mathsf{Mfd}_n$ in the remainder of this section.

Let $\mathcal{O}$ be a symmetric monoidal ∞-category. We shall refer to $\mathbb{E}_{\mathrm{M}}^{\otimes}$-algebras in $\mathcal{O}$ simply as $\mathbb{E}_{\mathrm{M}}$-*algebras*. They form an ∞-category $\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O})$.

Since the ∞-category underlying $\mathbb{E}_{\mathrm{M}}^{\otimes}$ is $\mathsf{BTop}(n)_{/\mathrm{M}}$, or equivalently $\mathrm{Sing}\,\mathrm{M}$ (*cf.* Remark 1.1.3), we have a forgetful functor

$$\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O}) \to \mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathcal{O}). \tag{1.2}$$

Given $\mathcal{A} \in \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O})$ and $x \in \operatorname{Sing} \mathrm{M}$, we write $\mathcal{A}_x \in \mathcal{O}$ for the image of x under the functor underlying $\mathcal{A}$ and refer to it as the *fiber* of $\mathcal{A}$ at x.

1.1.7 *Factorization homology***.** When $\mathcal{O}$ is sifted-complete, we shall construct a functor

$$\int_{\mathrm{M}} : \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O}) \to \mathcal{O}, \tag{1.3}$$

whose value at $\mathcal{A} \in \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O})$ is called the *factorization homology* of $\mathcal{A}$ over M.

Denote by Disk_n the full subcategory of Mfd_n consisting of objects homeomorphic to $\mathrm{S} \times \mathbf{R}^n$ for some finite set S.

The construction of (1.3) relies on a functor of ∞-categories

$$\mathsf{Disk}_{n/\mathrm{M}} \to \mathbb{E}_{\mathrm{M}}^{\otimes}, \tag{1.4}$$

which we shall define presently.[1]

1.1.8 *Construction of* (1.4)**.** It suffices to construct functors

$$\mathsf{Disk}_{n/\mathrm{M}} \to \mathsf{BTop}(n)^{\otimes}, \tag{1.5}$$

$$\mathsf{Disk}_{n/\mathrm{M}} \to (\mathsf{BTop}(n)_{/\mathrm{M}})^{\sqcup}, \tag{1.6}$$

and identify their compositions with the functors to $\mathsf{BTop}(n)^{\sqcup}$.

We follow the notation of [Lur17, §2.1.1] and write Fin_* for the category of pointed finite sets. We express any $\mathrm{S} \in \mathsf{Fin}_*$ as $\mathrm{S}^{\circ} \sqcup \{*\}$, where $*$ is the distinguished element. Observe that $\mathsf{Disk}_{n/\mathrm{M}}$ admits a natural functor to Fin_*, sending $\mathrm{U} \to \mathrm{M}$ to the pointed finite set $\pi_0 \mathrm{U} \sqcup \{*\}$. The functors (1.5) and (1.6), which we shall construct, intertwine this functor to Fin_* with the structural functors of the ∞-operads $\mathsf{BTop}(n)^{\otimes}$ and $(\mathsf{BTop}(n)_{/\mathrm{M}})^{\sqcup}$.

The functor (1.5) is defined as the composition of the forgetful functor $\mathsf{Disk}_{n/\mathrm{M}} \to \mathsf{Disk}_n$ with the faithful embedding of *topological* categories[2]

$$\mathsf{Disk}_n \to \mathsf{BTop}(n)^{\otimes}. \tag{1.7}$$

The functor (1.6) is adjoint to a functor

$$\mathsf{Disk}_{n/\mathrm{M}} \times_{\mathsf{Fin}_*} \Gamma^* \to \mathsf{BTop}(n)_{/\mathrm{M}}, \tag{1.8}$$

where Γ^* denotes the category of pairs (S, i), with $\mathrm{S} \in \mathsf{Fin}_*$ and $i \in \mathrm{S}^{\circ}$ (*cf.* [Lur17, Construction 2.4.3.1]). The functor (1.8) sends $(\mathrm{U} \to \mathrm{M}, i)$ to the restriction of $\mathrm{U} \to \mathrm{M}$ to the connected component of U corresponding to i.

To identify the compositions of (1.5) and (1.6) with the natural functors to $\mathsf{BTop}(n)^{\sqcup}$, we observe that the composition of (1.8) with the forgetful functor to $\mathsf{BTop}(n)$ factors as in the following commutative diagram

$$\begin{array}{ccc} \mathsf{Disk}_{n/\mathrm{M}} \times_{\mathsf{Fin}_*} \Gamma^* & \to & \mathsf{BTop}(n)_{/\mathrm{M}} \\ \downarrow & & \downarrow \\ \mathsf{Disk}_n \times_{\mathsf{Fin}_*} \Gamma^* & \longrightarrow & \mathsf{BTop}(n) \end{array}$$

Here, the bottom horizontal arrow is defined by evaluation at i. By construction, it is adjoint to the composition of (1.7) with the functor $\mathsf{BTop}(n)^{\otimes} \to \mathsf{BTop}(n)^{\sqcup}$.

[1] The functor (1.4) appears implicitly in the proof of [Lur17, Theorem 5.5.2.5], but we could not locate its definition in *op.cit.*.

[2] The topological category defining $\mathsf{BTop}(n)^{\otimes}$ is denoted by ${}^t\mathbb{E}^{\otimes}_{\mathsf{BTop}(n)}$ in [Lur17, Definition 5.4.2.1] and (1.7) identifies Disk_n with its faithful subcategory consisting of all objects and active morphisms.

1.1.9 *Construction of* (1.3)**.** The functor (1.3) is the composition of the restriction along (1.4) with the functor of taking colimits over $\mathsf{Disk}_{n/\mathrm{M}}$. In other words, we have

$$\int_{\mathrm{M}} \mathcal{A} := \operatorname*{colim}_{\mathrm{U}\in\mathsf{Disk}_{n/\mathrm{M}}} \mathcal{A}(\mathrm{U}).$$

The fact that this colimit exists (assuming that $\mathcal{O}$ is sifted-complete) is because the ∞-category $\mathsf{Disk}_{n/\mathrm{M}}$ is sifted (*cf.* [Lur17, Proposition 5.5.2.15]).

Remark 1.1.10. By [Lur17, Proposition 5.5.2.17(2)], (1.3) is functorial in $\mathcal{O}$: Given sifted-complete symmetric monoidal ∞-categories $\mathcal{O}_1$, $\mathcal{O}_2$ and a symmetric monoidal functor $\mathcal{O}_1 \to \mathcal{O}_2$ commuting with sifted colimits, we have a commutative square

$$\begin{array}{ccc} \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O}_1) & \xrightarrow{\int_{\mathrm{M}}} & \mathcal{O}_1 \\ \downarrow & & \downarrow \\ \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O}_2) & \xrightarrow{\int_{\mathrm{M}}} & \mathcal{O}_2 \end{array}$$

Remark 1.1.11. We shall endow the ∞-category $\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O})$ with the symmetric monoidal structure defined by pointwise tensor product (*cf.* [Lur17, Example 3.2.4.4]).

Suppose that $\mathcal{O}$ is sifted-complete. Then the functor of factorization homology (1.3) is symmetric monoidal by [Lur17, Theorem 5.5.3.2].

1.1.12. Let $\mathcal{O}$ be a sifted-complete symmetric monoidal ∞-category. We shall prove that (1.3) commutes with the *relative* tensor product.

Namely, given an associative algebra $\mathcal{A}$ in $\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O})$ and right (respectively, left) $\mathcal{A}$-module $\mathcal{B}_1$ (respectively, $\mathcal{B}_2$), we may form the relative tensor product as geometric realization of the Bar complex (*cf.* [Lur17, Definition 4.4.2.10])

$$\mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2 := \operatorname{colim} \operatorname{Bar}_{\mathcal{A}}(\mathcal{B}_1, \mathcal{B}_2)_\bullet.$$

Lemma 1.1.13. *There is a natural isomorphism*

$$\int_{\mathrm{M}} \mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2 \simeq \int_{\mathrm{M}} \mathcal{B}_1 \otimes_{\int_{\mathrm{M}} \mathcal{A}} \int_{\mathrm{M}} \mathcal{B}_2. \tag{1.9}$$

Proof. Since (1.3) is symmetric monoidal (*cf.* Remark 1.1.11), it suffices to show that (1.3) commutes with sifted colimits. By construction, it suffices to show that the functor of pre-composition with (1.4)

$$\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O}) \to \mathrm{Fun}(\mathsf{Disk}_{n/\mathrm{M}}, \mathcal{O})$$

commutes with sifted colimits.

Since colimits in $\mathrm{Fun}(\mathsf{Disk}_{n/\mathrm{M}}, \mathcal{O})$ are formed pointwise, this assertion follows from [Lur17, Proposition 3.2.3.1]. □

1.1.14. Finally, we recall the computation of factorization homology of (families of) commutative algebras.

Let $\mathcal{O}$ be a symmetric monoidal ∞-category. Write $\mathsf{CAlg}(\mathcal{O})$ for the ∞-category of commutative algebras in $\mathcal{O}$. The pointwise tensor structure on $\mathsf{CAlg}(\mathcal{O})$ coincides with the co-Cartesian symmetric monoidal structure (*cf.* [Lur17, Proposition 3.2.4.7]).

Since the underlying ∞-category of $\mathbb{E}_{\mathrm{M}}^{\otimes}$ is identified with $\operatorname{Sing} \mathrm{M}$ (*cf.* Remark 1.1.3), we have a canonical equivalence of ∞-categories (*cf.* [Lur17, Proposition 2.4.3.9])

$$\mathrm{Fun}(\operatorname{Sing} \mathrm{M}, \mathsf{CAlg}(\mathcal{O})) \simeq \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{CAlg}(\mathcal{O})). \tag{1.10}$$

Given a functor $\mathcal{A} : \operatorname{Sing} \mathrm{M} \to \mathsf{CAlg}(\mathcal{O})$, we shall denote its image under (1.10) by $\mathcal{A}_{\mathrm{M}}$.

Lemma 1.1.15. *Suppose that $\mathcal{O}$ is cocomplete. Given a functor $\mathcal{A} : \operatorname{Sing} \mathrm{M} \to \mathsf{CAlg}(\mathcal{O})$, there is a canonical isomorphism in $\mathsf{CAlg}(\mathcal{O})$:*

$$\int_{\mathrm{M}} \mathcal{A}_{\mathrm{M}} \simeq \operatorname{colim} \mathcal{A}. \tag{1.11}$$

Proof. This is a reformulation of (the proof of) [Lur17, Theorem 5.5.3.8]. □

1.2. Nonabelian Poincaré duality.

1.2.1. Denote by Spc the ∞-category of ∞-groupoids, endowed with the Cartesian symmetric monoidal structure. Denote by Spc_* the ∞-category of pointed ∞-groupoids.

By unstraightening, each functor $\mathcal{X} : \operatorname{Sing} \mathrm{M} \to \mathsf{Spc}_*$ may be regarded as a Kan fibration over $\operatorname{Sing} \mathrm{M}$, endowed with a neutral section. For any ∞-groupoid Y over $\operatorname{Sing} \mathrm{M}$, we write $\Gamma(\mathrm{Y}, \mathcal{X})$ for the pointed space $\operatorname{Maps}_{/ \operatorname{Sing} \mathrm{M}}(\mathrm{Y}, \mathcal{X})$.

Given a functor $\mathcal{X} : \operatorname{Sing} \mathrm{M} \to \mathsf{Spc}_*$ and an open subset $\mathrm{U} \subset \mathrm{M}$, we have the ∞-groupoid of *compactly supported sections* of $\mathcal{X}$ over U:

$$\Gamma_c(\mathrm{U}, \mathcal{X}) := \operatorname*{colim}_{\mathrm{K} \subset \mathrm{U}} \Gamma(\operatorname{Sing} \mathrm{M}, \mathcal{X}) \times_{\Gamma(\operatorname{Sing} \mathrm{M} \smallsetminus \mathrm{K}, \mathcal{X})} *, \tag{1.12}$$

where the colimit is taken over the poset of compact subsets K of U. The expression (1.12) depends (covariantly) functorially on U and on $\mathcal{X}$.

1.2.2. By [Lur17, Definition 5.5.6.2, Remark 5.5.6.3], there is a functor of ∞-categories

$$\Omega_{\mathrm{M}} : \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathsf{Spc}_*) \to \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Spc}) \tag{1.13}$$

extending (1.12) in the following sense: Given any $x \in \operatorname{Sing} \mathrm{M}$, corresponding to an object $\mathrm{U} \to \mathrm{M}$ of the underlying ∞-category of $\mathbb{E}_{\mathrm{M}}^{\otimes}$ (*cf.* Remark 1.1.3), the functor ev_x of taking fiber at x (*cf.* §1.1.6) renders the diagram below commute:

$$\begin{array}{ccc} \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathsf{Spc}_*) & \xrightarrow{\Omega_{\mathrm{M}}} & \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Spc}) \\ & \searrow^{\Gamma_c(\mathrm{U}, \cdot)} & \downarrow \operatorname{ev}_x \\ & & \mathsf{Spc} \end{array}$$

Lemma 1.2.3. *The functor* (1.13) *commutes with finite limits.*

Proof. The functors ev_x preserve limits and are jointly conservative when taken over all $x \in \operatorname{Sing} \mathrm{M}$. Therefore, it suffices to prove that each functor

$$\Gamma_c(\mathrm{U}, \cdot) : \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathsf{Spc}_*) \to \mathsf{Spc}$$

preserves finite limits.

This holds because filtered colimits and finite limits commute in Spc. □

Remark 1.2.4. It follows from Lemma 1.2.3 that the functor (1.13) is symmetric monoidal with respect to the Cartesian symmetric monoidal structures.

In particular, it induces a functor

$$\Omega_{\mathrm{M}} : \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathsf{CAlg}(\mathsf{Spc})) \to \mathsf{CAlg}(\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Spc})). \tag{1.14}$$

The target of (1.14) is equivalent to $\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{CAlg}(\mathsf{Spc}))$, as both ∞-categories consist of algebra objects over the tensor product ∞-operad (*cf.* [Lur17, Proposition 2.2.5.6]).

1.2.5. Next, we shall recall the statement of nonabelian Poincaré duality, due to Salvatore, Segal, and Lurie (*cf.* [Sal01, Seg10, Lur17]).

Proposition 1.2.6 (Nonabelian Poincaré duality)**.** *Let* $\mathcal{X} : \operatorname{Sing} \mathrm{M} \to \mathsf{Spc}_*$ *be a functor valued in* n*-connective* ∞*-groupoids. Then there is a canonical isomorphism*

$$\int_{\mathrm{M}} \Omega_{\mathrm{M}}(\mathcal{X}) \simeq \Gamma_c(\mathrm{M}, \mathcal{X}). \tag{1.15}$$

Proof. This is [Lur17, Theorem 5.5.6.6]. □

1.2.7 *Local trace map***.** In the remainder of this subsection, we will explain how (1.15) interacts with the trace maps in *abelian* Poincaré duality, when an orientation is provided. For a more complete treatment, see [AF20, §4].

Denote by $\mathbf{Z}$-mod the stable ∞-category of $\mathrm{H}\mathbf{Z}$-module spectra. Forgetting the $\mathrm{H}\mathbf{Z}$-action and applying connective truncation, we obtain a functor

$$\mathbf{Z}\text{-}\mathsf{mod} \to \mathsf{CAlg}(\mathsf{Spc}). \tag{1.16}$$

Let $\mathcal{A} : \operatorname{Sing} \mathrm{M} \to \mathbf{Z}$-$\mathsf{mod}$ be a functor. Applying (1.14) and (1.10) to the composition of $\mathcal{A}$ with (1.16), we obtain $\mathbb{E}_{\mathrm{M}}$-algebras $\Omega_{\mathrm{M}}(\mathcal{A})$, respectively $\mathcal{A}_{\mathrm{M}}$ in $\mathsf{CAlg}(\mathsf{Spc})$. When M is equipped with a ($\mathbf{Z}$-)orientation, they are related as follows:

$$\tau^{\mathrm{loc}}_{\mathrm{M}} : \Omega_{\mathrm{M}}(\mathcal{A}) \simeq (\Omega^n \mathcal{A})_{\mathrm{M}}, \tag{1.17}$$

We shall refer to (1.17) as the *local trace map*.

1.2.8 *Construction of* (1.17)**.** We shall use a linear version of the functor (1.13) (*cf.* the proof of [Lur17, Proposition 5.5.6.16]).

Namely, for any stable ∞-category $\mathcal{O}$ admitting limits and colimits, there is a functor

$$\Omega_{\mathrm{M}} : \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathcal{O}) \to \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O}), \tag{1.18}$$

where $\mathcal{O}$ is endowed with the Cartesian symmetric monoidal structure. For $\mathcal{O} := \mathbf{Z}$-$\mathsf{mod}$, the same-named functors (1.18) and (1.14) are related by the commutative square

$$\begin{array}{ccc} \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathbf{Z}\text{-}\mathsf{mod}) & \xrightarrow{\Omega_{\mathrm{M}}} & \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathbf{Z}\text{-}\mathsf{mod}) \\ \downarrow{\scriptstyle (1.16)} & & \downarrow{\scriptstyle (1.16)} \\ \operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathsf{CAlg}(\mathsf{Spc})) & \xrightarrow{\Omega_{\mathrm{M}}} & \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{CAlg}(\mathsf{Spc})) \end{array}$$

Since the symmetric monoidal structure on $\mathcal{O}$ is also co-Cartesian, (1.18) may be viewed as an endofunctor of $\operatorname{Fun}(\operatorname{Sing} \mathrm{M}, \mathcal{O})$ (*cf.* [Lur17, Proposition 2.4.3.9]). It remains to identify (1.18) with $[-n]$ for $\mathcal{O} := \mathbf{Z}$-$\mathsf{mod}$, given an orientation of M.

The desired identification is obtained from the natural isomorphisms

$$\begin{aligned} \Gamma_c(\mathrm{U}, \mathcal{A}) &\simeq \Gamma_c(\mathrm{U}, \mathbf{Z}[n]) \otimes \mathcal{A}_x[-n] \\ &\simeq \mathbf{Z} \otimes \mathcal{A}_x[-n] \simeq \mathcal{A}_x[-n] \end{aligned}$$

for any $x \in \operatorname{Sing} \mathrm{M}$, with corresponding object $\mathrm{U} \to \mathrm{M}$ in $\mathsf{BTop}(n)_{/\mathrm{M}}$ (*cf.* Remark 1.1.3). Here, the identification $\Gamma_c(\mathrm{U}, \mathbf{Z}[n]) \simeq \mathbf{Z}$ is provided by the orientation of M, which is evidently functorial in $\mathrm{U} \to \mathrm{M}$.

1.2.9. In the context of §1.2.7, we may apply factorization homology (1.3) to the local trace map (1.17). This yields the integrated local trace map $\int_{\mathrm{M}} \tau^{\mathrm{loc}}_{\mathrm{M}}$.

Suppose that $\mathcal{A}$ is n-connective as a Spc_*-valued functor. Then we may apply Lemma 1.1.15 and Proposition 1.2.6 to obtain a commutative diagram in $\mathsf{CAlg}(\mathsf{Spc})$:

$$\begin{array}{ccc} \int_{\mathrm{M}} \Omega_{\mathrm{M}}(\mathcal{A}) & \xrightarrow{\int_{\mathrm{M}} \tau^{\mathrm{loc}}_{\mathrm{M}}} & \int_{\mathrm{M}} (\Omega^n \mathcal{A})_{\mathrm{M}} \\ \big\downarrow {\scriptstyle (1.15)} & & \big\downarrow {\scriptstyle (1.11)} \\ \Gamma_c(\mathrm{M}, \mathcal{A}) & \xrightarrow{\simeq} & \operatorname{colim} \Omega^n \mathcal{A} \end{array} \tag{1.19}$$

where all arrows are isomorphisms. Here, the fact that (1.15) lifts to an isomorphism in $\mathsf{CAlg}(\mathsf{Spc})$ comes from the commutation of $\int_{\mathrm{M}} \Omega_{\mathrm{M}}(\cdot)$ and $\Gamma_c(\mathrm{M}, \cdot)$ with finite products, by Lemma 1.2.3 and the commutation of sifted colimits with finite products in Spc.

1.2.10 *Global trace map***.** Consider the constant functor $\mathcal{A}$ with values in $\mathrm{A}[k] \in \mathbf{Z}\text{-}\mathsf{mod}$, for an abelian group A and an integer $k \geq n$. In this case, we have a map in $\mathsf{CAlg}(\mathsf{Spc})$:

$$\operatorname{colim} \Omega^n (\mathsf{B}^k \mathrm{A}) \simeq \operatorname{colim} \mathsf{B}^{k-n} \mathrm{A} \to \mathsf{B}^{k-n} \mathrm{A}. \tag{1.20}$$

We define the *global trace map* to be the composition of the lower horizontal isomorphism in (1.19) with (1.20):

$$\tau^{\mathrm{glob}}_{\mathrm{M}} : \Gamma_c(\mathrm{M}, \mathsf{B}^k \mathrm{A}) \to \mathsf{B}^{k-n} \mathrm{A}. \tag{1.21}$$

From the commutative square (1.19), we deduce the following compatibility between the local and global trace maps:

$$\begin{array}{ccc} \int_{\mathrm{M}} \Omega_{\mathrm{M}}(\mathsf{B}^k \mathrm{A}) & \xrightarrow{\int_{\mathrm{M}} \tau^{\mathrm{loc}}_{\mathrm{M}}} & \int_{\mathrm{M}} (\mathsf{B}^{k-n} \mathrm{A})_{\mathrm{M}} \\ \big\downarrow {\scriptstyle (1.15)} & & \big\downarrow {\scriptstyle (1.20)\circ(1.11)} \\ \Gamma_c(\mathrm{M}, \mathsf{B}^k \mathrm{A}) & \xrightarrow{\tau^{\mathrm{glob}}_{\mathrm{M}}} & \mathsf{B}^{k-n} \mathrm{A} \end{array} \tag{1.22}$$

1.3. Coefficients.

1.3.1. Denote by Pr^{L} the ∞-category of presentable ∞-categories with colimit-preserving functors. It admits a symmetric monoidal structure given by the Lurie tensor product.

Denote by Vect the ∞-category of $\mathrm{H}\mathbf{C}$-module spectra, viewed as a commutative algebra object in Pr^{L}. Its module objects in Pr^{L} are called *DG categories*:

$$\mathsf{DGCat} := \mathsf{Vect}\text{-}\mathsf{mod}(\mathsf{Pr}^{\mathrm{L}}).$$

Being a module ∞-category, DGCat inherits a symmetric monoidal structure given by the tensor product relative to Vect.

1.3.2. Given an ∞-groupoid Y, we define the ∞-category of $\mathbf{C}$*-local systems* over Y to be the limit of the constant diagram

$$\mathrm{LS}(\mathrm{Y}) := \lim_{\mathrm{Y}} \mathsf{Vect}.$$

The assignment $\mathrm{Y} \mapsto \mathrm{LS}(\mathrm{Y})$ organizes into a functor

$$\mathsf{Spc}^{\mathrm{op}} \to \mathsf{DGCat}. \tag{1.23}$$

We denote the image of a morphism $f : \mathrm{Y}_1 \to \mathrm{Y}_2$ under (1.23) by $f^\dagger : \mathrm{LS}(\mathrm{Y}_2) \to \mathrm{LS}(\mathrm{Y}_1)$.

In fact, the functor (1.23) is right adjoint to the functor $\mathsf{DGCat} \to \mathsf{Spc}^{\mathrm{op}}$ sending $\mathcal{C}$ to $\mathrm{Maps}_{\mathsf{DGCat}}(\mathcal{C}, \mathsf{Vect})$. This implies that (1.23) preserves limits.

Remark 1.3.3. For each $\mathrm{M} \in \mathsf{Mfd}_n$, we may consider the DG category $\mathsf{Shv}_{\{0\}}(\mathrm{M})$ of sheaves of $\mathbf{C}$-vector spaces over M whose cohomology sheaves are locally constant. Then there is a canonical equivalence of DG categories

$$\mathsf{Shv}_{\{0\}}(\mathrm{M}) \simeq \mathrm{LS}(\operatorname{Sing} \mathrm{M}),$$

according to [GKRV22, Lemma A.4.2].

1.3.4. By [GKRV22, §1.4], the DG category $\mathrm{LS}(\mathrm{Y})$ is also *covariantly* functorial in $\mathrm{Y} \in \mathsf{Spc}$. More precisely, for each morphism $f : \mathrm{Y}_1 \to \mathrm{Y}_1$ in Spc, the functor $f^\dagger$ admits a left adjoint $f_\dagger$. Thus, we obtain a functor

$$\mathsf{Spc} \to \mathsf{DGCat} \tag{1.24}$$

by passing to left adjoints in (1.23).

Moreover, the canonical self-duality of Vect induces a canonical self-duality of $\mathrm{LS}(\mathrm{Y})$ for each $\mathrm{Y} \in \mathsf{Spc}$. Under this self-duality, $f_\dagger$ is identified with the dual of $f^\dagger$ for every morphism f in Spc. Thus, (1.24) may also be obtained from (1.23) by passing to duals and applying the canonical self-duality.

Lemma 1.3.5. *The functor* (1.24) *commutes with colimits.*

Proof. Given a diagram $\mathrm{I} \to \mathsf{Spc}$, $i \mapsto \mathrm{Y}_i$, the DG category $\operatorname{colim}_{i \in \mathrm{I}} \mathrm{LS}(\mathrm{Y}_i)$ is dualizable (*cf.* [GKRV22, Lemma 1.4.8(d)]). Moreover, its dual is canonically identified with $\lim_{i \in \mathrm{I}} \mathrm{LS}(\mathrm{Y}_i)$, with transition functors given by pullbacks.

The latter is identified wtih $\mathrm{LS}(\operatorname{colim}_{i \in \mathrm{I}} \mathrm{Y}_i)$ since (1.23) commutes with limits. □

1.3.6. Note that (1.23) admits a lax symmetric monoidal structure given by the external tensor product construction.

For $\mathrm{Y}_1, \mathrm{Y}_2 \in \mathsf{Spc}$, the lax symmetric monoidal structure supplies a functor

$$\mathrm{LS}(\mathrm{Y}_1) \otimes \mathrm{LS}(\mathrm{Y}_2) \to \mathrm{LS}(\mathrm{Y}_1 \times \mathrm{Y}_2), \quad \mathcal{A}_1, \mathcal{A}_2 \mapsto \mathcal{A}_1 \boxtimes \mathcal{A}_2. \tag{1.25}$$

Lemma 1.3.7. *The functor* (1.23) *(hence* (1.24)*) is symmetric monoidal.*

Proof. It suffices to prove that (1.25) is an equivalence.

Under the canonical identifications

$$\mathrm{LS}(\mathrm{Y}_1 \times \mathrm{Y}_2) \simeq \mathrm{LS}(\lim_{\mathrm{Y}_2} \mathrm{Y}_1) \simeq \lim_{\mathrm{Y}_2} \mathrm{LS}(\mathrm{Y}_1),$$

the functor (1.25) corresponds to [GKRV22, (1.17)] for $\mathcal{C} \coloneqq \mathrm{LS}(\mathrm{Y}_1)$ and $\mathrm{Y} \coloneqq \mathrm{Y}_2$. Thus the result follows from [GKRV22, Proposition 1.4.10]. □

1.3.8 *Tautological character***.** Finally, we shall define a categorical analogue of the tautological character local system on the classifying space of $\mathbf{C}^\times$.

Let us view $\mathbf{C}^\times$ as a discrete abelian group and its deloop $\mathsf{B}\mathbf{C}^\times$ as a commutative algebra in Spc. Since (1.23) is symmetric monoidal (*cf.* Lemma 1.3.7), the tautological character local system on $\mathsf{B}\mathbf{C}^\times$ may be viewed as a morphism $\mathsf{Vect} \to \mathrm{LS}(\mathsf{B}\mathbf{C}^\times)$ of cocommutative coalgebras in DGCat.

Dualizing, we obtain a morphism in $\mathsf{CAlg}(\mathsf{DGCat})$:

$$\chi : \mathrm{LS}(\mathsf{B}\mathbf{C}^\times) \to \mathsf{Vect}, \tag{1.26}$$

which we call the *(categorical) tautological character*.

2. Local constructions

In this section, we define the "quantum torus", or rather its category of representations $\mathrm{Rep}_q(\check{\mathrm{T}})$ as an $\mathbb{E}_{\mathrm{M}}$-algebra in DGCat, for an oriented 2-manifold M (*cf.* §2.2). The definition is simple, but not quite explicit. In §2.3-2.4, we make $\mathrm{Rep}_q(\check{\mathrm{T}})$ more explicit by showing that it is the derived ∞-category of its heart $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$, which may be viewed as a *twisted* family of braided monoidal categories over M, and computing its local invariants.

The material of §2.3 and §2.4 will *not* be used later, so the reader only is only interested in the factorization homology of $\mathrm{Rep}_q(\check{\mathrm{T}})$ may safely skip them.

2.1. Betti levels.

2.1.1. Given a $\mathbf{C}$-scheme Y of finite type, we write $\mathrm{Y}^{\mathrm{top}}$ for the topological space underlying the analytification of Y and $\mathrm{Sing}\,\mathrm{Y}^{\mathrm{top}}$ for its homotopy type (*cf.* §1.1.1).

The association $\mathrm{Y} \mapsto \mathrm{Sing}\,\mathrm{Y}^{\mathrm{top}}$ determines a functor

$$\mathrm{Sing}(\cdot)^{\mathrm{top}} : \mathsf{Sch}_{\mathrm{ft}} \to \mathsf{Spc}, \tag{2.1}$$

where $\mathsf{Sch}_{\mathrm{ft}}$ denotes the category of $\mathbf{C}$-schemes of finite type.

Remark 2.1.2. The functor (2.1) does *not* commute with finite limits. More precisely, the functor $\mathrm{Y} \mapsto \mathrm{Y}^{\mathrm{top}}$ from $\mathsf{Sch}_{\mathrm{ft}}$ to the category of topological spaces commutes with finite limits (*cf.* [GR03, Exposé XII, §1.2]), but the functor Sing does not.

However, given morphisms $\mathrm{Y}_1 \to \mathrm{Y} \leftarrow \mathrm{Y}_2$ in $\mathsf{Sch}_{\mathrm{ft}}$ where $\mathrm{Y}_1 \to \mathrm{Y}$ induces a Serre fibration $(\mathrm{Y}_1)^{\mathrm{top}} \to \mathrm{Y}^{\mathrm{top}}$, then the natural map in Spc is an isomorphism:

$$\mathrm{Sing}(\mathrm{Y}_1 \times_{\mathrm{Y}} \mathrm{Y}_2)^{\mathrm{top}} \simeq \mathrm{Sing}(\mathrm{Y}_1)^{\mathrm{top}} \times_{\mathrm{Sing}\,\mathrm{Y}^{\mathrm{top}}} \mathrm{Sing}(\mathrm{Y}_2)^{\mathrm{top}}.$$

2.1.3. Let X be a smooth $\mathbf{C}$-curve and G be a reductive group X-scheme. We shall define the ∞-groupoid Level(G) of "Betti levels" of G in this context.

Since $\mathrm{G}^{\mathrm{top}} \to \mathrm{X}^{\mathrm{top}}$ is a Serre fibration, $\mathrm{Sing}\,\mathrm{G}^{\mathrm{top}} \to \mathrm{Sing}\,\mathrm{X}^{\mathrm{top}}$ inherits a group structure from G (*cf.* Remark 2.1.2). We view $\mathsf{B}\,\mathrm{Sing}\,\mathrm{G}^{\mathrm{top}}$ as an ∞-groupoid over $\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}}$, equipped with a neutral section $\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}} \to \mathsf{B}\,\mathrm{Sing}\,\mathrm{G}^{\mathrm{top}}$.

Define the ∞-groupoid Level(G) of *Betti levels* of G to be:

$$\mathrm{Level}(\mathrm{G}) := \mathrm{Maps}_{\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}}/}(\mathsf{B}\,\mathrm{Sing}\,\mathrm{G}^{\mathrm{top}}, \mathsf{B}^4\mathbf{C}^{\times}), \tag{2.2}$$

where $\mathsf{B}^4\mathbf{C}^{\times}$ is viewed as an ∞-groupoid under $\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}}$ via the neutral point.

Remark 2.1.4. Denote by $\mathrm{K}(\mathrm{G}^{\mathrm{top}}, 1)$ the classifying space of $\mathrm{G}^{\mathrm{top}}$ relative to $\mathrm{X}^{\mathrm{top}}$.

The definition of Level(G) renders it a "categorification" of the reduced cohomology group $\mathrm{H}^4_*(\mathrm{K}(\mathrm{G}^{\mathrm{top}}, 1), \mathbf{C}^{\times})$. Namely, for each $n \geq 0$, we have

$$\pi_n \mathrm{Level}(\mathrm{G}) \simeq \mathrm{H}^{4-n}_*(\mathrm{K}(\mathrm{G}^{\mathrm{top}}, 1), \mathbf{C}^{\times}).$$

This definition of Level(G) is inspired by the analogous notions in the étale and de Rham cohomological contexts (*cf.* [GL18], [Zha22]).

To our knowledge, the first authors to suggest that $\mathrm{H}^4_*(\mathrm{K}(\mathrm{G}^{\mathrm{top}}, 1), \mathbf{C}^{\times})$ plays the role of levels for quantum groups are Dijkgraaf and Witten (*cf.* [DW90]).

2.1.5. Let us now specialize to the case where G = T is an X-torus. Denote by Λ the smooth X-scheme representing the sheaf of cocharacters of T.

There is a short exact sequence of topological groups relative to $\mathrm{X}^{\mathrm{top}}$:

$$0 \to \Lambda^{\mathrm{top}} \to \mathrm{Lie}(\mathrm{T})^{\mathrm{top}} \xrightarrow{\exp} \mathrm{T}^{\mathrm{top}} \to 1, \tag{2.3}$$

where Lie(T) denotes the Lie algebra of T and exp the exponential map. The short exact sequence (2.3) identifies $\operatorname{Sing}\mathrm{T}^{\mathrm{top}}$ with $\mathsf{B}\operatorname{Sing}\Lambda^{\mathrm{top}}$. In particular, we obtain

$$\mathrm{Level}(\mathrm{T}) \simeq \mathrm{Maps}_{\operatorname{Sing}\mathrm{X}^{\mathrm{top}}/}(\mathsf{B}^2\operatorname{Sing}\Lambda^{\mathrm{top}}, \mathsf{B}^4\mathbf{C}^\times). \tag{2.4}$$

Remark 2.1.6. The locally constant sheaf of abelian groups Λ^{top} corresponds to a functor

$$\operatorname{Sing}\mathrm{X}^{\mathrm{top}} \to \mathbf{Z}\text{-}\mathsf{mod}. \tag{2.5}$$

To each $x \in \operatorname{Sing}\mathrm{X}^{\mathrm{top}}$, (2.5) assigns a finite free $\mathbf{Z}$-module, which we view as the fiber of Λ at x. (This holds literally when x is defined by a $\mathbf{C}$-point of X.)

Note that the double deloop of (2.5) classifies $\mathsf{B}^2\operatorname{Sing}\Lambda^{\mathrm{top}}$ in the following sense: Its composition with the forgetful functor $\mathbf{Z}\text{-}\mathsf{mod} \to \mathsf{Spc}$ corresponds to $\mathsf{B}^2\operatorname{Sing}\Lambda^{\mathrm{top}}$ under unstraightening. In particular, Level(T) is equivalent to the mapping space from the double deloop of (2.5) to $\mathsf{B}^4\mathbf{C}^\times$ in the ∞-category $\mathrm{Fun}(\operatorname{Sing}\mathrm{X}^{\mathrm{top}}, \mathsf{Spc}_*)$.

2.1.7 *Topological context***.** In view of Remark 2.1.6, we find it convenient to change our context from algebra to topology.

Namely, we shall fix an *oriented* 2-manifold M and a functor $\Lambda : \operatorname{Sing}\mathrm{M} \to \mathbf{Z}\text{-}\mathsf{mod}$ valued in finite free $\mathbf{Z}$-modules. By a “Betti level”, we shall mean a morphism

$$q : \mathsf{B}^2\Lambda \to \mathsf{B}^4\mathbf{C}^\times$$

in the ∞-category $\mathrm{Fun}(\operatorname{Sing}\mathrm{M}, \mathsf{Spc}_*)$.

Applications to the algebraic context (*cf.* §2.1.3) will be obtained by setting $\mathrm{M} := \mathrm{X}^{\mathrm{top}}$, $\Lambda :=$ the functor (2.5), and q a Betti level in the sense of §2.1.3.

2.1.8 *Digression: cohomology of* $\mathsf{B}^2\Gamma$**.** Let Γ be a finite free $\mathbf{Z}$-module.

Write $\mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times)$ for the mapping space in Spc_* and $\mathrm{Maps}_{\mathbf{Z}}(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times)$ for the mapping space in $\mathbf{Z}\text{-}\mathsf{mod}$. Write $\mathrm{Quad}(\Gamma, \mathbf{C}^\times)$ for the abelian group

$$\mathrm{Quad}(\Gamma, \mathbf{C}^\times) := \mathrm{Sym}^2(\check{\Gamma}) \otimes_{\mathbf{Z}} \mathbf{C}^\times.$$

Its elements can be viewed as $\mathbf{C}^\times$-valued quadratic forms on Γ: Given $c\otimes\zeta \in (\check{\Gamma})^{\otimes 2} \otimes_{\mathbf{Z}} \mathbf{C}^\times$, we obtain a quadratic form $\Gamma \to \mathbf{C}^\times$ by the expression $\gamma \mapsto \zeta^{c(\gamma,\gamma)}$, and this expression depends only on the symmetrization of c.

The ∞-groupoid $\mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times)$ has nontrivial homotopy groups in degrees 0 and 2. Its postnikov tower may be identified explicitly as follows:

$$\begin{array}{ccccc}
\mathsf{B}^2\pi_2\,\mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times) & \to & \mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times) & \to & \pi_0\,\mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times) \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq \\
\mathrm{Maps}_{\mathbf{Z}}(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times) & \xrightarrow{\alpha} & \mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times) & \xrightarrow{\beta} & \mathrm{Quad}(\Gamma, \mathbf{C}^\times)
\end{array} \tag{2.6}$$

where α coincides with the map induced by the forgetful functor $\mathbf{Z}\text{-}\mathsf{mod} \to \mathsf{Spc}_*$.

Remark 2.1.9. The map β in (2.6) admits a splitting over the abelian group of $\mathbf{C}^\times$-valued *bilinear forms* on Γ:

$$\begin{array}{ccc}
 & & (\check{\Gamma})^{\otimes 2} \otimes_{\mathbf{Z}} \mathbf{C}^\times \\
 & \swarrow & \downarrow c\otimes\zeta \mapsto (\gamma\mapsto\zeta^{c(\gamma,\gamma)}) \\
\mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{C}^\times) & \xrightarrow{\beta} & \mathrm{Quad}(\Gamma, \mathbf{C}^\times)
\end{array} \tag{2.7}$$

Namely, there is a canonical $\mathbf{Z}$-linear map given by cup product

$$\check{\Gamma}^{\otimes 2} \to \mathrm{Maps}_*(\mathsf{B}^2\Gamma, \mathsf{B}^4\mathbf{Z}), \quad y \otimes z \mapsto (\mathsf{B}^2 y) \cup (\mathsf{B}^2 z),$$

which defines the dotted arrow in (2.7) by tensoring with $\mathbf{C}^\times$.

2.1.10 *Quadratic form.* In the context of §2.1.7, the fiber sequence (2.6) gives rise to fiber sequences functorial in $x \in \mathrm{Sing}\,\mathrm{M}$:

$$\mathrm{Maps}_{\mathbf{Z}}(\mathsf{B}^2\Lambda_x, \mathsf{B}^4\mathbf{C}^\times) \to \mathrm{Maps}_*(\mathsf{B}^2\Lambda_x, \mathsf{B}^4\mathbf{C}^\times) \to \mathrm{Quad}(\Lambda_x, \mathbf{C}^\times),$$

where Λ_x denote the image of x under Λ.

Taking limit over $\mathrm{Sing}\,\mathrm{M}$, we obtain a fiber sequence

$$\mathrm{Maps}_{\mathbf{Z}}(\mathsf{B}^2\Lambda, \mathsf{B}^4\mathbf{C}^\times) \to \mathrm{Maps}_*(\mathsf{B}^2\Lambda, \mathsf{B}^4\mathbf{C}^\times) \to \mathrm{Quad}(\Lambda, \mathbf{C}^\times) \tag{2.8}$$

where the first term denotes the mapping space in $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathbf{Z}\text{-}\mathsf{mod})$, the second term the mapping space in $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathsf{Spc}_*)$, and $\mathrm{Quad}(\Lambda, \mathbf{C}^\times)$ consists of locally constant $\mathbf{C}^\times$-valued quadratic form on Λ.

In particular, every Betti level q induces a locally constant $\mathbf{C}^\times$-valued quadratic form Q on Λ via the second map of (2.8). We call Q the *associated quadratic form* of q. The fiber sequence (2.8) shows that Q is the obstruction of q to be $\mathbf{Z}$-linear.

2.1.11 *Symmetric form.* To each $\mathbf{C}^\times$-valued quadratic form Q on a finite free $\mathbf{Z}$-module Γ, we may associate the symmetric bilinear form

$$b : \Gamma \otimes \Gamma \to \mathbf{C}^\times, \quad \gamma_1, \gamma_2 \mapsto \mathrm{Q}(\gamma_1 + \gamma_2)\mathrm{Q}(\gamma_1)^{-1}\mathrm{Q}(\gamma_2)^{-1}.$$

The symmetric form b vanishes if and only if Q is linear. When this happens, Q must take values in $\{\pm 1\}$.

Thus, every Betti level q of T also has an associated symmetric form $b : \Lambda \otimes \Lambda \to \mathbf{C}^\times$. We shall see that b is the obstruction of q to be commutative, *i.e.* $\mathbb{E}_\infty$-monoidal.

Indeed, write $\mathrm{Maps}_{\mathbb{E}_\infty}(\mathsf{B}^2\Lambda, \mathsf{B}^4\mathbf{C}^\times)$ for the mapping space in $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathsf{CAlg}(\mathsf{Spc}))$. The forgetful functor defines a map of ∞-groupoids

$$\mathrm{Maps}_{\mathbb{E}_\infty}(\mathsf{B}^2\Lambda, \mathsf{B}^4\mathbf{C}^\times) \to \mathrm{Maps}_*(\mathsf{B}^2\Lambda, \mathsf{B}^4\mathbf{C}^\times) \tag{2.9}$$

Lemma 2.1.12. *(1) The map* (2.9) *is fully faithful;*
(2) The essential image of (2.9) *consists precisely of Betti levels q whose symmetric forms b vanish.*

Proof. This is [GL18, Remark 4.6.7]. ☐

2.2. The $\mathbb{E}_\mathrm{M}$-category $\mathrm{Rep}_q(\check{\mathrm{T}})$.

2.2.1. We remain in the context of §2.1.7 and fix a Betti level

$$q : \mathsf{B}^2\Lambda \to \mathsf{B}^4\mathbf{C}^\times. \tag{2.10}$$

In this subsection, we construct the *quantum torus* as an $\mathbb{E}_\mathrm{M}$-algebra (*cf.* §1.1.6) in the symmetric monoidal ∞-category DGCat (*cf.* §1.3.1):

$$\mathrm{Rep}_q(\check{\mathrm{T}}) \in \mathsf{Alg}_{\mathbb{E}_\mathrm{M}}(\mathsf{DGCat}). \tag{2.11}$$

2.2.2. We first recall a special case of a construction in 1.1.14: The equivalence (1.10), when restricted to *constant* functors, yields a functor

$$(\cdot)_\mathrm{M} : \mathsf{CAlg}(\mathcal{O}) \to \mathsf{CAlg}(\mathsf{Alg}_{\mathbb{E}_\mathrm{M}}(\mathcal{O})) \tag{2.12}$$

for any symmetric monoidal ∞-category $\mathcal{O}$. (We identified the target with $\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{CAlg}(\mathcal{O}))$, *cf.* Remark 1.2.4.)

The functor (2.12) is easy to describe: It is the composition of the canonical equivalence $\mathsf{CAlg}(\mathcal{O}) \simeq \mathsf{CAlg}(\mathsf{CAlg}(\mathcal{O}))$ (*cf.* [Lur17, Example 3.2.4.5]) with the forgetful functor.

2.2.3 *Construction of* (2.11)**.** Denote by $\mathcal{H}_q$ the fiber of (2.10), viewed as a functor $\operatorname{Sing}\mathrm{M} \to \mathsf{Spc}_*$ equipped with an action of $\mathsf{B}^3\mathbf{C}^\times \in \mathsf{CAlg}(\mathsf{Spc})$.

Applying the functor (1.13), we obtain

$$\mathcal{H}_q^{\mathrm{loc}} := \Omega_{\mathrm{M}}(\mathcal{H}_q) \in \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Spc})$$

equipped with an action of the commutative algebra (*cf.* Remark 1.2.4)

$$\Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times) \in \mathsf{CAlg}(\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Spc})).$$

We shall identify $\Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times)$ with $(\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}}$ via the local trace map (1.17). (This identification depends on the orientation of M.)

On the other hand, the tautological character (1.26) induces a morphism

$$\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}}) \simeq \mathrm{LS}((\mathsf{B}\mathbf{C}^\times))_{\mathrm{M}} \xrightarrow{\chi_{\mathrm{M}}} \mathsf{Vect}_{\mathrm{M}} \tag{2.13}$$

in $\mathsf{CAlg}(\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{DGCat}))$, where the isomorphism uses the symmetric monoidal structure on LS (*cf.* Lemma 1.3.7) and the second map uses functoriality of (2.12).

We define the $\mathbb{E}_{\mathrm{M}}$-algebra in DGCat:

$$\mathrm{Rep}_q(\check{\mathrm{T}}) := \mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}}) \otimes_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})} \mathsf{Vect}_{\mathrm{M}} \tag{2.14}$$

where $\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})$ is acted on by $\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})$ via the $(\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}}$-action on $\mathcal{H}_q^{\mathrm{loc}}$, and $\mathsf{Vect}_{\mathrm{M}}$ is acted on by $\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})$ via the tautological character (2.13).

Remark 2.2.4. Suppose that q is trivial. Then we have $\mathcal{H}_q \simeq \mathsf{B}^3\mathbf{C}^\times \times \mathsf{B}^2\Lambda$. In particular, $\mathcal{H}_q$ lifts to a functor $\operatorname{Sing}\mathrm{M} \to \mathsf{CAlg}(\mathsf{Spc})$. Applying Ω_{M} and using the local trace map (1.17), we obtain an identification

$$\mathcal{H}_q^{\mathrm{loc}} \simeq (\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}} \times \Lambda_{\mathrm{M}}.$$

This shows that $\mathrm{Rep}_q(\check{\mathrm{T}})$ is identified with $\mathrm{LS}(\Lambda)_{\mathrm{M}}$, *i.e.* the $\mathbb{E}_{\mathrm{M}}$-algebra associated to the functor $\mathrm{LS}(\Lambda) : \operatorname{Sing}\mathrm{M} \to \mathsf{CAlg}(\mathsf{DGCat})$ under (1.10). The latter corresponds to the locally constant sheaf $\mathrm{Rep}(\check{\mathrm{T}})$ of representations of the torus $\check{\mathrm{T}}$ with sheaf of *characters* Λ.

Remark 2.2.5. The formula (2.14) expresses $\mathrm{Rep}_q(\check{\mathrm{T}})$ as the geometric realization of the Bar complex associated to the $\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})$-modules $\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})$ and $\mathsf{Vect}_{\mathrm{M}}$.

By passing to right adjoints, we may also realize $\mathrm{Rep}_q(\check{\mathrm{T}})$ as the totalization of the co-Bar complex associated to the $\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})$-comodules $\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})$ and $\mathsf{Vect}_{\mathrm{M}}$, defined using the pullback functoriality (1.23) of local systems:

$$\begin{aligned}\mathrm{Rep}_q(\check{\mathrm{T}}) &\simeq \operatorname{colim} \mathrm{Bar}_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})}(\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}}), \mathsf{Vect}_{\mathrm{M}}) \\ &\simeq \lim \mathrm{coBar}_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})}(\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}}), \mathsf{Vect}_{\mathrm{M}}).\end{aligned} \tag{2.15}$$

2.2.6 *Fibers of* $\mathrm{Rep}_q(\check{\mathrm{T}})$**.** The limit presentation (2.15) gives a concrete description of the fiber of $\mathrm{Rep}_q(\check{\mathrm{T}})$ at any $x \in \operatorname{Sing}\mathrm{M}$.

Indeed, the fiber of $\mathcal{H}_q^{\mathrm{loc}}$ at x is a $\mathbf{C}^\times$-gerbe over the set Λ_x:

$$\begin{array}{c} \mathcal{H}_{q,x}^{\mathrm{loc}} \\ \big\downarrow \mathsf{B}\mathbf{C}^\times \\ \Lambda_x \end{array}$$

The expression (2.15) identifies $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ as the DG category of local systems on $\mathcal{H}_{q,x}^{\mathrm{loc}}$ which are $\mathsf{B}\mathbf{C}^\times$-equivariant against the tautological character local system.

In particular, $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ admits a Λ_x-grading determined by the support:

$$\mathrm{Rep}_q(\check{\mathrm{T}})_x \simeq \bigoplus_{\lambda\in\Lambda_x} \mathrm{Rep}_q(\check{\mathrm{T}})_x^\lambda,$$

where each summand $\mathrm{Rep}_q(\check{\mathrm{T}})_x^\lambda$ is *non-canonically* equivalent to Vect.

Remark 2.2.7. Denote by b the symmetric form associated to q (*cf.* §2.1.11).

When b vanishes, q lifts to a morphism in $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathsf{CAlg}(\mathsf{Spc}))$ (*cf.* Lemma 2.1.12). This equips $\mathcal{H}_q^{\mathrm{loc}}$ with the structure of a commutative algebra in $\mathsf{Alg}_{\mathbb{E}_\mathrm{M}}(\mathsf{Spc})$, so $\mathrm{Rep}_q(\check{\mathrm{T}})$ also lifts to a commutative algebra in $\mathsf{Alg}_{\mathbb{E}_\mathrm{M}}(\mathsf{DGCat})$.

In §2.4, we shall establish a converse to this assertion: When b is nonvanishing at a point $x \in \mathrm{Sing}\,\mathrm{M}$, the fiber $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ does *not* lift to a commutative algebra.

2.3. t-structure.

2.3.1. We remain in the context of §2.1.7.

In this subsection, we shall explain in what sense $\mathrm{Rep}_q(\check{\mathrm{T}})$ is the derived category of its heart compatibly with the $\mathbb{E}_\mathrm{M}$-algebra structure.

Denote by $\mathsf{Vect}^{\le 0} \subset \mathsf{Vect}$ the full subcategory of connective objects. Denote by $\mathsf{Vect}^\heartsuit$ the heart of $\mathsf{Vect}^{\le 0}$, *i.e.* the abelian category of $\mathbf{C}$-vector spaces.

2.3.2. The ∞-categories $\mathsf{Vect}^{\le 0}$ and $\mathsf{Vect}^\heartsuit$ inherit symmetric monoidal structures from Vect, so we may view them as objects of $\mathsf{CAlg}(\mathsf{Pr}^\mathrm{L})$.

Consider the morphisms in $\mathsf{CAlg}(\mathsf{Pr}^\mathrm{L})$:

$$\mathsf{Vect}^\heartsuit \leftarrow \mathsf{Vect}^{\le 0} \rightarrow \mathsf{Vect}.$$

They induce symmetric monoidal functors

$$\mathsf{Vect}^{\le 0}\text{-}\mathsf{mod}(\mathsf{Pr}^\mathrm{L}) \rightarrow \mathsf{Vect}^\heartsuit\text{-}\mathsf{mod}(\mathsf{Pr}^\mathrm{L}), \quad \mathcal{C} \mapsto \mathcal{C} \otimes_{\mathsf{Vect}^{\le 0}} \mathsf{Vect}^\heartsuit, \tag{2.16}$$

$$\mathsf{Vect}^{\le 0}\text{-}\mathsf{mod}(\mathsf{Pr}^\mathrm{L}) \rightarrow \mathsf{DGCat}, \quad \mathcal{C} \mapsto \mathcal{C} \otimes_{\mathsf{Vect}^{\le 0}} \mathsf{Vect}. \tag{2.17}$$

Remark 2.3.3. Since $\mathsf{Vect}^\heartsuit$ is identified with $\mathsf{Vect}^{\le 0} \otimes \mathsf{Set}$, the functor (2.16) coincides with the functor of taking discrete objects (*cf.* [Lur17, Example 4.8.1.22]).

Since Vect is identified with $\mathsf{Vect}^{\le 0} \otimes \mathsf{Sptr}$, the functor (2.17) coincides with the functor of taking spectrum objects (*cf.* [Lur17, Example 4.8.1.23]).

2.3.4. Given $\mathrm{Y} \in \mathsf{Spc}$, the DG category $\mathrm{LS}(\mathrm{Y})$ carries a t-structure, with connective and coconnective parts defined by

$$\mathrm{LS}(\mathrm{Y})^{\le 0} := \lim_{\mathrm{Y}} \mathsf{Vect}^{\le 0}, \quad \mathrm{LS}(\mathrm{Y})^{\ge 0} := \lim_{\mathrm{Y}} \mathsf{Vect}^{\ge 0}.$$

Note that $f^\dagger$ is t-exact. In particular, its left adjoint $f_\dagger$ is right t-exact. This implies that the functor (1.24) restricts to a functor

$$\mathsf{Spc} \to \mathsf{Vect}^{\le 0}\text{-}\mathsf{mod}(\mathsf{Pr}^{\mathrm{L}}), \quad \mathrm{Y} \mapsto \mathrm{LS}(\mathrm{Y})^{\le 0}. \tag{2.18}$$

Furthermore, (2.18) inherits a symmetric monoidal structure from (1.24) (*cf.* Lemma 2.18) and we recover (1.24) as the composition of (2.18) with the functor of taking spectrum objects (2.17).

2.3.5 *The heart of* $\mathrm{Rep}_q(\check{\mathrm{T}})$**.** The construction of $\mathrm{Rep}_q(\check{\mathrm{T}})$ (*cf.* §2.2.3) may thus be repeated with (2.18) instead of (1.24). This yields an $\mathbb{E}_{\mathrm{M}}$-algebra in $\mathsf{Vect}^{\le 0}\text{-}\mathsf{mod}(\mathsf{Pr}^{\mathrm{L}})$

$$\mathrm{Rep}_q(\check{\mathrm{T}})^{\le 0} := \mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})^{\le 0} \otimes_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})^{\le 0}} (\mathsf{Vect}^{\le 0})_{\mathrm{M}}$$

together with an isomorphism of $\mathbb{E}_{\mathrm{M}}$-algebras in DGCat:

$$\mathrm{Rep}_q(\check{\mathrm{T}}) \simeq \mathrm{Rep}_q(\check{\mathrm{T}})^{\le 0} \otimes_{\mathsf{Vect}^{\le 0}} \mathsf{Vect}. \tag{2.19}$$

Likewise, we may form the symmetric monoidal functor

$$\mathsf{Spc} \to \mathsf{Vect}^{\heartsuit}\text{-}\mathsf{mod}(\mathsf{Pr}^{\mathrm{L}}), \quad \mathrm{Y} \mapsto \mathrm{LS}(\mathrm{Y})^{\heartsuit} \tag{2.20}$$

by composing (2.18) with the functor of taking discrete objects (2.16). The construction of §2.2.3 then yields an $\mathbb{E}_{\mathrm{M}}$-algebra in $\mathsf{Vect}^{\heartsuit}\text{-}\mathsf{mod}(\mathsf{Pr}^{\mathrm{L}})$

$$\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit} := \mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})^{\heartsuit} \otimes_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})^{\heartsuit}} (\mathsf{Vect}^{\heartsuit})_{\mathrm{M}},$$

together with an isomorphism of such:

$$\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit} \simeq \mathrm{Rep}_q(\check{\mathrm{T}})^{\le 0} \otimes_{\mathsf{Vect}^{\le 0}} \mathsf{Vect}^{\heartsuit}.$$

Remark 2.3.6. The observation of Remark 2.2.5 also applies to $\mathrm{Rep}_q(\check{\mathrm{T}})^{\le 0}$ and $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$. Namely, by passing to right adjoints, we obtain equivalences

$$\mathrm{Rep}_q(\check{\mathrm{T}})^{\le 0} \simeq \lim \mathrm{coBar}_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})^{\le 0}}(\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})^{\le 0}, (\mathsf{Vect}^{\le 0})_{\mathrm{M}}), \tag{2.21}$$

$$\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit} \simeq \lim \mathrm{coBar}_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})^{\heartsuit}}(\mathrm{LS}(\mathcal{H}_q^{\mathrm{loc}})^{\heartsuit}, (\mathsf{Vect}^{\heartsuit})_{\mathrm{M}}). \tag{2.22}$$

As in §2.2.6, the expression (2.22) allows us to identify the fiber $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}_x$ at $x \in \mathrm{Sing}\,\mathrm{M}$ as the *abelian* category of local systems on $\mathcal{H}_{q,x}^{\mathrm{loc}}$ which are $\mathsf{B}\mathbf{C}^\times$-equivariant against the tautological character local system.

2.3.7. Next, we shall show that $\mathrm{Rep}_q(\check{\mathrm{T}})^{\le 0}$ (hence $\mathrm{Rep}_q(\check{\mathrm{T}})$, by (2.19)) is completely determined by $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$ as an $\mathbb{E}_{\mathrm{M}}$-algebra. This requires some formalism of derived ∞-categories established by Lurie (*cf.* [Lur18, Appendix C])

Denote by $\mathsf{Groth}_\infty^{\mathrm{sep}}$ the 1-full subcategory of Pr^{L} whose objects are separated Grothendieck prestable ∞-categories and whose morphisms are exact functors.[3]

Denote by Groth_0 the 1-full subcategory of Pr^{L} whose objects are Grothendieck *abelian* categories and whose morphisms are exact functors. (By definition, morphisms in $\mathsf{Groth}_\infty^{\mathrm{sep}}$ and Groth_0 commute with colimits.)

By [Lur18, Theorem C.5.4.9], the functor of taking discrete object

$$(\cdot)^{\heartsuit} : \mathsf{Groth}_\infty^{\mathrm{sep}} \to \mathsf{Groth}_0 \tag{2.23}$$

admits a left adjoint, whose value at $\mathrm{A} \in \mathsf{Groth}_0$ is identified with the connective part of the derived ∞-category $\mathrm{D}(\mathrm{A})^{\le 0}$ (*cf.* [Lur18, Proposition C.5.3.2, Proposition C.5.4.5]).

[3] In *op.cit.*, this ∞-category is denoted by $\mathsf{Groth}_\infty^{\mathrm{lex,sep}}$.

2.3.8. By [Lur18, Theorem C.5.4.16], Groth_0 inherits a symmetric monoidal structure from Pr^{L}. We endow $\mathsf{Groth}_\infty^{\mathrm{sep}}$ with the symmetric monoidal structure given by the *separated* Lurie tensor product (*cf.* [Lur18, Corollary C.4.6.2]).[4]

Consider $\mathsf{Vect}^{\leq 0}$ (respectively $\mathsf{Vect}^{\heartsuit}$) as a commutative algebra in $\mathsf{Groth}_\infty^{\mathrm{sep}}$ (respectively, Groth_0). Since (2.23) is symmetric monoidal, it lifts to a functor

$$(\cdot)^{\heartsuit} : \mathsf{Vect}^{\leq 0}\text{-}\mathsf{mod}(\mathsf{Groth}_\infty^{\mathrm{sep}}) \to \mathsf{Vect}^{\heartsuit}\text{-}\mathsf{mod}(\mathsf{Groth}_0). \tag{2.24}$$

Since (2.23) admits a left adjoint, so does (2.24).

2.3.9. Since $\mathrm{LS}(\mathrm{Y})$ is left complete (in particular, left separated) for any $\mathrm{Y} \in \mathsf{Spc}$ and the pullback functor $f^{\dagger} : \mathrm{LS}(\mathrm{Y}_2) \to \mathrm{LS}(\mathrm{Y}_1)$, for any morphism $f : \mathrm{Y}_1 \to \mathrm{Y}_2$ in Spc, is t-exact, the functor (1.23) induces functors

$$\mathsf{Spc}^{\mathrm{op}} \to \mathsf{Vect}^{\leq 0}\text{-}\mathsf{mod}(\mathsf{Groth}_\infty^{\mathrm{sep}}), \quad \mathrm{Y} \mapsto \mathrm{LS}(\mathrm{Y})^{\leq 0}$$
$$\mathsf{Spc}^{\mathrm{op}} \to \mathsf{Vect}^{\heartsuit}\text{-}\mathsf{mod}(\mathsf{Groth}_0), \quad \mathrm{Y} \mapsto \mathrm{LS}(\mathrm{Y})^{\heartsuit}$$

The expressions (2.21), (2.22) define $\mathrm{Rep}_q(\check{\mathrm{T}})^{\leq 0}$, respectively $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$ as $\mathbb{E}_{\mathrm{M}}$-algebras in $\mathsf{Vect}^{\leq 0}\text{-}\mathsf{mod}(\mathsf{Groth}_\infty^{\mathrm{sep}})$, respectively $\mathsf{Vect}^{\heartsuit}\text{-}\mathsf{mod}(\mathsf{Groth}_0)$.

2.3.10. Denote by L the left adjoint of (2.24). Being the left adjoint of a symmetric monoidal functor, L is *oplax* symmetric monoidal.

We do not know if L is symmetric monoidal in general. Thus, we do not know if the image $\mathrm{L}(\mathcal{A})$ of an $\mathbb{E}_{\mathrm{M}}$-algebra $\mathcal{A}$ in $\mathsf{Vect}^{\heartsuit}\text{-}\mathsf{mod}(\mathsf{Groth}_0)$ automatically carries an $\mathbb{E}_{\mathrm{M}}$-algebra structure. This *does* happen, however, if the leftward arrow in the correspondence

$$\bigotimes_{i\in \mathrm{I}} \mathrm{L}(\mathcal{A}(\mathrm{U}_i)) \leftarrow \mathrm{L}(\bigotimes_{i\in \mathrm{I}} \mathcal{A}(\mathrm{U}_i)) \to \mathrm{L}(\mathcal{A}(\mathrm{U}))$$

associated to any active morphism $\{\mathrm{U}_i\}_{i\in\mathrm{I}} \to \mathrm{U}$ of $\mathbb{E}_{\mathrm{M}}^{\otimes}$ is an isomorphism.

In particular, if $\mathcal{A}$ is *fiberwise semisimple*, *i.e.* $\mathcal{A}_x$ is semisimple for every $x \in \mathrm{Sing}\,\mathrm{M}$, then $\mathrm{L}(\mathcal{A})$ carries a natural $\mathbb{E}_{\mathrm{M}}$-algebra structure.

Since $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$ is fiberwise semisimple (*cf.* §2.2.6), we obtain

$$\mathrm{L}(\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}) \in \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Vect}^{\leq 0}\text{-}\mathsf{mod}(\mathsf{Groth}_\infty^{\mathrm{sep}})).$$

Lemma 2.3.11. *There is a natural isomorphism in* $\mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathsf{Vect}^{\leq 0}\text{-}\mathsf{mod}(\mathsf{Groth}_\infty^{\mathrm{sep}}))$*:*

$$\mathrm{L}(\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}) \simeq \mathrm{Rep}_q(\check{\mathrm{T}})^{\leq 0}. \tag{2.25}$$

Proof. The morphism in one direction is defined by the counit of the adjunction between L and (2.24).

The fact that it is an equivalence may be checked fiberwise, where it follows immediately from the description of §2.2.6. □

Remark 2.3.12. Informally, Lemma 2.3.11 and (2.19) express the fact that $\mathrm{Rep}_q(\check{\mathrm{T}})$ is the derived ∞-category of $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$ "compatibly with the $\mathbb{E}_{\mathrm{M}}$-algebra structures."

Namely, for each $x \in \mathrm{Sing}\,\mathrm{M}$, the DG category $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ is equivalent to the derived ∞-category $\mathrm{D}(\mathrm{Rep}_q(\check{\mathrm{T}})_x^{\heartsuit})$, and the $\mathbb{E}_{\mathrm{M}}$-algebra structure on $\mathrm{Rep}_q(\check{\mathrm{T}})$ is determined by the $\mathbb{E}_{\mathrm{M}}$-algebra structure on $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$.

[4]For both assertions, we invoked the fact that exactness of functors in Pr^{L} is preserved by tensor product (*cf.* [Lur18, Proposition C.4.4.1]).

2.4. The ribbon structure.

2.4.1. We remain in the context of §2.1.7.

The goal of this subsection is to provide interpretations of the discrete invariants Q and b of q (*cf.* §2.1.10, §2.1.11) in terms of the $\mathbb{E}_{\mathrm{M}}$-algebra $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$ (*cf.* §2.3.5).

Informally, $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$ may be viewed as a family of "twisted" braided monoidal categories parametrized by M. Our results say that b controls the square of the commutativity constraint on $\mathrm{Rep}_q(\check{\mathrm{T}})^{\heartsuit}$, while Q controls an additional ribbon structure (*cf.* Proposition 2.4.4, Proposition 2.4.9).

2.4.2. Given an object $\tau \in \mathsf{BTop}(2)$, we define the *$\tau$-twisted $\mathbb{E}_2$-operad* to be

$$\mathbb{E}_{\tau}^{\otimes} := \mathsf{BTop}(2)^{\otimes} \times_{\mathsf{BTop}(2)^{\sqcup}} *^{\sqcup},$$

using the map $*^{\sqcup} \to \mathsf{BTop}(2)^{\sqcup}$ induced from τ.

Any *neutralization* of τ, *i.e.* isomorphism with the neutral point of $\mathsf{BTop}(2)$, induces an isomorphism between $\mathbb{E}_{\tau}^{\otimes}$ and the operad $\mathbb{E}_2^{\otimes}$ (*cf.* [Lur17, Example 5.4.2.15]).

Given $x \in \mathrm{Sing}\,\mathrm{M}$, the map $x : * \to \mathrm{Sing}\,\mathrm{M}$ induces a morphism of ∞-operads

$$\mathbb{E}_{\tau_x}^{\otimes} \to \mathbb{E}_{\mathrm{M}}^{\otimes}, \tag{2.26}$$

where τ_x denotes the image of x under the map $\tau_{\mathrm{M}} : \mathrm{Sing}\,\mathrm{M} \to \mathsf{BTop}(2)$ classifying the tangent microbundle of M (*cf.* Remark 1.1.3).

In particular, given any symmetric monoidal ∞-category $\mathcal{O}$ and $\mathcal{A} \in \mathsf{Alg}_{\mathbb{E}_{\mathrm{M}}}(\mathcal{O})$, restriction along (2.26) defines $\mathcal{A}_x \in \mathsf{Alg}_{\mathbb{E}_{\tau_x}}(\mathcal{O})$, whose underlying object of $\mathcal{O}$ is the fiber of $\mathcal{A}$ at x.

2.4.3. Recall that $\mathbb{E}_2$-algebras in the ∞-category Cat_0 of (1-)categories are precisely braided monoidal categories (*cf.* [Lur17, Example 5.1.2.4]).

Thus, given any $x \in \mathrm{Sing}\,\mathrm{M}$ and a neutralization of τ_x, the fiber $\mathrm{Rep}_q(\check{\mathrm{T}})_x^{\heartsuit}$ admits the structure of a braided monoidal category.

Given two objects $\mathrm{V}^{\lambda_1}, \mathrm{V}^{\lambda_2}$ of $\mathrm{Rep}_q(\check{\mathrm{T}})_x^{\heartsuit}$ with gradings $\lambda_1, \lambda_2 \in \Lambda_x$ (*cf.* §2.2.6), the square of the commutativity constraint

$$\mathrm{V}^{\lambda_1} \otimes \mathrm{V}^{\lambda_2} \simeq \mathrm{V}^{\lambda_2} \otimes \mathrm{V}^{\lambda_1} \simeq \mathrm{V}^{\lambda_1} \otimes \mathrm{V}^{\lambda_2} \tag{2.27}$$

is an automorphism of $\mathrm{V}^{\lambda_1} \otimes \mathrm{V}^{\lambda_2}$.

Proposition 2.4.4. *The automorphism* (2.27) *equals multiplication by* $b(\lambda_1, \lambda_2)$.

2.4.5. We shall prove Proposition 2.4.4 along with an assertion describing a "twisted" ribbon structure on $\mathrm{Rep}_q(\check{\mathrm{T}})_x^{\heartsuit}$ in terms of the quadratic form Q.

We fix a smooth structure on M. This allows us to reduce the tangent microbundle of M to its tangent bundle, classified by a map $\tau_{\mathrm{M}} : \mathrm{Sing}\,\mathrm{M} \to \mathsf{B}\mathrm{GL}_2^+$, where GL_2^+ is the group of orientation-preserving automorphisms of $\mathbf{R}^2$.

Consider the ∞-operad (*cf.* [Lur17, Example 5.4.2.16])

$$\mathbb{E}_{\mathsf{B}\mathrm{GL}_2^+}^{\otimes} := \mathsf{BTop}(2)^{\otimes} \times_{\mathsf{BTop}(2)^{\sqcup}} (\mathsf{B}\mathrm{GL}_2^+)^{\sqcup}.$$

For any $x \in \mathrm{Sing}\,\mathrm{M}$, the map $\tau_x : * \to \mathsf{B}\mathrm{GL}_2^+$ induces a morphism of ∞-operads

$$\mathbb{E}_{\tau_x}^{\otimes} \to \mathbb{E}_{\mathsf{B}\mathrm{GL}_2^+}^{\otimes}. \tag{2.28}$$

We shall lift the $\mathbb{E}_{\tau_x}$-algebra $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ to an $\mathbb{E}_{\mathsf{B}\mathrm{GL}_2^+}$-algebra along (2.28).

Remark 2.4.6. Denote by T_x the oriented 2-dimensional vector space classified by $\tau_x \in \mathsf{BGL}_2^+$. Then its orientation-preserving automorphisms form a topological group $\mathrm{GL}^+(\mathrm{T}_x)$ and we have a canonical isomorphism of ∞-groupoids over $\mathsf{BTop}(2)$:

$$\tau_x/\mathrm{GL}^+(\mathrm{T}_x) \simeq \mathsf{BGL}_2^+.$$

Given a symmetric monoidal ∞-category $\mathcal{O}$, we write $\mathsf{Alg}_{\mathbb{E}_{\tau_x}}(\mathcal{O})^{\mathrm{GL}^+(\mathrm{T}_x)}$ for the ∞-category of $\mathrm{GL}^+(\mathrm{T}_x)$-invariants of $\mathsf{Alg}_{\mathbb{E}_{\tau_x}}(\mathcal{O})$. Since $\mathbb{E}_{\tau_x}^{\otimes}$ defines a family of ∞-operads over $\mathsf{BGL}^+(\mathrm{T}_x)$ with assembly is $\mathbb{E}^{\otimes}_{\mathrm{BGL}_2^+}$ (*cf.* [Lur17, Remark 5.4.2.13]), we obtain

$$\mathsf{Alg}_{\mathbb{E}_{\tau_x}}(\mathcal{O})^{\mathrm{GL}^+(\mathrm{T}_x)} \simeq \mathsf{Alg}_{\mathbb{E}_{\mathrm{BGL}_2^+}}(\mathcal{O}).$$

2.4.7 *Construction of the* $\mathbb{E}_{\mathsf{BGL}_2^+}$*-algebra structure***.** Applying the construction of the quantum torus (*cf.* §2.2.3) with T_x instead of M and the *constant* Betti level $q_x : \mathsf{B}^2\Lambda_x \to \mathsf{B}^4\mathbf{C}^\times$ instead of q, we obtain

$$\mathrm{Rep}_{q_x}(\check{\mathrm{T}}) \in \mathsf{Alg}_{\mathbb{E}_{\mathrm{T}_x}}(\mathsf{DGCat}) \tag{2.29}$$

which recovers $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ under the canonical isomorphism $\mathbb{E}^{\otimes}_{\tau_x} \simeq \mathbb{E}^{\otimes}_{\mathrm{T}_x}$ of ∞-operads.

To lift $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ to $\mathsf{Alg}_{\mathbb{E}_{\mathrm{BGL}_2^+}}(\mathsf{DGCat})$, it thus suffices to endow $\mathrm{Rep}_{q_x}(\check{\mathrm{T}})$ with a $\mathrm{GL}^+(\mathrm{T}_x)$-equivariance structure (*cf.* Remark 2.4.6).

Consider the commutative diagram of $\mathbb{E}_{\mathrm{T}_x}$-algebras in Spc:

$$\begin{array}{ccc} \Omega_{\mathrm{T}_x}(\mathsf{B}^2\Lambda_x) & \xrightarrow{\simeq} & (\Lambda_x)_{\mathrm{T}_x} \\ \downarrow{\scriptstyle \Omega_{\mathrm{T}_x}(q_x)} & & \downarrow \\ \Omega_{\mathrm{T}_x}(\mathsf{B}^4\mathbf{C}^\times) & \xrightarrow{\simeq} & (\mathsf{B}^2\mathbf{C}^\times)_{\mathrm{T}_x} \end{array} \tag{2.30}$$

where the horizontal isomorphisms are given by the local trace maps (*cf.* §1.2.7). By construction, it suffices to endow the right vertical map in (2.30) with a $\mathrm{GL}^+(\mathrm{T}_x)$-equivariance structure, with respect to the natural $\mathrm{GL}^+(\mathrm{T}_x)$-equivariance on $(\Lambda_x)_{\mathrm{T}_x}$ and $(\mathsf{B}^2\mathbf{C}^\times)_{\mathrm{T}_x}$.

Note that the left vertical arrow of (2.30) admits an $\mathrm{GL}^+(\mathrm{T}_x)$-equivariance structure by functoriality with respect to T_x. The desired structure follows because the local trace map (1.17) for T_x is naturally $\mathrm{GL}^+(\mathrm{T}_x)$-equivariant.

2.4.8. We return to the context of §2.4.5.

Since the underlying ∞-category of $\mathbb{E}_{\mathrm{BGL}_2^+}$ is BGL_2^+, the $\mathbb{E}_{\mathrm{BGL}_2^+}$-algebra structure on $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ yields a functor

$$\mathsf{BGL}_2^+ \to \mathsf{DGCat}, \tag{2.31}$$

sending the point τ_x (*not* the neutral point!) to $\mathrm{Rep}_q(\check{\mathrm{T}})_x$.

Let us identify BGL_2^+ with $\mathsf{B}^2\mathbf{Z}$ as objects of Spc_*. This identification is determined by the homotopy equivalences

$$\mathsf{B}\mathbf{Z} \simeq \mathrm{S}^1 \simeq \mathrm{SO}(2) \simeq \mathrm{GL}_2^+.$$

Once a neutralization of $\tau_x \in \mathsf{BGL}_2^+$ is chosen, the generator $1 \in \mathbf{Z}$ defines an automorphism θ of the identity endofunctor on $\mathrm{Rep}_q(\check{\mathrm{T}})_x$ under (2.31). Given $\mathrm{V}^\lambda \in \mathrm{Rep}_q(\check{\mathrm{T}})_x^{\heartsuit}$ with grading $\lambda \in \Lambda_x$, the automorphism θ specializes to an automorphism

$$\theta_{\mathrm{V}^\lambda} : \mathrm{V}^\lambda \simeq \mathrm{V}^\lambda. \tag{2.32}$$

Proposition 2.4.9. *The automorphism* (2.32) *equals multiplication by* $\mathrm{Q}(\lambda)$.

2.4.10. Before we prove Proposition 2.4.9, we relate the local trace map (*cf.* §1.2.7) for $\mathbf{R}^2$ to the SO(2)-action. Namely, consider the (inverse of the) local trace map

$$\mathbf{Z} \simeq \Gamma_c(\mathbf{R}^2, \mathsf{B}^2\mathbf{Z}), \quad 1 \mapsto [\mathbf{R}^2]. \tag{2.33}$$

Writing $\mathrm{B}_r \subset \mathbf{R}^2$ for the closed disk of radius $r \in \mathbf{R}_{>0}$ centered at the origin, we may express $\Gamma_c(\mathbf{R}^2, \mathsf{B}^2\mathbf{Z})$ as the fiber of the map

$$\Gamma(\mathbf{R}^2, \mathsf{B}^2\mathbf{Z}) \to \operatorname*{colim}_{r\to\infty} \Gamma(\mathbf{R}^2 \smallsetminus \mathrm{B}_r, \mathsf{B}^2\mathbf{Z}). \tag{2.34}$$

The inclusion $\mathbf{R}^2 \smallsetminus \mathrm{B}_r \subset \mathbf{R}^2$ is SO(2)-equivariant. Taking the quotient by SO(2) and passing to homotopy types, it gives rise to the neutral map $e : * \to \mathsf{BSO}(2)$. In particular, the fiber of (2.34) is identified with the fiber of $e^* : \mathrm{Maps}(\mathsf{BSO}(2), \mathsf{B}^2\mathbf{Z}) \to \mathsf{B}^2\mathbf{Z}$ via pullback:

$$\mathrm{Maps}_*(\mathsf{BSO}(2), \mathsf{B}^2\mathbf{Z}) \simeq \Gamma_c(\mathbf{R}^2, \mathsf{B}^2\mathbf{Z}). \tag{2.35}$$

Under the isomorphism (2.35), the class $[\mathbf{R}^2]$ corresponds to the canonical isomorphism $\mathsf{BSO}(2) \simeq \mathsf{B}^2\mathbf{Z}$ in Spc_* (*cf.* §2.4.8).

2.4.11. We now turn to the proof of Proposition 2.4.9.

Proof of Proposition 2.4.9. We identify T_x with $\mathbf{R}^2$ using the chosen neutralization of τ_x.

The commutative square (2.30) specializes to a commutative square of ∞-groupoids endowed with SO(2)-action:

$$\begin{array}{ccc} \Gamma_c(\mathbf{R}^2, \mathsf{B}^2\Lambda_x) & \xrightarrow{\simeq} & \Lambda_x \\ \downarrow{\scriptstyle \Gamma_c(\mathbf{R}^2, q_x)} & & \downarrow \\ \Gamma_c(\mathbf{R}^2, \mathsf{B}^4\mathbf{C}^\times) & \xrightarrow{\simeq} & \mathsf{B}^2\mathbf{C}^\times \end{array} \tag{2.36}$$

Here, the horizontal isomorphisms are the local trace maps. The group SO(2) acts on the left column of (2.36) via its action on $\mathbf{R}^2$ and acts trivially on the right column.

Let us express $\theta_{\mathrm{V}^\lambda}$ in terms of the SO(2)-equivariant morphism $\Lambda_x \to \mathsf{B}^2\mathbf{C}^\times$ in (2.36). Indeed, evaluating the latter at λ yields an SO(2)-invariant object of $\mathsf{B}^2\mathbf{C}^\times$, *i.e.* an object of the ∞-groupoid

$$\Gamma_c(\mathbf{R}^2, q_x)(\lambda) \in \mathrm{Maps}(\mathsf{BSO}(2), \mathsf{B}^2\mathbf{C}^\times) \tag{2.37}$$

Under the identification $\mathsf{BSO}(2) \simeq \mathsf{B}^2\mathbf{Z}$ (*cf.* §2.4.8), the class of (2.37) is an element $\theta(\lambda) \in \mathbf{C}^\times$. By construction, the automorphism $\theta_{\mathrm{V}^\lambda}$ acts as multiplication by $\theta(\lambda)$.

It remains to prove the following equality for each $\lambda \in \Lambda_x$:

$$\theta(\lambda) = \mathrm{Q}(\lambda). \tag{2.38}$$

First, we observe that both sides of (2.38) depend linearly on q: For $\theta(\lambda)$, this holds because (2.37) depends linearly on q, while for $\mathrm{Q}(\lambda)$, this holds because (2.6) comes from a fiber sequence in $\mathbf{Z}$-mod. By Remark 2.1.9, we may assume that q is of the form

$$q \simeq (\mathsf{B}^2 y) \cup (\mathsf{B}^2 z),$$

for characters $y : \Lambda_x \to \mathbf{Z}$ and $z : \Lambda_x \to \mathbf{C}^\times$. The value $\mathrm{Q}(\lambda)$ then equals $z(\lambda)^{y(\lambda)}$.

Let us determine (2.37) for this choice of q. Indeed, as an SO(2)-invariant object of $\Gamma_c(\mathbf{R}^2, \mathsf{B}^4\mathbf{C}^\times)$, it is given by the cup product

$$(y(\lambda) \cdot [\mathbf{R}^2]) \cup (z(\lambda) \cdot [\mathbf{R}^2]). \tag{2.39}$$

We may view $z(\lambda)\cdot[\mathbf{R}^2]$ as an SO(2)-invariant object of $\Gamma(\mathbf{R}^2, \mathsf{B}^2\mathbf{C}^\times)$ (*i.e.* forgetting that it is compactly supported) and consequently as an SO(2)-equivariant morphism

$$z(\lambda)\cdot[\mathbf{R}^2] : \mathbf{Z} \to \mathsf{B}^2\mathbf{C}^\times. \tag{2.40}$$

If we forget the SO(2)-equivariance structure on (2.40), then it is simply the H$\mathbf{Z}$-linear morphism $z(\lambda) : \mathbf{Z} \to \mathsf{B}^2\mathbf{C}^\times$. The SO(2)-equivariance structure, however, is determined by $[\mathbf{R}^2]$: It says that the image of the generator $1 \in \mathbf{Z}$ is an SO(2)-invariant object of $\mathsf{B}^2\mathbf{C}^\times$ whose class equals $z(\lambda)$ (*cf.* §2.4.10).

The cup product (2.39) is isomorphic to a Yoneda product, *i.e.* the image of the SO(2)-invariant object $y(\lambda)\cdot[\mathbf{R}^2] \in \Gamma_c(\mathbf{R}^2, \mathsf{B}^2\mathbf{Z})$ under the double deloop of (2.40). By naturality of the local trace map (*cf.* §1.2.7), we have a commutative square

$$\begin{array}{ccc} \Gamma_c(\mathbf{R}^2, \mathsf{B}^2\mathbf{Z}) & \xrightarrow{\simeq} & \mathbf{Z} \\ \downarrow{\scriptstyle z(\lambda)\cdot[\mathbf{R}^2]} & & \downarrow{\scriptstyle z(\lambda)\cdot[\mathbf{R}^2]} \\ \Gamma_c(\mathbf{R}^2, \mathsf{B}^4\mathbf{C}^\times) & \xrightarrow{\simeq} & \mathsf{B}^2\mathbf{C}^\times \end{array}$$

Thus, (2.37) is isomorphic to the image of $y(\lambda) \in \mathbf{Z}$ under the SO(2)-equivariant morphism (2.40). In particular, its class equals $z(\lambda)^{y(\lambda)}$, as desired. □

2.4.12. Finally, we shall deduce Proposition 2.4.4 from Proposition 2.4.9 and standard facts about braided monoidal categories.

Proof of Proposition 2.4.4. Choose a smooth structure on M and a neutralization of τ_x as a point of BGL_2^+.

The construction of §2.4.7 lifts the braided monoidal category $\mathrm{Rep}_q(\check{\mathrm{T}})_x^\heartsuit$ lifts to an $\mathbb{E}_{\mathsf{BGL}_2^+}$-algebra in Cat_0. The latter is precisely a ribbon structure on $\mathrm{Rep}_q(\check{\mathrm{T}})_x^\heartsuit$, whose ribbon twist is provided by the automorphism (2.32) (*cf.* [SW01, §4]).

In particular, this implies that the automorphism (2.27) equals

$$\theta_{\mathrm{V}^{\lambda_1}\otimes\mathrm{V}^{\lambda_2}} \circ (\theta^{-1}_{\mathrm{V}^{\lambda_1}} \otimes \theta^{-1}_{\mathrm{V}^{\lambda_2}}).$$

By Proposition 2.4.9, this is the multiplication by $b(\lambda_1, \lambda_2)$, as desired. □

Remark 2.4.13. In the proof of Proposition 2.4.4, we used the term "ribbon" as in [SW01, Definition 4.9], referring only to additional structure of the twist.

Some authors call this structure "balanced" and reserve the term "ribbon" for *rigid* balanced braided monoidal categories. The subcategory of *compact objects* in $\mathrm{Rep}_q(\check{\mathrm{T}})_x^\heartsuit$ is indeed rigid, hence "ribbon" in the stronger sense.

3. Global constructions

In this section, we calculate the factorization homology of $\mathrm{Rep}_q(\check{\mathrm{T}})$ over an oriented 2-manifold M (*cf.* Theorem 3.1.3) and use it to prove a version of the quantum Betti geometric Langlands conjecture for tori (*cf.* Corollary 3.2.8).

3.1. Calculation of $\int_{\mathrm{M}} \mathrm{Rep}_q(\check{\mathrm{T}})$.

3.1.1. Let M be an oriented 2-manifold. Let $\Lambda : \mathrm{Sing}\,\mathrm{M} \to \mathbf{Z}\text{-}\mathsf{mod}$ taking values in finite free $\mathbf{Z}$-modules and fix a morphism in $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{M}, \mathsf{Spc}_*)$:

$$q : \mathsf{B}^2\Lambda \to \mathsf{B}^4\mathbf{C}^\times. \tag{3.1}$$

In this context, we have defined the $\mathbb{E}_{\mathrm{M}}$-algebra $\mathrm{Rep}_q(\check{\mathrm{T}})$ in DGCat (*cf.* §2.2.3). The goal of this subsection is to compute the factorization homology of $\mathrm{Rep}_q(\check{\mathrm{T}})$ (*cf.* §1.1.9).

3.1.2. Applying the functor $\Gamma_c(\mathrm{M}, \cdot)$ of compactly supported sections (*cf.* §1.2.1) to (3.1), we obtain a morphism in Spc:

$$\Gamma_c(\mathrm{M}, q) : \Gamma_c(\mathrm{M}, \mathsf{B}^2\Lambda) \to \Gamma_c(\mathrm{M}, \mathsf{B}^4\mathbf{C}^\times). \tag{3.2}$$

Composing (3.2) with the global trace map $\tau^{\mathrm{glob}}_{\mathrm{M}}$ (*cf.* §1.2.10, for $\mathrm{A} := \mathbf{C}^\times$ and $k := 4$), we obtain a morphism in Spc:

$$\Gamma_c(\mathrm{M}, \mathsf{B}^2\Lambda) \to \mathsf{B}^2\mathbf{C}^\times \tag{3.3}$$

Denote by $\mathcal{H}^{\mathrm{glob}}_q$ the fiber of (3.3). Thus $\mathcal{H}^{\mathrm{glob}}_q$ is an ∞-groupoid equipped with a $\mathsf{B}\mathbf{C}^\times$-action. In particular, $\mathrm{LS}(\mathcal{H}^{\mathrm{glob}}_q)$ carries an action of $\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)$, using the symmetric monoidal structure on LS (*cf.* Lemma 1.3.7). Viewing Vect as an $\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)$-module via the tautological character χ (1.26), we form

$$\mathrm{LS}_q(\Gamma_c(\mathrm{M}, \mathsf{B}^2\Lambda)) := \mathrm{LS}(\mathcal{H}^{\mathrm{glob}}_q) \otimes_{\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)} \mathsf{Vect}.$$

Theorem 3.1.3. *There is a canonical equivalence in* DGCat*:*

$$\int_{\mathrm{M}} \mathrm{Rep}_q(\check{\mathrm{T}}) \simeq \mathrm{LS}_q(\Gamma_c(\mathrm{M}, \mathsf{B}^2\Lambda)).$$

Proof. Recall that $\mathrm{Rep}_q(\check{\mathrm{T}})$ is defined as the tensor product $\mathrm{LS}(\mathcal{H}^{\mathrm{loc}}_q) \otimes_{\mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}})} \mathsf{Vect}_{\mathrm{M}}$, where $\mathcal{H}^{\mathrm{loc}}_q$ is defined as $\Omega_{\mathrm{M}}(\mathcal{H}_q)$ for $\mathcal{H}_q$ the fiber of (3.1) (*cf.* §2.2.3). Let us rewrite it as the tensor product

$$\mathrm{Rep}_q(\check{\mathrm{T}}) \simeq \mathrm{LS}(\Omega_{\mathrm{M}}(\mathcal{H}_q)) \otimes_{\mathrm{LS}(\Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times))} \mathsf{Vect}_{\mathrm{M}}, \tag{3.4}$$

where $\mathsf{Vect}_{\mathrm{M}}$ is viewed as a $\mathrm{LS}(\Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times))$-module via the composition

$$\mathrm{LS}(\Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times)) \simeq \mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}}) \xrightarrow{\chi_{\mathrm{M}}} \mathsf{Vect}_{\mathrm{M}}, \tag{3.5}$$

where the isomorphism is $\mathrm{LS}(\tau^{\mathrm{loc}}_{\mathrm{M}})$ for the local trace map $\tau^{\mathrm{loc}}_{\mathrm{M}}$ (*cf.* §1.2.7).

Using the presentation (3.4), we compute:

$$\begin{aligned}
\int_{\mathrm{M}} \mathrm{Rep}_q(\check{\mathrm{T}}) &\simeq \int_{\mathrm{M}} \mathrm{LS}(\Omega_{\mathrm{M}}(\mathcal{H}_q)) \otimes_{\int_{\mathrm{M}} \mathrm{LS}(\Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times))} \int_{\mathrm{M}} \mathsf{Vect}_{\mathrm{M}} && \text{(Lemma 1.1.13)} \\
&\simeq \mathrm{LS}(\int_{\mathrm{M}} \Omega_{\mathrm{M}}(\mathcal{H}_q)) \otimes_{\mathrm{LS}(\int_{\mathrm{M}} \Omega_{\mathrm{M}}(\mathsf{B}^3\mathbf{C}^\times))} \mathsf{Vect} && \text{(Remark 1.1.10)} \\
&\simeq \mathrm{LS}(\Gamma_c(\mathrm{M}, \mathcal{H}_q)) \otimes_{\mathrm{LS}(\Gamma_c(\mathrm{M}, \mathsf{B}^3\mathbf{C}^\times))} \mathsf{Vect} && \text{(Proposition 1.2.6)}
\end{aligned}$$

Here, the identification $\int_{\mathrm{M}} \mathsf{Vect}_{\mathrm{M}} \simeq \mathsf{Vect}$ follows from the symmetric monoidal structure on $\int_{\mathrm{M}}$ (*cf.* Remark 1.1.11).

Claim: The $\mathrm{LS}(\Gamma_c(\mathrm{M}, \mathsf{B}^3\mathbf{C}^\times))$-module structure on Vect, appearing in the above expression, is induced from the morphism in $\mathsf{CAlg}(\mathsf{DGCat})$:

$$\mathrm{LS}(\Gamma_c(\mathrm{M}, \mathsf{B}^3\mathbf{C}^\times)) \to \mathrm{LS}(\mathsf{B}\mathbf{C}^\times) \xrightarrow{\chi} \mathsf{Vect} \tag{3.6}$$

where the first morphism is $\mathrm{LS}(\tau^{\mathrm{glob}}_{\mathrm{M}})$ for the global trace map $\tau^{\mathrm{glob}}_{\mathrm{M}}$ (*cf.* §1.2.10).

Indeed, assuming the claim, we obtain the desired isomorphism:

$$\begin{aligned}\int_{\mathrm{M}} \mathrm{Rep}_q(\check{\mathrm{T}}) &\simeq \mathrm{LS}(\Gamma_c(\mathrm{M},\mathcal{H}_q)) \otimes_{\mathrm{LS}(\Gamma_c(\mathrm{M},\mathsf{B}^3\mathbf{C}^\times))} \mathrm{LS}(\mathsf{B}\mathbf{C}^\times) \otimes_{\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)} \mathsf{Vect} \\ &\simeq \mathrm{LS}(\mathcal{H}_q^{\mathrm{glob}}) \otimes_{\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)} \mathsf{Vect} \\ &\simeq \mathrm{LS}_q(\Gamma_c(\mathrm{M},\mathsf{B}^2\Lambda)),\end{aligned}$$

using the fact that $\mathcal{H}_q^{\mathrm{glob}}$ is the quotient of $\Gamma_c(\mathrm{M},\mathcal{H}_q)\times\mathsf{B}\mathbf{C}^\times$ by the anti-diagonal action of $\Gamma_c(\mathrm{M},\mathsf{B}^3\mathbf{C}^\times)$ and that LS commutes with colimits (*cf.* Lemma 1.3.5).

To prove the claim, it suffices to identify (3.6) with the factorization homology of (3.5) under nonabelian Poincaré duality (*cf.* Proposition 1.2.6). This amounts to the solid commutative diagram in $\mathsf{CAlg}(\mathsf{DGCat})$ below:

$$\begin{array}{ccccc} \int_{\mathrm{M}} \mathrm{LS}(\Omega_{\mathrm{M}}(\mathsf{B}^2\mathbf{C}^\times)) & \xrightarrow{\int_{\mathrm{M}}\mathrm{LS}(\tau_{\mathrm{M}}^{\mathrm{loc}})} & \int_{\mathrm{M}} \mathrm{LS}((\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}}) & \xrightarrow{\int_{\mathrm{M}}\chi_{\mathrm{M}}} & \int_{\mathrm{M}} \mathsf{Vect}_{\mathrm{M}} \\ \downarrow\simeq & & \vdots & & \downarrow\simeq \\ \mathrm{LS}(\Gamma_c(\mathrm{M},\mathsf{B}^3\mathbf{C}^\times)) & \xrightarrow{\mathrm{LS}(\tau_{\mathrm{M}}^{\mathrm{glob}})} & \mathrm{LS}(\mathsf{B}\mathbf{C}^\times) & \xrightarrow{\chi} & \mathsf{Vect} \end{array} \tag{3.7}$$

We shall supply the dotted arrow in (3.7) making both squares commute. Indeed, we let it be the composite

$$\int_{\mathrm{M}} \mathrm{LS}(\mathsf{B}\mathbf{C}^\times)_{\mathrm{M}} \simeq \operatorname{colim} \mathrm{LS}(\mathsf{B}\mathbf{C}^\times) \to \mathrm{LS}(\mathsf{B}\mathbf{C}^\times), \tag{3.8}$$

where the isomorphism is (1.11) (applied to $\mathcal{A} := \mathrm{LS}(\mathsf{B}\mathbf{C}^\times)$) and the second map is induced from the identity on $\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)$, as it determines a constant functor out of $\mathrm{Sing}\,\mathrm{M}$.

The right square of (3.7) commutes by naturality of the construction of (3.8) with respect to χ. The left square of (3.8) commutes because it is the image of (1.22) under LS. ☐

3.2. The Ben-Zvi–Nadler conjecture for T.

3.2.1. Let X be a smooth **C**-curve, assumed projective and connected. Let T be an X-torus and q be a Betti level for T in the sense of §2.1.3.

Our goal is to interpret Theorem 3.1.3 as a version of a conjecture of Ben-Zvi and Nadler (*cf.* [BZN18, Conjecture 4.27]) for T.

To do so, we invoke the passage from the algebraic to the topological context (*cf.* §2.1.7): We regard Λ as a functor $\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}}\to\mathbf{Z}\text{-}\mathsf{mod}$ and q as a morphism $\mathsf{B}^2\Lambda\to\mathsf{B}^4\mathbf{C}^\times$ in the ∞-category $\mathrm{Fun}(\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}},\mathsf{Spc}_*)$. To ease the notation, we shall denote factorization homology over $\mathrm{X}^{\mathrm{top}}$ by $\int_{\mathrm{X}}$.

3.2.2. We also need to extend the construction of the underlying homotopy type of a **C**-scheme of finite type (*cf.* §2.1.1) to **C**-prestacks.

Indeed, we write $\mathsf{PStk}(\mathsf{Sch}_{\mathrm{ft}})$ for the ∞-category of functors $(\mathsf{Sch}_{\mathrm{ft}})^{\mathrm{op}}\to\mathsf{Spc}$ and consider the left Kan extension of (2.1) along the Yoneda embedding. This yields a functor

$$\mathsf{PStk}(\mathsf{Sch}_{\mathrm{ft}})\to\mathsf{Spc}, \tag{3.9}$$

which we will still denote by $\mathrm{Sing}(\cdot)^{\mathrm{top}}$.

Remark 3.2.3. Let $\mathcal{Y}$ be an algebraic stack with a smooth cover $f:\mathrm{Z}\to\mathcal{Y}$ with $\mathrm{Z}\in\mathsf{Sch}_{\mathrm{ft}}$ such that for any $\mathrm{S}\in\mathsf{Sch}_{\mathrm{ft}}$ over $\mathcal{Y}$ the base change $f_{\mathrm{S}}:\mathrm{Z}\times_{\mathcal{Y}}\mathrm{S}\to\mathrm{S}$ induces a Serre fibration $(\mathrm{Z}\times_{\mathcal{Y}}\mathrm{S})^{\mathrm{top}}\to\mathrm{S}^{\mathrm{top}}$, then we have a canonical isomorphism in Spc:

$$\operatorname{colim}\mathrm{Sing}(\mathrm{Z}^\bullet_{/\mathcal{Y}})^{\mathrm{top}}\simeq\mathrm{Sing}\,\mathcal{Y}^{\mathrm{top}}, \tag{3.10}$$

where $Z^{\bullet}_{/\mathcal{Y}}$ denotes the Čech nerve of f.

To see this, we note that (3.9) commutes with colimits, so it suffices to prove that given a morphism $Z \to Y$ in $\mathsf{Sch}_{\mathrm{ft}}$ inducing a Serre fibration $Z^{\mathrm{top}} \to Y^{\mathrm{top}}$, its Čech nerve induces an isomorphism

$$\operatorname{colim} \operatorname{Sing}(Z^{\bullet}_{/Y})^{\mathrm{top}} \simeq \operatorname{Sing} Y^{\mathrm{top}}.$$

By Remark 2.1.2, $\operatorname{Sing}(Z^{\bullet}_{/Y})^{\mathrm{top}}$ is identified with the Čech nerve of the effective epimorphism $\operatorname{Sing} Z^{\mathrm{top}} \to \operatorname{Sing} Y^{\mathrm{top}}$, so this follows from [Lur09, Corollary 6.2.3.5].

Notably, for the algebraic stack $\mathsf{B}T$ with the smooth cover $X \to \mathsf{B}T$, (3.10) yields an isomorphism in Spc:

$$\mathsf{B} \operatorname{Sing} T^{\mathrm{top}} \simeq \operatorname{Sing}(\mathsf{B}T)^{\mathrm{top}}. \tag{3.11}$$

3.2.4. Denote by Bun_T the moduli stack of T-bundles over X, whose S-points (for $S \in \mathsf{Sch}_{\mathrm{ft}}$) are T-bundles over $X \times S$. Let us construct a morphism in Spc:

$$\operatorname{Sing}(\mathrm{Bun}_T)^{\mathrm{top}} \to \Gamma(\operatorname{Sing} X^{\mathrm{top}}, \mathsf{B}^2\Lambda). \tag{3.12}$$

Indeed, applying $\operatorname{Sing}(\cdot)^{\mathrm{top}}$ to the universal T-bundle

$$X \times \mathrm{Bun}_T \to \mathsf{B}T$$

and using its commutation with finite products (*cf.* Remark 2.1.2), we obtain a morphism:

$$\begin{aligned} \operatorname{Sing} X^{\mathrm{top}} \times \operatorname{Sing}(\mathrm{Bun}_T)^{\mathrm{top}} &\to \operatorname{Sing}(\mathsf{B}T)^{\mathrm{top}} \\ &\simeq \mathsf{B} \operatorname{Sing} T^{\mathrm{top}} \simeq \mathsf{B}^2\Lambda, \end{aligned} \tag{3.13}$$

where the two isomorphism are (3.11), respectively the one of §2.1.5. The desired map (3.12) now follows from adjunction.

Lemma 3.2.5. *The morphism* (3.12) *is an isomorphism.*

Proof. It suffices to prove that (3.12) induces an isomorphism on homotopy groups.

The connected components of $\operatorname{Sing}(\mathrm{Bun}_T)^{\mathrm{top}}$ are classified by the first Chern class of a T-bundle, thus in natural bijection with $H^2(X^{\mathrm{top}}, \Lambda)$.

Denote by $\mathrm{Bun}_T^0 \subset \mathrm{Bun}_T$ the substack of T-bundles with vanishing first Chern class. Then the homotopy groups of $\operatorname{Sing}(\mathrm{Bun}_T^0)^{\mathrm{top}}$ are computed by Atiyah–Bott uniformization, which yields $H^1(X^{\mathrm{top}}, \Lambda)$, respectively $H^0(X^{\mathrm{top}}, \Lambda)$ in degrees 1 and 2. ☐

3.2.6. By post-composing (3.13) with the Betti level q, we obtain a map $\operatorname{Sing}(\mathrm{Bun}_T)^{\mathrm{top}} \to \Gamma(\operatorname{Sing} X^{\mathrm{top}}, \mathsf{B}^4\mathbf{C}^{\times})$ by adjunction.

Post-composing with the global trace map $\tau_{\mathrm{M}}^{\mathrm{glob}}$ (*cf.* §1.2.10, for $\mathrm{M} := X^{\mathrm{top}}$, $\mathrm{A} := \mathbf{C}^{\times}$ and $k := 4$), we obtain a map in Spc:

$$\operatorname{Sing}(\mathrm{Bun}_T)^{\mathrm{top}} \to \mathsf{B}^2\mathbf{C}^{\times}, \tag{3.14}$$

which can be viewed as a Betti $\mathbf{C}^{\times}$-gerbe over Bun_T.

By construction, (3.3) and (3.14) are intertwined by the isomorphism (3.12):

$$\begin{array}{ccc} \operatorname{Sing}(\mathrm{Bun}_T)^{\mathrm{top}} & \xrightarrow{(3.14)} & \mathsf{B}^2\mathbf{C}^{\times} \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ \Gamma(\operatorname{Sing} X^{\mathrm{top}}, \mathsf{B}^2\Lambda) & \xrightarrow{(3.3)} & \mathsf{B}^2\mathbf{C}^{\times} \end{array} \tag{3.15}$$

3.2.7. The authors of [BZN18] propose to study the ∞-category $\mathsf{Shv}_{\mathcal{N},q}(\mathrm{Bun}_{\mathrm{T}})$ of sheaves of $\mathbf{C}$-vector space on $\mathrm{Bun}_{\mathrm{T}}$ twisted by the $\mathbf{C}^\times$-gerbe (3.14), satisfying the condition of having *nilpotent* singular support.

Since the global nilpotent cone of $\mathrm{Bun}_{\mathrm{T}}$ is the zero section, this condition is equivalent to having locally constant cohomology sheaves. By Remark 1.3.3, we may identify

$$\mathsf{Shv}_{\mathcal{N},q}(\mathrm{Bun}_{\mathrm{T}}) \simeq \mathrm{LS}_q(\mathrm{Bun}_{\mathrm{T}}),$$

where $\mathrm{LS}_q(\mathrm{Bun}_{\mathrm{T}})$ denotes the ∞-category of local systems over $\mathrm{Sing}(\mathrm{Bun}_{\mathrm{T}})^{\mathrm{top}}$ twisted by the $\mathbf{C}^\times$-gerbe (3.14). In other words, we have an equivalence

$$\mathsf{Shv}_{\mathcal{N},q}(\mathrm{Bun}_{\mathrm{T}}) \simeq \mathrm{LS}(\mathcal{G}_q^{\mathrm{glob}}) \otimes_{\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)} \mathsf{Vect}, \tag{3.16}$$

where $\mathcal{G}_q^{\mathrm{glob}}$ denotes the fiber of (3.14), endowed with the natural $\mathsf{B}\mathbf{C}^\times$-action, and Vect is viewed as an $\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)$-module via the tautological character χ.

The following is a version of [BZN18, Conjecture 4.27] for X-tori.

Corollary 3.2.8. *There is a canonical equivalence in* DGCat*:*

$$\int_{\mathrm{X}} \mathrm{Rep}_q(\check{\mathrm{T}}) \simeq \mathsf{Shv}_{\mathcal{N},q}(\mathrm{Bun}_{\mathrm{T}}).$$

Proof. Using (3.16), we reduce to constructing an equivalence

$$\int_{\mathrm{X}} \mathrm{Rep}_q(\check{\mathrm{T}}) \simeq \mathrm{LS}(\mathcal{G}_q^{\mathrm{glob}}) \otimes_{\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)} \mathsf{Vect}. \tag{3.17}$$

In view of (3.15), we may replace $\mathcal{G}_q^{\mathrm{glob}}$ by the fiber $\mathcal{H}_q^{\mathrm{glob}}$ of (3.3). Thus, the right-hand-side of (3.17) is canonically equivalent to $\mathrm{LS}(\mathcal{H}_q^{\mathrm{glob}}) \otimes_{\mathrm{LS}(\mathsf{B}\mathbf{C}^\times)} \mathsf{Vect}$, which is by definition the DG category $\mathrm{LS}_q(\Gamma(\mathrm{Sing}\,\mathrm{X}^{\mathrm{top}}, \mathsf{B}^2\Lambda))$ (*cf.* §3.1.2).

The desired equivalence (3.17) is now provided by Theorem 3.1.3. □

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