REPRESENTATIONS OF LOOP GROUPS AS FACTORIZATION MODULE CATEGORIES

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Contents

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Introduction

In this paper we show that the (2-)category of categorical representations of the loop group embeds fully faithfully into the (2-)category of factorization module categories with respect to the affine Grassmannian.

0.1. **Why might one expect this kind of thing to be true?**

0.1.1. Let X be a smooth curve and $x_0 \in X$ a point on it. Let G be a reductive group.

A construction going back to A. Beilinson (and probably first fleshed out in [Ga0]) says that there exists a family parameterized by X, whose fiber at $x \neq x_0$ is the product $\mathrm{Gr}_{G,x} \times \mathfrak{L}(G)_{x_0}$, and whose fiber at x_0 is $\mathfrak{L}(G)_{x_0}$, where:

- $\mathrm{Gr}_{G,x}$ is the affine Grassmannian of G associated with the formal disc $\mathcal{D}_x$ around x;
- $\mathfrak{L}(G)_{x_0}$ is the loop group of G associated with the formal punctured disc $\overset{\circ}{\mathcal{D}}_{x_0}$ around x_0.

One can generalize this construction slightly, and construct a similar family parameterized by X^n for any n.

0.1.2. In modern language, this construction says that we can regard Gr_G as a factorization space, and $\mathfrak{L}(G)_{x_0}$ as a factorization module space at x_0 with respect to Gr_G (see Sect. 1.2, where the relevant definitions are recalled).

Another insight of Beilinson's, articulated in the early 2000's, says that this factorization module space is universal in the following (imprecise) sense: $\mathfrak{L}(G)_{x_0}$ should be isomorphic to the factorization homology of Gr_G over $\overset{\circ}{\mathcal{D}}_{x_0}$:

$$\int\limits_{\overset{\circ}{\mathcal{D}}_{x_0}} \mathrm{Gr}_G \simeq \mathfrak{L}(G)_{x_0}, \tag{0.1}$$

whatever this means.

0.1.3. The above principle can be made precise in the topological setting:

The affine Grassmannian Gr_G is homotopy-equivalent to $\Omega(G)$, the loop space of G. Hence, applying Lurie's non-abelian Poincaré duality, we obtain that the factorization homology of Gr_G over the circle is homotopy-equivalent to

$$\mathrm{Maps}_{\mathrm{cont}}(S^1, G),$$

which can be regarded as a topological counterpart of $\mathfrak{L}(G)_{x_0}$.

0.1.4. The goal of this paper is to give an articulation of this principle in algebraic geometry. We do so by finding an appropriate linearized statement.

There are (at least) two ways to linearize the above principle: 0-categorical and 1-categorical.

The 0-categorical way is straightforward: it says that the factorization homology of $\mathrm{C}_\cdot(\mathrm{Gr}_G)$ on $\overset{\circ}{\mathcal{D}}_{x_0}$ (which can be made precise sense of) maps isomorphically to $\mathrm{C}_\cdot(\mathfrak{L}(G)_{x_0})$.

This is a true statement, and it will serve as an ingredient in the proof of our main theorem, see Sect. 0.2.6.

0.1.5. The 1-categorical linearization is richer:

We can consider the category of D-modules on Gr_G as a factorization category, and we can take its factorization homology over $\overset{\circ}{\mathcal{D}}_{x_0}$. This is a monoidal category that maps to $\mathrm{D}(\mathfrak{L}(G)_{x_0})$ (the latter is viewed as a monoidal category under convolution), and we can ask whether this functor is an equivalence.

The answer is "no" and that is for a simple reason: factorization homology of a category in the de Rham setting is too loose a construction; we rarely expect it to be equivalent to something sensible. However, we do expect that it has the "right category" as a quotient.

And indeed, this happens to be this case. Our main result, Theorem 2.1.6, is equivalent to saying that the functor

$$\int\limits_{\overset{\circ}{\mathcal{D}}_{x_0}} \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{D}(\mathfrak{L}(G)_{x_0})$$

is a quotient.

0.1.6. Finally, we would like to draw a (loose) analogy between the above statement and the contractibility result of [Ga2]:

The latter says that for a complete curve X, the pullback functor

$$\mathrm{D}(\mathrm{Bun}_G(X)) \to \mathrm{D}(\mathrm{Gr}_{G,\mathrm{Ran}})$$

is fully faithful.

Note also that in the global setting, the corresponding statement in topology is that the factorization homology of $\mathrm{Gr}_G \simeq \Omega^2(BG)$ over X is homotopy equivalent to

$$\mathrm{Maps}_{\mathrm{cont}}(X, BG),$$

which is in turn homotopy-equivalent to $\mathrm{Bun}_G(X)$.

0.2. **What is actually done in this paper?**

0.2.1. In the main body of the paper we do not talk about factorization homology of categories over the punctured disc. Rather, we formulate our main result as follows:

We can view $\mathrm{D}(\mathfrak{L}(G)_{x_0})$ as a factorization module category at x_0 with respect to the factorization category $\mathrm{D}(\mathrm{Gr}_G)$; as such it carries a commuting action of $\mathfrak{L}(G)_{x_0}$ "on the right". This structure allows us to construct a functor

$$\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod} \to \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}, \tag{0.2}$$

where:

- $\mathfrak{L}(G)_{x_0}$-**mod** is the 2-category of categorical representations of $\mathfrak{L}(G)_{x_0}$ (see Sect. B.3, where the definition is recalled);
- $\mathrm{D}(\mathrm{Gr}_G)$-$\mathbf{mod}^{\mathrm{fact}}_{x_0}$ is the 2-category of factorization module categories at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$ (see Sect. 1.4).

Our main result, Theorem 2.1.6, says that the functor (0.2) is fully faithful.

0.2.2. Concretely, Theorem 2.1.6 says that for $\mathbf{C}_1, \mathbf{C}_2 \in \mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}$, the functor

$$\mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2) \to \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathbf{C}_1^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}_2^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}), \tag{0.3}$$

induced by (0.2), is an equivalence.

It is easy to see that when proving this statement, one can assume that the source category, i.e., $\mathbf{C}_1$, is a copy of Vect, equipped with the trivial action of $\mathfrak{L}(G)_{x_0}$. I.e., we have to show that for $\mathbf{C} \in \mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}$, the functor

$$\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathbf{C}) \to \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}), \tag{0.4}$$

induced by (0.2), is an equivalence.

0.2.3. The first step in the proof of Theorem 2.1.6 consists of rewriting the right-hand side of (0.4) in terms of *factorization modules* over a *factorization algebra*.

Namely, we show that for any $\widetilde{\mathbf{C}} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$, we have a canonical equivalence

$$\mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}, \widetilde{\mathbf{C}}) \simeq \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\widetilde{\mathbf{C}})_{x_0},$$

where:

- ω_{Gr_G} is the dualizing sheaf on Gr_G, viewed as a factorization algebra in the factorization category $\mathrm{D}(\mathrm{Gr}_G)$ (see Sect. 1.5.1);
- $\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\widetilde{\mathbf{C}})_{x_0}$ denotes the category of factorization modules at x_0 in $\widetilde{\mathbf{C}}$ with respect to ω_{Gr_G} (see Sect. 1.5.3).

Thus, we obtain that Theorem 2.1.6 is equivalent to the following statement, which appears as Theorem 3.1.7 in the main body of the paper:

The functor

$$\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathbf{C}) \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}\left(\mathbf{C}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}\right)_{x_0}, \tag{0.5}$$

induced by (0.2)*, is an equivalence.*

0.2.4. We now briefly indicate the main steps involved in the proof of Theorem 3.1.7.

The first step, which is the geometric core of the argument says that when proving (0.5), one can replace $\mathbf{C}$ by its maximal subcategory $\text{alm-inv}_{\mathfrak{L}(G)_{x_0}}(\mathbf{C})$, on which the action of $\mathfrak{L}(G)_{x_0}$ is *almost trivial* (see Sect. 4.6 for what this means).

This step is carried out in Sect. 8 and it involves playing with the geometry of the fusion construction.

0.2.5. Once we assume that the action of $\mathfrak{L}(G)_{x_0}$ is almost trivial, there is no more "de Rham complexity" in the game, and the idea is to try to mimic the topological argument.

The second step in the proof of Theorem 3.1.7 consists of replacing "almost trivial" by "trivial". This is done in Sect. 4.7, by a categorical Koszul duality type argument.

This reduces the proof that (0.5) is an equivalence to the case when $\mathbf{C} = \mathrm{Vect}$. I.e., we have to show that the functor

$$\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}\left(\mathrm{Vect}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}\right)_{x_0} \tag{0.6}$$

is an equivalence.

0.2.6. The third step in the proof of Theorem 0.6 consists of establishing the equivalence (0.6). We show that both sides admit monadic forgetful functors to Vect, and we show that the corresponding monads are isomorphic. This is done in Sect. 6.

That said, one could view/prove the assertion that (0.6) is an isomorphism differently:

The left-hand side in (0.6) identifies with the category of modules over $\mathrm{C}_\cdot(\mathfrak{L}(G)_{x_0})$, where the structure of associative algebra on it is induced by the group structure on $\mathfrak{L}(G)_{x_0}$.

The right-hand side in (0.6) identifies with the category of factorization modules with respect to the factorization algebra $\mathrm{C}_\cdot(\mathrm{Gr}_G)$. Hence, it can be further identified with

$$\Big(\int\limits_{\overset{\circ}{\mathcal{D}}_{x_0}} \mathrm{C}_\cdot(\mathrm{Gr}_G)\Big)\text{-mod}.$$

Hence, the assertion that (0.6) is an isomorphism is equivalent to the assertion that the map

$$\Big(\int\limits_{\overset{\circ}{\mathcal{D}}_{x_0}} \mathrm{C}_\cdot(\mathrm{Gr}_G)\Big)\text{-mod} \to \mathrm{C}_\cdot(\mathfrak{L}(G)_{x_0})\text{-mod}$$

is an isomorphism, which is the linearization statement from Sect. 0.1.4.

0.3. **Extensions, applications and relation to prior work.**

0.3.1. Recall that our main result, Theorem 2.1.6, has the following form: it says that a certain functor from the 2-category of modules over a given monoidal category to the 2-category of factorization modules over a factorization category is fully faithful.

As far as we know, this is the *second-of-its-kind* result of this form. The first such result was established in [Bogd]. There, the main theorem says that a certain naturally defined functor

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_G(\overset{\circ}{\mathcal{D}}_{x_0}))\text{-}\mathbf{mod} \to \mathrm{Rep}(G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$$

is fully faithful, where:

- $\mathrm{LS}^{\mathrm{restr}}_G(\overset{\circ}{\mathcal{D}}_{x_0})$ is the stack of local systems with restricted variation on $\overset{\circ}{\mathcal{D}}_{x_0}$ with respect to G (defined as in [AGKRRV, Sect. 1.4]).

This result paves a way to questions of spectral decomposition in the *restricted* local geometric Langlands theory, see [Ga5, Sect. 2.6].

0.3.2. That said, one expects a stronger result to be true. Namely, we expect that (a similarly defined functor)

$$\mathrm{QCoh}(\mathrm{LS}_G(\overset{\circ}{\mathcal{D}}_{x_0}))\text{-}\mathbf{mod} \to \mathrm{Rep}(G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \tag{0.7}$$

is fully faithful, where:

- $\mathrm{LS}_G(\overset{\circ}{\mathcal{D}}_{x_0})$ is the stack of de Rham local systems on $\overset{\circ}{\mathcal{D}}_{x_0}$ with respect to G.

A result of this form would be of crucial importance for the full (i.e., unrestricted) local geometric Langlands theory.

0.3.3. Note, however, that when $G = T$ is a torus, the functor (0.7) identifies with the functor (0.2) for the dual torus.

So, thanks to our Theorem 3.1.7, the fully-faithfulness of (0.7) is known for tori.

0.3.4. For a non-commutative G, one can formulate the following conjecture:

Recall that to a category $\mathbf{C}$ acted on by the loop group, one can associate its Whittaker model, $\mathrm{Whit}(\mathbf{C})$, see [GLC2, Sect. 1.3.3]. Moreover, this construction works in the factorization setting.

Thus, on the one hand, we can consider

$$\mathrm{Whit}(G) := \mathrm{Whit}(\mathrm{D}(\mathrm{Gr}_G))$$

as a factorization category.

On the other hand, we can consider the monoidal category

$$\text{bi-Whit}(\mathfrak{L}(G)_{x_0}) := \mathrm{End}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathrm{Whit}(\mathrm{D}(\mathfrak{L}(G)_{x_0})).$$

A construction similar to (0.2) gives rise to a functor

$$\text{bi-Whit}(\mathfrak{L}(G)_{x_0})\text{-}\mathbf{mod} \to \mathrm{Whit}(G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}. \tag{0.8}$$

We conjecture that the functor (0.8) is fully faithful. (Note that when G is a torus, the Whittaker operation is the identity functor, and the functor (0.8) is just the functor (0.2)).

0.3.5. Now, the geometric Casselman-Shalika equivalence says that we have an equivalence of factorization categories

$$\mathrm{Whit}(G) \simeq \mathrm{Rep}(\check{G}),$$

where $\check{G}$ is the Langlands dual group of G.

And one of the conjectures in local geometric Langlands says that

$$\text{bi-Whit}(\mathfrak{L}(G)_{x_0}) \simeq \mathrm{QCoh}(\mathrm{LS}_G(\overset{\circ}{\mathcal{D}}_{x_0}))$$

as monoidal categories.

Under this equivalence, the functor (0.2) is supposed to correspond to the functor (0.8). This is the basis for believing that (0.8) is fully faithful.

0.3.6. Up until now, we have discussed the idea that (0.1) is an equivalence, when we linearize our algebro-geometric objects by applying the functor $\mathrm{D}(-)$.

One may wonder, however, whether one could expect a similar behavior when we linearize by means of $\mathrm{QCoh}(-)$ instead.

The answer is that an analog of Theorem 2.1.6 will fail in this case. We explain a counterexample in Sect. 10.6.

That said, this failure (at least, in our example) happens for subtle homological algebra reasons (it takes place, so to say, at the cohomological $-\infty$). It is not impossible that one could modify the definitions around the objects involved and make an analog of Theorem 2.1.6 hold.

0.3.7. We now explain one concrete application of our Theorem 2.1.6, rather in its incarnation as Theorem 3.1.7, to usual representation theory.

Let κ be a non-negative integral Kac-Moody level. To it we can associate a chiral algebra $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}$.

For example, when G is semi-simple and simply-connected, $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}$ is the "maximal integrable quotient" of the vacuum chiral algebra $\mathbb{V}_{\mathfrak{g},\kappa}$. When G is a torus, $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}$ is the lattice chiral algebra.

It is known that *at the level of abelian categories*, the category $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$ is equivalent to the category $\mathrm{Rep}(\mathfrak{L}(G)_{x_0},\kappa)$ of *integrable* Kac-Moody modules at level κ, i.e., representations of the central extension

$$1 \to \mathbb{G}_m \to \widehat{\mathfrak{L}}(G)_{\kappa,x_0} \to \mathfrak{L}(G)_{x_0} \to 1,$$

corresponding to κ, on which the central $\mathbb{G}_m$ acts by the standard character.

In Theorem 10.1.8, we show that this equivalence continues to hold at the derived level.

It is known that the category $\mathrm{Rep}(\mathfrak{L}(G)_{x_0},\kappa)$ is semi-simple (even at the derived level). So the content of Theorem 10.1.8 is that there are no higher Exts between irreducible objects of $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$.

0.4. **Structure of the paper.** We now explain the structure of the paper section-by-section.

0.4.1. In Sect. 1 we supply some background in factorization, mostly borrowed from [GLC2, Sects. B and C].

0.4.2. In Sect. 2 we construct the functor (0.2), state our main result (Theorem 2.1.6) and reduce it to the case when the source category is Vect.

0.4.3. In Sect. 3 we state Theorem 3.1.7, which says that (0.5) is an equivalence. We prove that Theorem 3.1.7 is logically equivalent to Theorem 2.1.6.

The rest of the paper (up until Sect. 10) is devoted to the proof of Theorem 3.1.7.

0.4.4. In Sect. 4 we discuss the notion of *almost trivial* action of a group (in particular, a loop group) on a category.

We state Theorem 4.7.3 that says that for the proof of Theorem 3.1.7 we can assume that the action of $\mathfrak{L}(G)_{x_0}$ on $\mathbf{C}$ is trivial.

We show that Theorem 4.7.3 allows us to reduce Theorem 3.1.7 to the case when $\mathbf{C} = \text{Vect}$.

Theorem 4.7.3 will be proved in Sects. 7-9.

0.4.5. In Sect. 5 we prove several technical statements formulated in Sect. 4.

0.4.6. In Sect. 6 we prove Theorem 3.1.7 for $\mathbf{C} = \text{Vect}$ by a direct calculation, which amounts to an algebro-geometric incarnation of a particular case of Lurie's non-abelian Poincaré duality.

0.4.7. In Sect. 7 we prove Theorem 4.7.3 for a torus using local geometric class field theory.

0.4.8. In Sect. 8 we supply a key geometric argument that tackles Theorem 4.7.3 in the non-abelian case.

0.4.9. In Sect. 9 we finish the proof of Theorem 4.7.3.

0.4.10. In Sect. 10 we discuss the application of our Theorem 3.1.7 to integrable Kac-Moody representation. In addition, we *disprove* a coherent version of Theorem 2.1.6.

0.4.11. In Sect. A we (re)collect some material pertaining to the theory of D-modules on algebro-geometric objects of infinite type.

0.4.12. In Sect. B we (re)collect some material pertaining to categorical representations of groups and in particular, loop groups.

0.4.13. In Sect. C we supply proofs of statements pertaining to factorization categories and modules.

0.5. **Conventions and notation.**

0.5.1. Throughout the paper, we will be working over a ground field k, assumed algebraically closed and of characteristic 0.

0.5.2. We will be working with k-linear *higher algebra*. The basic object of study for us is the ∞-category of k-linear DG categories, denoted DGCat (see [GR1, Sect. 1], where the relevant definitions are discussed in detail).

The ∞-category DGCat carries a symmetric monoidal structure, the Lurie tensor product. Its unit is the category Vect of chain complexes of k-vector spaces.

In particular, all objects of DGCat are automatically enriched over Vect; for $\mathbf{C} \in \text{DGCat}$ and $\mathbf{c}_1, \mathbf{c}_2 \in \text{DGCat}$, we will denote by $\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)$ the corresponding object of Vect.

The category DGCat has an internal Hom, denoted

$$\text{Funct}_{\text{DGCat}}(\mathbf{C}_1, \mathbf{C}_2), \quad \mathbf{C}_1, \mathbf{C}_2 \in \text{DGCat}\,.$$

0.5.3. As we will be interested in D-modules[1], we can stay in the world of *classical* (as opposed to *derived*) algebraic geometry. By a prestack we will mean a (presentable) functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \infty\text{-Grpds},$$

where $\mathrm{Sch}^{\mathrm{aff}}$ is the category of classical affine schemes over k.

All specific classes of algebro-geometric objects (e.g., schemes, ind-schemes, algebraic stacks, etc.) are full subcategories in the category PreStk of prestacks.

0.5.4. The theory of D-modules on prestacks locally of finite type is built in [GR2, Chapter 4]. An extension to relevant algebro-geometric objects of infinite type is discussed in Sect. A.

0.5.5. The material in this paper that has to do with factorization relies to a large extent on [GLC2]. We make a brief review in Sect. 1, and refer the reader to *loc. cit.* for details.

0.5.6. The main result of this paper, Theorem 2.1.6, talks about comparing 2-categories.

In this paper, by a (k-linear) 2-category, we will mean an ∞-category enriched over DGCat. In particular, given a 2-category $\mathfrak{C}$, and $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathfrak{C}$, we will denote by

$$\mathrm{Funct}_{\mathfrak{C}}(\mathfrak{c}_1, \mathfrak{c}_2)$$

the corresponding object of DGCat.

0.5.7. The main source of 2-categories will be of the form

$$\mathbf{A}\text{-}\mathbf{mod},$$

where $\mathbf{A}$ is a monoidal DG category, i.e., an associative algebra object in DGCat.

Objects of $\mathbf{A}$-$\mathbf{mod}$ are $\mathbf{A}$-module categories, and for $\mathbf{C}_1, \mathbf{C}_2 \in \mathbf{A}\text{-}\mathbf{mod}$,

$$\mathrm{Funct}_{\mathbf{A}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2)$$

is the naturally defined category of $\mathbf{A}$-linear functors.

0.5.8. All other conventions and notations follow ones adopted in [AGKRRV] and [GLC2].

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1. Background and preliminaries

For the readers convenience, in this section, we will (re)collect the definitions of the main players appearing in the paper. The discussions will mostly repeat [GLC2, Sect. B and C].

1.1. **The arc and loop groups.** The material here follows [GLC2, Sects. B.3 and B.4].

1.1.1. The Ran space of X, denoted Ran is a prestack that attaches to a test affine scheme S the set of finite non-empty subsets of $\mathrm{Hom}(S, X)$.

We let Ran_{x_0} be a variant of Ran, where we consider finite subsets with a distinguished element corresponding to

$$S \to \mathrm{pt} \overset{x_0}{\to} X.$$

[1]Except Sect. 10.6.

1.1.2. For an S-point $\underline{x}$ of Ran, let $\widehat{\mathcal{D}}_{\underline{x}}$ be the *complete* formal disc around $\underline{x}$, i.e. the completion of $S \times X$ at the union of graphs of the maps comprising $\underline{x}$. This is ind-affine ind-scheme. We let $\mathcal{D}_{\underline{x}}$ be the colimit of $\widehat{\mathcal{D}}_{\underline{x}}$ taken in the category of affine schemes.

For example, for $\underline{x} = \{x_0\}$ and $S = \operatorname{Spec}(R)$, if t is a local coordinate at x_0, then

$$\widehat{\mathcal{D}}_{\underline{x}} = \operatorname{Spf}(R[\![t]\!]) := \text{“}\underset{n}{\operatorname{colim}}\text{”} \operatorname{Spec}(R[t]/t^n) \text{ and } \mathcal{D}_{\underline{x}} = \operatorname{Spec}(R[\![t]\!]).$$

The above union of graphs is naturally a closed subscheme of $\mathcal{D}_{\underline{x}}$. We denote by $\overset{\circ}{\mathcal{D}}_{\underline{x}}$ the open complement. This is also an affine scheme.

In the above example,

$$\overset{\circ}{\mathcal{D}}_{\underline{x}} = \operatorname{Spec}(R(\!(t)\!)).$$

1.1.3. We let $\mathfrak{L}^+(G)_{\operatorname{Ran}}$ be the group-scheme over Ran, whose points are pairs $(\underline{x}, g)$, where $\underline{x} \in \operatorname{Hom}(S, \operatorname{Ran})$ and g is a point of $\operatorname{Hom}(\mathcal{D}_{\underline{x}}, G)$. Note that the latter is the same as $\operatorname{Hom}(\widehat{\mathcal{D}}_{\underline{x}}, G)$, since G is affine.

We let $\mathfrak{L}(G)_{\operatorname{Ran}}$ be the group-scheme over Ran, whose points are pairs $(\underline{x}, g)$, where $\underline{x} \in \operatorname{Hom}(S, \operatorname{Ran})$ and g is a point of $\operatorname{Hom}(\overset{\circ}{\mathcal{D}}_{\underline{x}}, G)$.

We will denote by $(-)_{\operatorname{Ran}_{x_0}}$ and $(-)_{x_0}$ the base change of the above objects along $\operatorname{Ran}_{x_0} \to \operatorname{Ran}$ and $\operatorname{pt} \overset{\{x_0\}}{\to} \operatorname{Ran}$, respectively.

Explicitly, the group of S-points of $\mathfrak{L}^+(G)_{x_0}$ is

$$\operatorname{Maps}(\operatorname{Spf}(R[\![t]\!]), G) \simeq \operatorname{Maps}(\operatorname{Spec}(R[\![t]\!]), G) = G\left(R[\![t]\!]\right).$$

The group pf S-points of $\mathfrak{L}(G)_{x_0}$ is

$$\operatorname{Maps}(\operatorname{Spec}(R(\!(t)\!)), G) = G\left(R(\!(t)\!)\right).$$

1.1.4. The object of study of this paper is categories equipped with an action of $\mathfrak{L}(G)_{x_0}$. We refer the reader to Sect. B.3, where this notion is reviewed.

1.2. **Factorization spaces.** The material here follows [GLC2, Sects. B.1 and B.2].

1.2.1. A factorization space $\mathcal{T}$ is a prestack $\mathcal{T}_{\operatorname{Ran}}$ over Ran equipped with the datum of isomorphisms:

$$\mathcal{T}_{\operatorname{Ran}} \underset{\operatorname{Ran},\operatorname{union}}{\times} (\operatorname{Ran} \times \operatorname{Ran})_{\operatorname{disj}} \simeq (\mathcal{T}_{\operatorname{Ran}} \times \mathcal{T}_{\operatorname{Ran}}) \underset{\operatorname{Ran}\times\operatorname{Ran}}{\times} (\operatorname{Ran} \times \operatorname{Ran})_{\operatorname{disj}}, \tag{1.1}$$

where:

- $(\operatorname{Ran} \times \operatorname{Ran})_{\operatorname{disj}} \subset (\operatorname{Ran} \times \operatorname{Ran})$ is the disjoint locus, i.e., the open subfunctor consisting of pairs $(\underline{x}_1, \underline{x}_2) \in (\operatorname{Ran} \times \operatorname{Ran})$ such that $\operatorname{Graph}_{\underline{x}_1} \cap \operatorname{Graph}_{\underline{x}_2} = \emptyset$;
- union is the union map $\operatorname{Ran} \times \operatorname{Ran} \to \operatorname{Ran}$.

The isomorphisms (1.1) must be equipped with a homotopy-coherent data of commutativity and associativity.

Remark 1.2.2. We distinguish notationally $\mathcal{Y}$ and $\mathcal{Y}_{\operatorname{Ran}}$: the latter is a just prestack over Ran, and the former taken into account the factorization structure.

1.2.3. Note that for $(\underline{x}_1, \underline{x}_2) \in (\operatorname{Ran} \times \operatorname{Ran})_{\operatorname{disj}}$ and $\underline{x} = \underline{x}_1 \cup \underline{x}_2$, we have

$$\mathcal{D}_{\underline{x}} \simeq \mathcal{D}_{\underline{x}_1} \sqcup \mathcal{D}_{\underline{x}_2} \text{ and } \overset{\circ}{\mathcal{D}}_{\underline{x}} \simeq \overset{\circ}{\mathcal{D}}_{\underline{x}_1} \sqcup \overset{\circ}{\mathcal{D}}_{\underline{x}_2}.$$

These isomorphisms endow $\mathfrak{L}^+(G)_{\operatorname{Ran}}$ and $\mathfrak{L}(G)_{\operatorname{Ran}}$ with a factorization structure. We denote the resulting factorization spaces by $\mathfrak{L}^+(G)$ and $\mathfrak{L}(G)$, respectively.

1.2.4. A key geometric player is the factorization space

$$\mathrm{Gr}_G := \mathfrak{L}(G)/\mathfrak{L}^+(G).$$

Explicitly, for a point $\underline{x}$ of Ran, the fiber $\mathrm{Gr}_{G,\underline{x}}$ is the set of pairs $(\mathcal{P}_G, \alpha)$, where:

- $\mathcal{P}_G$ is a G-bundle on $\mathcal{D}_{\underline{x}}$;
- α is a trivialization of $\mathcal{P}_G|_{\overset{\circ}{\mathcal{D}}_{\underline{x}}}$.

Or equivalently, it is the set of pairs $(\mathcal{P}_G^{\mathrm{glob}}, \beta)$

- $\mathcal{P}_G^{\mathrm{glob}}$ is a G-bundle on X;
- β is a trivialization of $\mathcal{P}_G^{\mathrm{glob}}|_{X\setminus\underline{x}}$.

1.2.5. Let $\mathcal{T}$ be a factorization space. A factorization module space $\mathcal{T}_m$ at x_0 with respect to $\mathcal{T}$ is a prestack $(\mathcal{T}_m)_{\mathrm{Ran}_{x_0}}$ over Ran_{x_0}, equipped with the datum of isomorphisms:

$$(\mathcal{T}_m)_{\mathrm{Ran}_{x_0}} \underset{\mathrm{Ran}_{x_0},\mathrm{union}}{\times} (\mathrm{Ran}\times\mathrm{Ran}_{x_0})_{\mathrm{disj}} \simeq (\mathcal{T}_{\mathrm{Ran}} \times (\mathcal{T}_m)_{\mathrm{Ran}_{x_0}}) \underset{\mathrm{Ran}\times\mathrm{Ran}_{x_0}}{\times} (\mathrm{Ran}\times\mathrm{Ran}_{x_0})_{\mathrm{disj}}, \tag{1.2}$$

equipped with a homotopy-coherent datum of associativity against (1.1).

We will often think of $\mathcal{T}_m$ as a prestack $(\mathcal{T}_m)_{x_0}$, equipped with an additional datum of extension to a prestack over Ran_{x_0}, all of whose fibers are specified by (1.2).

1.2.6. For a factorization space $\mathcal{T}$, the pullback

$$\mathcal{T}_{\mathrm{Ran}_{x_0}} := \mathcal{T}_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_{x_0}$$

has a natural factorization structure against $\mathcal{T}$.

We denote the resulting factorization module space by $\mathcal{T}^{\mathrm{fact}_{x_0}}$. We refer to it as the *vacuum* factorization module space at x_0.

1.2.7. For a factorization space, one can talk about it having a *unital*, *counital* and *corr-unital* structure. See Sect. C.1 for their definitions.

For example, Gr_G is unital, $\mathfrak{L}^+(G)$ is counital, and $\mathfrak{L}(G)$ is corr-unital.

If $\mathcal{T}$ is factorization space that is *unital* (resp., *counital*, *corr-unital*) and $\mathcal{T}_m$ is a factorization module space at x_0 with respect to $\mathcal{T}$, one can talk about $\mathcal{T}_m$ being *unital* (resp., *counital*, *corr-unital*) against the corresponding structure on $\mathcal{T}$.

1.3. **Factorization categories.** The material here follows [GLC2, Sect. B.11]. The reader is referred to *loc. cit.* for more details.

1.3.1. Let $\mathcal{Y}$ be a prestack (assumed locally of finite type). A crystal of categories $\underline{\mathbf{C}}$ on $\mathcal{Y}$ is an assignment:

- $(S \xrightarrow{y} \mathcal{Y}) \mapsto \mathbf{C}_{S,y} \in \mathrm{D}(S)\text{-}\mathbf{mod}$, where S is an affine scheme of finite type, and $\mathrm{D}(S)$ is viewed as a symmetric monoidal category via the $\overset{!}{\otimes}$ operation;
- $(S_1 \xrightarrow{f} S_2) \mapsto \mathbf{C}_{S_1,y_1} \simeq \mathrm{D}(S_1) \underset{\mathrm{D}(S_2)}{\otimes} \mathbf{C}_{S_2,y_2}$, where $y_1 = y_2 \circ f$, and the symmetric monoidal functor $\mathrm{D}(S_2) \to \mathrm{D}(S_1)$ is $f^!$.
- A homotopy-coherent system of compatibilities for higher order compositions.

1.3.2. The most basic example of a crystal of categories is $\underline{\mathrm{D}}(\mathcal{Y})$, whose value on every $S \xrightarrow{y} \mathcal{Y}$ is $\mathrm{D}(S)$.

1.3.3. We let $\mathbf{CrysCat}(\mathcal{Y})$ denote the 2-category of crystals of categories over $\mathcal{Y}$.

We have a naturally defined functor

$$\mathbf{\Gamma}(\mathcal{Y},-):\mathbf{CrysCat}(\mathcal{Y})\to \mathrm{D}(\mathcal{Y})\text{-}\mathbf{mod},\quad \underline{\mathbf{C}}\mapsto \lim_{S\xrightarrow{y}\mathcal{Y}} \mathbf{C}_{S,y}. \tag{1.3}$$

Recall (see [Ga3, Definition 1.3.7]) that $\mathcal{Y}_{\mathrm{dR}}$ is said to be *1-affine* if the functor (1.3) is an equivalence.

For example $\mathrm{Ran}_{\mathrm{dR}}$ is 1-affine (see [GLC2, Lemma B.8.15]).

1.3.4. For a map $g:\mathcal{Y}_1\to\mathcal{Y}_2$, there is a tautologically defined pullback functor

$$g^*:\mathbf{CrysCat}(\mathcal{Y}_2)\to\mathbf{CrysCat}(\mathcal{Y}_1).$$

When there is no ambiguity for g, for $\underline{\mathbf{C}}\in\mathbf{CrysCat}(\mathcal{Y}_2)$, we will sometimes write

$$\underline{\mathbf{C}}|_{\mathcal{Y}_1}:=g^*(\underline{\mathbf{C}}).$$

We have a naturally defined functor

$$g^!:\mathbf{\Gamma}(\mathcal{Y}_2,\underline{\mathbf{C}})\to\mathbf{\Gamma}(\mathcal{Y}_1,g^*(\underline{\mathbf{C}})).$$

For $\underline{\mathbf{C}}_i\in\mathbf{CrysCat}(\mathcal{Y}_i)$, $i=1,2$, we define

$$\underline{\mathbf{C}}_1\boxtimes\underline{\mathbf{C}}_2\in\mathbf{CrysCat}(\mathcal{Y}_1\times\mathcal{Y}_2)$$

naturally: for $y_i\in\mathrm{Hom}(S,\mathcal{Y}_i)$,

$$(\underline{\mathbf{C}}_1\boxtimes\underline{\mathbf{C}}_2)_{S,(y_1,y_2)}:=(\underline{\mathbf{C}}_1)_{S,y_1}\underset{\mathrm{D}(S)}{\otimes}(\underline{\mathbf{C}}_2)_{S,y_2}.$$

We have a naturally defined functor,

$$\mathbf{\Gamma}(\mathcal{Y}_1,\underline{\mathbf{C}}_1)\otimes\mathbf{\Gamma}(\mathcal{Y}_2,\underline{\mathbf{C}}_2)\to\mathbf{\Gamma}(\mathcal{Y}_1\times\mathcal{Y}_2,\underline{\mathbf{C}}_1\boxtimes\underline{\mathbf{C}}_2),$$

to be denoted

$$\mathbf{c}_1,\mathbf{c}_2\mapsto\mathbf{c}_1\boxtimes\mathbf{c}_2.$$

1.3.5. A factorization category $\mathbf{A}$ is a crystal of categories $\underline{\mathbf{A}}$ over Ran, equipped with an equivalence

$$\mathrm{union}^*(\underline{\mathbf{A}})|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}}\simeq(\underline{\mathbf{A}}\boxtimes\underline{\mathbf{A}})|_{(\mathrm{Ran}\times\mathrm{Ran})_{\mathrm{disj}}} \tag{1.4}$$

equipped additionally with a homotopy-coherent datum of commutativity and associativity.

Remark 1.3.6. Even though $\mathrm{Ran}_{\mathrm{dR}}$ is 1-affine, we distinguish notationally between $\mathbf{A}$, $\underline{\mathbf{A}}$ and

$$\mathbf{A}_{\mathrm{Ran}}:=\mathbf{\Gamma}(\mathrm{Ran},\underline{\mathbf{A}}).$$

That said, the data of (1.4), can be equivalently spelled out in terms of $\mathbf{A}_{\mathrm{Ran}}$.

1.3.7. The most basic example of a factorization category, denoted Vect, is when the corresponding crystal of categories over Ran is $\underline{\mathrm{D}}(\mathrm{Ran})$ itself.

1.3.8. Let $\mathcal{T}$ be a factorization space. Assume that $\mathcal{T}_{\mathrm{Ran}}$ is locally of finite type. Then the crystal of categories

$$(\underline{x}\in\mathrm{Hom}(S,\mathrm{Ran}))\mapsto\mathrm{D}(\mathcal{T}_{\underline{x}})$$

has a natural factorization structure.

We denote the resulting factorization category by $\mathrm{D}(\mathcal{T})$.

1.3.9. A prime example of this is when $\mathcal{T} = \mathrm{Gr}_G$. This way, we obtain a factorization category $\mathrm{D}(\mathrm{Gr}_G)$, which is the second main player in this paper.

Remark 1.3.10. Let us weaken the hypothesis that $\mathcal{T}_{\mathrm{Ran}}$ is locally of finite type. Instead, let us require that for every $\underline{x} \in \mathrm{Hom}(S, \mathrm{Ran})$ (with S an affine scheme of finite type), the prestack $\mathcal{T}_{\underline{x}}$ is an *ind-placid ind-scheme*, see Sect. A.4.7 for what this means.

In this case, the assignments

$$(\underline{x} \in \mathrm{Hom}(S, \mathrm{Ran})) \mapsto \mathrm{D}^!(\mathcal{T}_{\underline{x}}) \text{ and } (\underline{x} \in \mathrm{Hom}(S, \mathrm{Ran})) \mapsto \mathrm{D}_*(\mathcal{T}_{\underline{x}})$$

are crystals of categories equipped with natural factorization structures.

We denote the resulting factorization categories by

$$\mathrm{D}^!(\mathcal{T}) \text{ and } \mathrm{D}_*(\mathcal{T}),$$

respectively.

1.3.11. Given a factorization category, one can talk about a *unital* structure on it. We refer the reader to [GLC2, Sect. C.11]. Some of this material will be reviewed in Sect. C of this paper.

1.4. **Factorization *module* categories.** The material here follows [GLC2, Sect. B.12]. The reader is referred to *loc. cit.* for more details.

1.4.1. Let $\mathbf{A}$ be a factorization category. A factorization module category $\mathbf{C}$ at x_0 with respect to $\mathbf{A}$ is a crystal of categories $\underline{\mathbf{C}}$ over Ran_{x_0} equipped with an equivalence

$$\mathrm{union}^*(\underline{\mathbf{C}})|_{(\mathrm{Ran}\times\mathrm{Ran}_{x_0})_{\mathrm{disj}}} \simeq (\underline{\mathbf{A}} \boxtimes \underline{\mathbf{C}})|_{(\mathrm{Ran}\times\mathrm{Ran}_{x_0})_{\mathrm{disj}}}, \tag{1.5}$$

equipped with a homotopy-coherent data of associativity against (1.4).

The totality of factorization module categories at x_0 with respect to $\mathbf{A}$ naturally forms a 2-category, to be denoted

$$\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}.$$

1.4.2. We have a tautological forgetful functor

$$\mathrm{oblv}_{\mathbf{A}} : \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \to \mathrm{DGCat}, \quad \mathbf{C} \mapsto \mathbf{C}_{x_0}. \tag{1.6}$$

Note, however, that the functor (1.6) is *not* conservative. Rather, it induces conservative functors between Funct categories. As a formal corollary, we obtain the following:

Lemma 1.4.3. *Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a 1-morphism in $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$ that admits an adjoint (on either side). Then if*

$$\mathrm{oblv}_{\mathbf{A}}(F) : \mathrm{oblv}_{\mathbf{A}}(\mathbf{C}_1) \to \mathrm{oblv}_{\mathbf{A}}(\mathbf{C}_2)$$

is an equivalence, then so is F.

1.4.4. The category $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$ has colimits that commute with the forgeftul functors

$$\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \to \mathrm{D}(S)\text{-}\mathbf{mod} \tag{1.7}$$

for any $\underline{x} : S \to \mathrm{Ran}$.

Assume now that $\mathbf{A}$ is such that for any $\underline{x} : S \to \mathrm{Ran}$, the category $\mathbf{A}_{S,\underline{x}}$ is dualizable. Then $\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$ also contains all limits that commute with the forgetful functors (1.7).

1.4.5. Let $\mathcal{T}$ be as in Sect. 1.3.8, and let $\mathcal{T}_m$ be a factorization module space at x_0 with respect to $\mathcal{T}$, also assumed locally of finite type.

Then the assignment

$$(\underline{x} \in \mathrm{Hom}(S, \mathrm{Ran}_{x_0})) \mapsto \mathrm{D}((\mathcal{T}_m)_{\underline{x}})$$

is a crystal of categories that carries a natural factorization structure against $\mathrm{D}(\mathcal{T})$.

We denote the resulting object of $\mathrm{D}(\mathcal{T})\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$ by

$$\mathrm{D}(\mathcal{T}_m) \in \mathrm{D}(\mathcal{T})\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}.$$

1.4.6. Let $\mathbf{A}$ be a factorization category. The pullback of $\underline{\mathbf{A}}$ along $\mathrm{Ran}_{x_0} \to \mathrm{Ran}$ has a natural factorization structure against $\mathbf{A}$.

We denote the resulting object by

$$\mathbf{A}^{\mathrm{fact}_{x_0}} \in \mathbf{A}\text{-mod}^{\mathrm{fact}}_{x_0},$$

and refer to it as the *vacuum factorization module category* at x_0.

Note that in the example of Sect. 1.3.8, we have

$$\mathrm{D}(\mathcal{T})^{\mathrm{fact}_{x_0}} \simeq \mathrm{D}(\mathcal{T}^{\mathrm{fact}_{x_0}}),$$

where $\mathcal{T}^{\mathrm{fact}_{x_0}}$ is as in Sect. 1.2.6.

1.4.7. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a homomorphism of factorization categories. For $\mathbf{C}_2 \in \mathbf{A}_2\text{-mod}^{\mathrm{fact}}_{x_0}$ one attaches an object

$$\mathbf{Res}_\Phi(\mathbf{C}_2) \in \mathbf{A}_1\text{-mod}^{\mathrm{fact}}_{x_0},$$

which is the universal object such that there is a morphism $\mathbf{Res}_\Phi(\mathbf{C}_2) \to \mathbf{C}_2$ compatible with Φ, see [GLC2, Sect. B.12.11].

This construction will be reviewed in Sect. C of the present paper.

1.4.8. Given a unital factorization category $\mathbf{A}$, one can talk about *unital factorization module categories* at x_0 with respect to $\mathbf{A}$, see [GLC2, Sect. C.14]. We denote the resulting 2-category by

$$\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}.$$

When we want to ignore the unital structure on $\mathbf{A}$, we will denote the resulting 2-category of *non-unital* factorization module categories by

$$\mathbf{A}\text{-}\mathbf{mod}^{\text{fact-n.u.}}_{x_0}.$$

There exists an obvious forgetful functor

$$\mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \to \mathbf{A}\text{-}\mathbf{mod}^{\text{fact-n.u.}}_{x_0}. \tag{1.8}$$

Note, however, that (1.8) is *not* fully faithful. Yet, it is 1-fully faithful, i.e., it is fully faithful on the Funct-categories.

1.5. **Factorization algebras and modules.**

1.5.1. Let $\mathbf{A}$ be a factorization category. A factorization algebra $\mathcal{A}$ in $\mathbf{A}$ is an object $\mathcal{A}_{\mathrm{Ran}} \in \mathbf{A}_{\mathrm{Ran}}$, equipped with an identification between

$$\mathrm{union}^!(\mathcal{A}) \in \mathbf{\Gamma}\left((\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}, \mathrm{union}^*(\underline{\mathbf{A}})\right) \text{ and } \mathcal{A} \boxtimes \mathcal{A} \in \mathbf{\Gamma}\left((\mathrm{Ran} \times \mathrm{Ran})_{\mathrm{disj}}, \mathbf{A} \boxtimes \mathbf{A}\right)$$

with respect to the equivalence (1.4), and further equipped with a homotopy-coherent system of compatibilities.

Taking $\mathbf{A} = \mathrm{Vect}$, we recover the usual notion of factorization algebra.

1.5.2. Let $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a homomorphism of factorization categories. Then Φ sends factorization algebras in $\mathbf{A}_1$ to factorization algebras in $\mathbf{A}_2$.

1.5.3. Let $\mathbf{C}$ be a factorization module category at x_0 with respect to $\mathbf{A}$. Given a factorization algebra $\mathcal{A}$ in $\mathbf{A}$, we define a factorization $\mathcal{A}$-module at x_0 in $\mathbf{C}$ to be an object $\mathcal{M} \in \mathbf{C}_{\mathrm{Ran}_{x_0}}$, equipped with an identification

$$\mathrm{union}^!(\mathcal{M}) \in \mathbf{\Gamma}\left((\mathrm{Ran} \times \mathrm{Ran}_{x_0})_{\mathrm{disj}}, \mathrm{union}^*(\underline{\mathbf{C}})\right) \text{ and } \mathcal{A} \boxtimes \mathcal{M} \in \mathbf{\Gamma}\left((\mathrm{Ran} \times \mathrm{Ran}_{x_0})_{\mathrm{disj}}, \mathbf{A} \boxtimes \mathbf{A}\right)$$

with respect to the equivalence (1.5), and further equipped with a homotopy-coherent system of compatibilities.

1.5.4. We let

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{x_0}$$

denote category of factorization $\mathcal{A}$-modules at x_0 in $\mathbf{C}$.

We have a naturally defined *conservative* forgetful functor

$$\text{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{x_0} \to \mathbf{C}_{x_0}.$$

1.5.5. In the particular case when $\mathbf{C} = \mathbf{A}^{\text{fact}_{x_0}}$, we will use a short-hand notation

$$\mathcal{A}\text{-mod}^{\text{fact}}_{x_0} := \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{A}^{\text{fact}_{x_0}})_{x_0}.$$

In the particular case when $\mathbf{A} = \text{Vect}$, and thus $\mathcal{A}$ is a usual factorization algebra we thus recover the usual category

$$\mathcal{A}\text{-mod}^{\text{fact}}_{x_0}$$

of factorization $\mathcal{A}$-modules at x_0.

1.5.6. Consider the pullback of $\mathcal{A}_{\text{Ran}}$ with respect to $\text{Ran}_{x_0} \to \text{Ran}$. It has a natural factorization module structure against $\mathcal{A}$. We denote the resulting object by

$$\mathcal{A}^{\text{fact}_{x_0}} \in \mathcal{A}\text{-mod}^{\text{fact}}_{x_0}.$$

We refer to it as the *vacuum* factorization $\mathcal{A}$-module at x_0.

1.5.7. Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism between factorization algebras in $\mathbf{A}$. Then to ϕ one attaches a restriction functor

$$\text{Res}_{\phi} : \mathcal{A}_2\text{-mod}^{\text{fact}}(\mathbf{C})_{x_0} \to \mathcal{A}_1\text{-mod}^{\text{fact}}(\mathbf{C})_{x_0},$$

see [GLC2, Sect. B.9.25].

This construction will be reviewed also in Sect. C of the present paper.

1.5.8. Let now $\Phi : \mathbf{A}_1 \to \mathbf{A}_2$ be a homomorphism between factorization categories. Let $\mathcal{A}_1$ be a factorization algebra in $\mathbf{A}_1$, and set $\mathcal{A}_2 := \Phi(\mathcal{A}_1)$, which we view as a factorization algebra in $\mathbf{A}_2$ (see Sect. 1.5.2 above).

Let $\mathbf{C}_2$ be an object of $\mathbf{A}_2\textbf{-mod}^{\text{fact}}_{x_0}$ and set

$$\mathbf{C}_1 := \mathbf{Res}_{\Phi}(\mathbf{C}_2),$$

see Sect. 1.4.7.

In this case, we have a canonical equivalence

$$\mathcal{A}_1\text{-mod}^{\text{fact}}(\mathbf{C}_1)_{x_0} \simeq \mathcal{A}_2\text{-mod}^{\text{fact}}(\mathbf{C}_2)_{x_0}, \tag{1.9}$$

see [GLC2, Lemma B.12.14], to be reviewed in Sect. C.

1.5.9. Assume that $\mathbf{A}$ is unital. In this case, one can talk about a factorization algebra being unital, see [GLC2, Sect. 7].

Let $\mathbf{C}$ be a unital factorization module category at x_0 with respect to $\mathbf{A}$, and let $\mathcal{A}$ be a unital factorization algebra in $\mathbf{A}$. In this case, the category of factorization $\mathcal{A}$-modules at x_0 in $\mathbf{C}$ contains a full subcategory of *unital* factorization modules, see [GLC2, C.11.19], to be reviewed in Sect. C of the present paper.

In this case, we denote this subcategory by $\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{x_0}$, and the category of not necessarily unital $\mathcal{A}$-modules by $\mathcal{A}\text{-mod}^{\text{fact-n.u.}}(\mathbf{C})_{x_0}$.

2. Statement of the result

In this section we state the main result of this paper, Theorem 2.1.6, and then reformulate it in terms of the computation of a category of factorization modules with respect to a certain factorization algebra.

2.1. **From loop group modules to factorization modules over** Gr_G**.** In this subsection we will construct a functor

$$\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod} \to \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \tag{2.1}$$

with the basic property that it preserves the forgetful functors from both sides to DGCat.

2.1.1. Recall that Gr_G is a unital factorization space and $\mathrm{D}(\mathrm{Gr}_G)$ is a unital factorization category, see Sect. 1.3.9 and Sect. C.1.16.

The functor (2.1) will be constructed by exhibiting a suitable bimodule category, to be denoted $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$, which is an object of $\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$, and as such carries an action of $\mathfrak{L}(G)_{x_0}$.

Moreover, the underlying DG category of $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$ (i.e., its fiber over $x_0 \in \mathrm{Ran}_{x_0}$) will identify with $\mathrm{D}(\mathfrak{L}(G)_{x_0})$ itself, with the $\mathfrak{L}(G)_{x_0}$-action given by right translations.

2.1.2. The category $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$ will be defined as the category of D-modules on a certain prestack, to be denoted $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$, which is a factorization module space *at* x_0 with respect to the factorization algebra space Gr_G (see Sect. 1.2.5), and as such equipped with an action of $\mathfrak{L}(G)_{x_0}$.

Moreover, the fiber of $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$ at $x_0 \in \mathrm{Ran}_{x_0}$ will identify with $\mathfrak{L}(G)_{x_0}$ itself, with the $\mathfrak{L}(G)_{x_0}$-action given by right translations.

2.1.3. The space $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$ is constructed as follows:

For an affine test scheme S and a given S-point $\underline{x}$ of Ran_{x_0}, the fiber

$$\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}}$$

of $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$ over it consists of the data of

$$(\mathcal{P}_G, \alpha, \epsilon),$$

where:

- $\mathcal{P}_G$ is a G-bundle on $S \times X$;
- α is a trivialization of $\mathcal{P}_G$ over $S \times X - \mathrm{Graph}_{\underline{x}}$;
- ϵ is a trivialization of $\mathcal{P}_G$ over the formal completion of $S \times X$ along $S \times x_0$.

The (unital) factorization module space structure on $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$ is contructed using Beauville-Laszlo theorem.

The action of $\mathfrak{L}(G)_{x_0}$ on $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$ is given by the standard regluing procedure. It is easy to see this action preserves the factorization module space structure in above.

2.1.4. As in Sect. A.5, one can show $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}}$ is ind-placid, and there is a canonical equivalence $\mathrm{D}^!(\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}}) \simeq \mathrm{D}_*(\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}})$. We will simply write $\mathrm{D}(\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}})$ for these categories.

Now the assignment

$$\underline{x} \mapsto \mathrm{D}(\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}})$$

is a crystal of categories over Ran_{x_0} that carries a natural (unital) factorization structure against $\mathrm{D}(\mathrm{Gr}_G)$. See Sect. 1.4.5. We define

$$\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$$

to be this object[2].

[2]Using Proposition C.10.20, one can show

$$\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \simeq \mathbf{Res}_{p^!}\big(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathfrak{L}(G))}\big),$$

where $p^! : \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{D}^!(\mathfrak{L}(G))$ is the unital factorization functor given by !-pullbacks and $\mathbf{Res}_{p^!}$ is the restriction functor along it (see Sect. 1.4.7).

By construction, $\mathrm{D}(\mathfrak{L}(G)_{x_0})$ acts on $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$. We define the functor in (2.1) by

$$\mathbf{C} \mapsto \mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \underset{\mathrm{D}(\mathfrak{L}(G)_{x_0})}{\otimes} \mathbf{C}. \tag{2.2}$$

By Corollary B.4.12 and Sect. 1.4.4, the functor (2.2) commutes with limits and colimits.

2.1.5. We can now state the main result of this paper:

Theorem 2.1.6. *The functor* (2.1) *is fully faithful.*

2.1.7. Before we proceed any further, let us remark that the functor (2.1) is *not* an equivalence. Indeed, it is easy to exhibit an object in $\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$ that which is not in the essential image of (2.1).

Namely, restriction to the unit section defines a factorization functor

$$\iota^! : \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{Vect}\,.$$

Hence, we obtain a restriction functor

$$\mathbf{Res}_{\iota^!} : \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \to \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0},$$

(see Sect. 1.4.7).

Nonzero objects in the essential image of this functor do not lie in the essential image of the functor (2.1) unless G is trivial [3].

Remark 2.1.8. For the trival group, it is easy to see the corresponding functor

$$\mathrm{DGCat} \to \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \tag{2.3}$$

is fully faithful. However, it is *not* an equivalence. To see this, consider the Ran space $\mathrm{Ran}_\circ$ for the punctured curve $X - x_0$. Write $\mathrm{Ran}_\circ := \mathrm{Ran}_\circ \sqcup \{\emptyset\}$, and consider the map

$$j : \mathrm{Ran}_\circ \to \mathrm{Ran}_{x_0},\ \underline{y} \mapsto \underline{y} \sqcup \{x_0\}.$$

One can show $\mathrm{Ran}_\circ$, viewed as a prestack over Ran_{x_0}, has a natural factorization structure against the factorization space Ran (see Sect. 1.2.5). By Sect. 1.4.5, we obtain a factorization module category with respect to Vect, such that $\mathbf{\Gamma}(\mathrm{Ran}_{x_0}, -)$ sends it to $\mathrm{D}(\mathrm{Ran}_\circ)$. Moreover, using the *unital* Ran spaces (see Sect. C.1.3), we can upgrade it to a *unital* factorization module category with respect to Vect. In other words, we obtain a bizarre object, denote by

$$\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{disj}} \in \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0},$$

which is not isomorphic to the vaccum factorization module category $\mathrm{Vect}^{\mathrm{fact}_{x_0}}$, but has the same fiber at x_0 (i.e. Vect). It is clear this obect is not in the essential image of (2.3).

Remark 2.1.9. At the moment we do not know how to characterize (even conjecturally) the full subcategory of $\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$ equal to the essential image of (2.1).

2.2. **Functoriality.** In this subsection we establish a functoriality property of the construction in Sect. 2.1 with respect to homomorphisms of reductive groups.

---

[3]Sketch of proof: suppose $\mathbf{Res}_{\iota^!}(\mathbf{C})$ is contained in the essential image of (2.1). By Theorem 3.1.7 below, we have

$$\mathbf{C}_{x_0}^{\mathfrak{L}(G)_{x_0}} \simeq \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{Res}_{\iota^!}(\mathbf{C}))_{x_0} \simeq \iota^!(\omega_{\mathrm{Gr}_G})\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{x_0} \simeq k\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{x_0} \simeq \mathbf{C}_{x_0},$$

which is impossible unless G is trivial or $\mathbf{C} \simeq 0$.

2.2.1. Let $\phi : G' \to G$ be a homomorphism between connected reductive groups. By a slight abuse of notation we will denote by the same symbol ϕ the resulting homomorphism

$$\mathfrak{L}(G')_{x_0} \to \mathfrak{L}(G)_{x_0}.$$

Let

$$\mathrm{Gr}_\phi : \mathrm{Gr}_{G'} \to \mathrm{Gr}_G$$

between (unital) factorization spaces.

Direct image with respect to Gr_ϕ has a natural structure of (unital) factorization functor

$$(\mathrm{Gr}_\phi)_* : \mathrm{D}(\mathrm{Gr}_{G'}) \to \mathrm{D}(\mathrm{Gr}_G).$$

2.2.2. Note that we also have a morphism between (unital) factorization module spaces

$$\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G',\mathrm{Ran}_{x_0}} \to \mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}, \tag{2.4}$$

compatible with Gr_ϕ and the right actions of $\mathfrak{L}(G')_{x_0}$ and $\mathfrak{L}(G)_{x_0}$, respectively.

In particular, (2.4) induces a morphism

$$\underset{\mathfrak{L}(G')_{x_0}}{\underline{\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G',\mathrm{Ran}_{x_0}} \times \mathfrak{L}(G)_{x_0}}} \to \mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}, \tag{2.5}$$

compatible with the actions of $\mathfrak{L}(G)_{x_0}$, where $\underset{H}{\underline{(-)}}$ means "divide by the diagonal action of H."

Moreover, the diagram

$$\begin{CD}
\underset{\mathfrak{L}(G')_{x_0}}{\underline{\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G',\mathrm{Ran}_{x_0}} \times \mathfrak{L}(G)_{x_0}}} @>>> \mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}} \\
@VVV @VVV \\
\mathrm{Gr}_{G',\mathrm{Ran}_{x_0}} @>>> \mathrm{Gr}_{G,\mathrm{Ran}_{x_0}}
\end{CD}$$

in Cartesian.

In particular, we obtain that the morphism (2.5) is ind-proper.

2.2.3. Let us denote by $\mathbf{Res}_\phi$ the restriction functor

$$\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \to \mathrm{D}(\mathrm{Gr}_{G'})\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0},$$

corresponding to the factorization functor $(\mathrm{Gr}_\phi)_*$, see Sect. 1.4.7.

From Sect. 2.2.2, for any $\mathfrak{L}(G)_{x_0}$-module category $\mathbf{C}$ and the $\mathfrak{L}(G')_{x_0}$-module category $\mathbf{C}'$ given by restriction along $\mathfrak{L}(G')_{x_0} \to \mathfrak{L}(G)_{x_0}$, we obtain a naturally defined functor

$$(\mathbf{C}')^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_{G'})} \to \mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}, \tag{2.6}$$

compatible with Gr_ϕ.

In particular, we obtain a 1-morphism

$$(\mathbf{C}')^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_{G'})} \to \mathbf{Res}_{\mathrm{Gr}_\phi}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}). \tag{2.7}$$

in $\mathrm{D}(\mathrm{Gr}_{G'})\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$.

We claim:

Lemma 2.2.4. *The 1-morphism* (2.7) *is an isomorphism.*

Proof. Follows from Proposition C.10.20.

□

2.3. **The first reduction step.** From now on, until Sect. 10, we will be occupied with the proof of Theorem 2.1.6. In this subsection we perform the first reduction case: we show that we can assume that the source object of $\mathfrak{L}(G)_{x_0}$-**mod** is Vect, equipped with the trivial action of $\mathfrak{L}(G)_{x_0}$.

2.3.1. Let $\mathbf{C}$ be a category acted on by $\mathfrak{L}(G)$. Let

$$\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \tag{2.8}$$

denote its image under the functor (2.2).

The assertion of Theorem 2.1.6 is that for $\mathbf{C}_1, \mathbf{C}_2$ as above, the functor

$$\mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2) \to \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_{G,\mathrm{Ran}})\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathbf{C}_1^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}_2^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}) \tag{2.9}$$

is an equivalence.

In this subsection we will perform the first reduction step in the proof of Theorem 2.1.6. Namely, we will show that we can assume that $\mathbf{C}_1 = \mathrm{Vect}$, equipped with the trivial action of $\mathfrak{L}(G)_{x_0}$.

2.3.2. First, since the functor (2.2) is compatible with *colimits*, both sides in (2.9) send colimits in $\mathbf{C}_1$ to limits in DGCat.

Hence, it is enough to show that (2.9) is an isomorphism on objects of $\mathfrak{L}(G)$-$\mathbf{mod}$ that generate this category under colimits.

2.3.3. Note that a collection of generating objects for $\mathfrak{L}(G)_{x_0}$-$\mathbf{mod}$ provided by

$$\mathrm{D}(\mathfrak{L}(G)_{x_0}) \otimes \mathbf{C}_0, \tag{2.10}$$

where $\mathbf{C}_0$ is a plain DG category (i.e., the action of $\mathfrak{L}(G)_{x_0}$ on (2.10) comes from the action by left translations on the first factor).

This reduces the assertion that (2.9) is an equivalence to the particular case when $\mathbf{C}_1$ is of the form $\mathrm{D}(\mathfrak{L}(G)_{x_0}) \otimes \mathbf{C}_0$.

2.3.4. It is easy to see that for $\mathbf{C}_1 := \mathbf{C}_1' \otimes \mathbf{C}_0$, where the action comes from the first factor, we have

$$\mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathbf{C}_1, \mathbf{C}_2) \simeq \mathrm{Funct}(\mathbf{C}_0, \mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathbf{C}_1', \mathbf{C}_2))$$

and

$$\mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathbf{C}_1^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}_2^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}) \simeq$$
$$\simeq \mathrm{Funct}(\mathbf{C}_0, \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathbf{C}_1'^{\,\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}_2^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})).$$

Hence, if (2.9) is an equivalence for $\mathbf{C}_1'$, then it is an equivalence for $\mathbf{C}_1$.

2.3.5. Thus we obtain a further reduction of the assertion that (2.9) is an equivalence to the case when $\mathbf{C}_1 = \mathrm{D}(\mathfrak{L}(G)_{x_0})$.

2.3.6. Note that both sides in (2.1) have a natural symmetric monoidal structure:

In the left-hand side, if $\mathbf{C}_1, \mathbf{C}_2$ are DG categories equipped with an action of $\mathfrak{L}(G)_{x_0}$, then the tensor product $\mathbf{C}_1 \otimes \mathbf{C}_2$ acquires a $\mathfrak{L}(G)_{x_0}$-action via the diagonal action map $\mathfrak{L}(G)_{x_0} \to \mathfrak{L}(G)_{x_0} \times \mathfrak{L}(G)_{x_0}$. Furthermore, an object in $\mathfrak{L}(G)_{x_0}$-$\mathbf{mod}$ is dualizable if and only if the underlying DG category is dualizable.

In the right-hand side, if $\widetilde{\mathbf{C}}_1$ and $\widetilde{\mathbf{C}}_2$ are factorization module categories at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$, then $\widetilde{\mathbf{C}}_1 \otimes \widetilde{\mathbf{C}}_2$ (i.e., the tensor product of the corresponding crystals of categories over Ran_{x_0}) is naturally a factorization module category at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G) \otimes \mathrm{D}(\mathrm{Gr}_G)$. We produce the sought-for factorization module category at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$ by applying the restriction functor (see Sect. 1.4.7) along the (unital) factorization functor

$$(\Delta_{\mathrm{Gr}_G})_* : \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{D}(\mathrm{Gr}_G \times \mathrm{Gr}_G) \simeq \mathrm{D}(\mathrm{Gr}_G) \otimes \mathrm{D}(\mathrm{Gr}_G).$$

The following assertion follows from Lemma 2.2.4:

Lemma 2.3.7. *The functor* (2.1) *has a canonical symmetric monoidal structure.*

Thus, we obtain that if $\mathbf{C}_1 \in \mathfrak{L}(G)$-$\mathbf{mod}$ is dualizable, then the functor (2.9) is an equivalence if and only if it is an equivalence for $\mathbf{C}_1$ replaced by Vect and $\mathbf{C}_2$ replaced by $\mathbf{C}_1^\vee \otimes \mathbf{C}_2$.

2.3.8. Since $\mathrm{D}(\mathfrak{L}(G))$ is dualizable as a DG category (and, hence, as an object of $\mathfrak{L}(G)_{x_0}$-**mod**), we obtain that in order to prove that (2.9) is an equivalence, it is enough to do so in the case when $\mathbf{C}^1 = \mathrm{Vect}$.

In other words, it suffices to show that for $\mathbf{C} \in \mathfrak{L}(G)_{x_0}$-**mod**, the functor

$$\mathbf{C}^{\mathfrak{L}(G)_{x_0}} \simeq \mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C}) \to \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}) \tag{2.11}$$

is an equivalence.

3. A reformulation: factorization modules for the dualizing sheaf

In the previous section we reduced the assertion of Theorem 2.1.6 to its particular case, when the source $\mathfrak{L}(G)_{x_0}$-module is Vect.

From now on we will focus on this particular case and we will formulate a statement, Theorem 3.1.7, which talks about calculating the category of factorization modules for a particular factorization algebra. We will show that Theorem 3.1.7 is equivalent to the above particular case of Theorem 2.1.6.

3.1. **Factorization modules for the dualizing sheaf.** In this subsection we state Theorem 3.1.7. In a sense, this theorem on its own is no less interesting than Theorem 2.1.6: it gives a recipe of how to calculate $\mathfrak{L}(G)_{x_0}$-invariants using factorization algebras.

3.1.1. Denote by π_{Ran} the projection

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Ran}.$$

It has a natural structure of map between factorization spaces, which we denote by

$$\pi : \mathrm{Gr}_G \to \mathrm{pt}\,.$$

We consider

$$\pi^! : \mathrm{Vect} \simeq \mathrm{D}(\mathrm{pt}) \to \mathrm{D}(\mathrm{Gr}_G) \tag{3.1}$$

as a *lax-unital* factorization functor between unital factorization categories (see Sect. C.5.1).

The image of the unit factorization algebra $k \in \mathrm{Vect}$ under $\pi^!$ is a *unital* factorization algebra, denoted ω_{Gr_G} in $\mathrm{D}(\mathrm{Gr}_G)$. The corresponding object

$$(\omega_{\mathrm{Gr}_G})_{\mathrm{Ran}} \in (\mathrm{D}(\mathrm{Gr}_G))_{\mathrm{Ran}} := \mathrm{D}(\mathrm{Gr}_{G,\mathrm{Ran}})$$

is $\omega_{\mathrm{Gr}_{G,\mathrm{Ran}}}$, equipped with its natural factorization structure.

3.1.2. Denote by $\pi^{\mathrm{level}^\infty_{x_0}}_{\mathrm{Ran}_{x_0}}$ the projection $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}} \to \mathrm{Ran}_{x_0}$. We can regard it as a map between factorization module spaces for Gr_G and pt, respectively, compatible with π. Denote by

$$(\pi^{\mathrm{level}^\infty_{x_0}})^! : \mathrm{Vect} \simeq \mathrm{D}(\mathrm{pt}) \to \mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}$$

the resulting (lax-unital) functor between the factorization module categories with respect to $\mathrm{Vect} \simeq \mathrm{D}(\mathrm{pt})$ and $\mathrm{D}(\mathrm{Gr}_G)$, respectively, compatible with $\pi^!$ (see Sect. C.7.1).

According to C.7.14, the functor $(\pi^{\mathrm{level}^\infty_{x_0}})^!$ induces a functor

$$\mathrm{Vect} \simeq k\text{-mod}^{\mathrm{fact}}(\mathrm{Vect})_{x_0} \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)})_{x_0}, \tag{3.2}$$

where

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)})_{x_0}, \tag{3.3}$$

is the category of (unital) factorization modules at x_0 with respect to

$$\omega_{\mathrm{Gr}_G} \in \mathrm{FactAlg}(\mathrm{D}(\mathrm{Gr}_G))$$

in the (unital) factorization module category $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}$ at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$, see Sect. 1.5.4 and Sect. 1.5.9.

3.1.3. The functor (3.2) sends the generator $\mathsf{e} \in \mathrm{Vect}$ to an object

$$(\omega_{\mathfrak{L}(G)_{x_0}})^{\mathrm{fact}_{x_0},\omega_{\mathrm{Gr}_G}} \in \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \tag{3.4}$$

in (3.3). Explicitly, this object is given by the dualizing sheaf

$$\omega_{\mathrm{Gr}_{G,\mathrm{Ran}_{x_0}}^{\mathrm{level}_{x_0}^\infty}} \in \mathrm{D}(\mathrm{Gr}_{G,\mathrm{Ran}_{x_0}}^{\mathrm{level}_{x_0}^\infty}),$$

equipped with its natural factorization structure with respect to ω_{Gr_G}.

3.1.4. Note that the object (3.4) is naturally $\mathfrak{L}(G)_{x_0}$-equivariant, with respect to the action of $\mathfrak{L}(G)_{x_0}$ on (3.3) induced from its action on $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$ by *right translations* (see Sect. 2.1.3).

Since

$$\left(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}\right)^{\mathfrak{L}(G)_{x_0}} \simeq \mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)},$$

the object (3.4) corresponds to an object

$$k^{\mathrm{fact}_{x_0},\omega_{\mathrm{Gr}_G}} \in \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}. \tag{3.5}$$

3.1.5. By functoriality, the object (3.5) gives rise to a functor

$$\mathbf{C}^{\mathfrak{L}(G)_{x_0}} \simeq \mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}}(\mathrm{Vect},\mathbf{C}) \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \tag{3.6}$$

for $\mathbf{C} \in \mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}$.

3.1.6. Over the course Sects. 4-9 we will prove:

Theorem 3.1.7. *The functor* (3.6) *is an equivalence.*

A particular case of Theorem 3.1.7 is:

Corollary 3.1.8. *The functor* (3.2) *is an equivalence.*

3.1.9. In Sect. 3.2 below we will show that the assertion of Theorem 3.1.7 is logically equivalent to the statement that the functor (2.11) is an equivalence. As we have concluded in Sect. 2.3.8, the latter is equivalent to the statement of Theorem 2.1.6.

3.2. **The implication Theorem 3.1.7 $\Rightarrow$ Theorem 2.1.6.** In this subsection we will show that Theorem 3.1.7 implies (the particular case of $\mathbf{C}_1 = \mathrm{Vect}$ of) Theorem 2.1.6.

The basic tool here is the adjunction of Proposition C.10.8.

3.2.1. Note that since the map $\pi_{\mathrm{Ran}} : \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Ran}$ is proper, the functor $\pi^!_{\mathrm{Ran}}$ admits a left adjoint, given by $(\pi_{\mathrm{Ran}})_!$, compatible with the factorization structure.

We will denote by $\pi_!$ the resulting (unital) factorization functor $\mathrm{D}(\mathrm{Gr}_G) \to \mathrm{D}(\mathrm{pt}) = \mathrm{Vect}$. The functors $(\pi_!, \pi^!)$ form an adjoint pair as factorization functors, where $\pi^!$ is *lax-unital*.

Denote by $\mathbf{Res}_{\pi_!}$ the resulting restriction operation on factorization categories, see Sect. 1.4.7.

3.2.2. Recall that for a factorization category $\mathbf{A}$, we denote by $\mathbf{A}^{\mathrm{fact}_{x_0}}$ the tautological (i.e., vacuum) factorization module at x_0 with respect to $\mathbf{A}$, see Sect. 1.4.6.

In particular, for $\mathbf{A} = \mathrm{Vect}$, we can consider the (unital) factorization module category $\mathrm{Vect}^{\mathrm{fact}_{x_0}}$ at x_0, which under the embedding

$$\mathrm{DGCat} \hookrightarrow \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$$

corresponds to $\mathrm{Vect} \in \mathrm{DGCat}$.

We claim:

Proposition 3.2.3. *We have a canonical identification*

$$\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \simeq \mathbf{Res}_{\pi_!}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}).$$

The proof will be given in Sect. 3.3.

3.2.4. Recall now that for the *lax-unital* factorization functor $\pi^!$ there is a *unital* restriction operation

$$\mathbf{Res}^{\mathrm{untl}}_{\pi^!} : \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0} \to \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0},$$

see Sect. C.10.6.

Moreover, we have the following basic facts:

For $\widetilde{\mathbf{C}} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$, there is a canonical identification:

$$\mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathbf{Res}_{\pi_!}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}), \widetilde{\mathbf{C}}) \simeq \mathrm{Funct}_{\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}, \mathbf{Res}^{\mathrm{untl}}_{\pi^!}(\widetilde{\mathbf{C}})), \tag{3.7}$$

see Proposition C.10.8.

For $\widetilde{\mathbf{C}} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$, we have

$$\mathbf{Res}^{\mathrm{untl}}_{\pi^!}(\widetilde{\mathbf{C}})_{x_0} \simeq \pi^!(\mathrm{unit}_{\mathrm{Vect}})\text{-}\mathrm{mod}^{\mathrm{fact}}(\widetilde{\mathbf{C}})_{x_0}, \tag{3.8}$$

see Proposition C.10.10.

3.2.5. Combining Proposition 3.2.3 with (3.7) and (3.8), we obtain an equivalence

$$\begin{aligned}\mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}, \widetilde{\mathbf{C}}) \simeq \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathbf{Res}_{\pi_!}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}), \widetilde{\mathbf{C}}) \simeq \\ \simeq \mathrm{Funct}_{\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}, \mathbf{Res}^{\mathrm{untl}}_{\pi^!}(\widetilde{\mathbf{C}})) \simeq \mathbf{Res}^{\mathrm{untl}}_{\pi^!}(\widetilde{\mathbf{C}})_{x_0} \simeq \pi^!(\mathrm{unit}_{\mathrm{Vect}})\text{-}\mathrm{mod}^{\mathrm{fact}}(\widetilde{\mathbf{C}})_{x_0} = \\ = \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\widetilde{\mathbf{C}})_{x_0}.\end{aligned} \tag{3.9}$$

3.2.6. Let now $\mathbf{C}$ be an object of $\mathfrak{L}(G)_{x_0}$-$\mathbf{mod}$. Unwinding the constructions, we obtain:

Lemma 3.2.7. *The functor*

$$\begin{aligned}\mathbf{C}^{\mathfrak{L}(G)_{x_0}} \overset{(2.11)}{\longrightarrow} \mathrm{Funct}_{\mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}, \mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}) \simeq \\ \overset{(3.9)}{\simeq} \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\end{aligned}$$

identifies canonically with the functor (3.6).

3.2.8. From Lemma 3.6, we obtain that the assertion of Theorem 3.1.7 is equivalent to the statement that the functor (2.11) is an equivalence, which in turn is equivalent to the statement of Theorem 2.1.6.

3.3. Proof of Proposition 3.2.3.

3.3.1. By construction, we can identify $\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$ with

$$(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})^{\mathfrak{L}(G)_{x_0}}, \tag{3.10}$$

where we take invariants with respect to the (right) action of $\mathfrak{L}(G)_{x_0}$ on $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$, viewed as a factorization module category at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$.

We will now rewrite (3.10) slightly differently.

3.3.2. Consider the vacuum factorization module space over Gr_G at x_0; denote it $(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}}$, see Sect. 1.2.6. The resulting factorization module category identifies with $\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}}$.

The $\mathfrak{L}(G)_{x_0}$-action on $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$ gives rise to an action of the Hecke groupoid Hecke_{x_0} at x_0 on $(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}}$.

Tautologically, we can rewrite (3.10) as

$$(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}})^{\mathrm{Hecke}_{x_0}}. \tag{3.11}$$

Since Hecke_{x_0} is proper, we can rewrite (3.11) also as

$$(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}})_{\mathrm{Hecke}_{x_0}}. \tag{3.12}$$

3.3.3. The functor of !-pushforward along

$$\pi_{\mathrm{Ran}_{x_0}} : (\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}} \to \mathrm{Ran}_{x_0}$$

gives rise to a functor

$$(\pi_{x_0})_! : \mathrm{D}(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}} \to \mathrm{Vect}^{\mathrm{fact}_{x_0}}$$

as factorization module categories with respect to $\mathrm{D}(\mathrm{Gr})$ and Vect, compatible with the factorization functor $\pi_! : \mathrm{D}(\mathrm{Gr}) \to \mathrm{Vect}$.

Moreover, the functor $(\pi_{x_0})_!$ canonically factors via a functor

$$(\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}})_{\mathrm{Hecke}_{x_0}} \to \mathrm{Vect}^{\mathrm{fact}_{x_0}}\,. \tag{3.13}$$

i.e., a functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \to \mathrm{Vect}^{\mathrm{fact}_{x_0}}, \tag{3.14}$$

as factorization module categories with respect to $\mathrm{D}(\mathrm{Gr})$ and Vect, compatible with the factorization functor $\pi_! : \mathrm{D}(\mathrm{Gr}) \to \mathrm{Vect}$.

By the definition of the restriction operation $\mathbf{Res}_{\pi_!}$, the functor (3.14) gives rise to a (unital) functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \to \mathbf{Res}_{\pi_!}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}) \tag{3.15}$$

as factorization module categories over $\mathrm{D}(\mathrm{Gr}_G)$.

The functor (3.15) is the sought-for functor in Proposition 3.2.3.

3.3.4. We will now show that (3.15) is an equivalence. In order to do so, we will apply Proposition C.10.20.

Condition (i) in this lemma is satisfied because the morphism π_{Ran} is proper. Condition (iii) is satisfied, since at the level of fibers at x_0, the functor (3.15) induces an identity endofunctor of Vect.

Hence, it remains to show that the functor (3.15) admits a right adjoint, viewed as a functor between sheaves of categories over Ran_{x_0}.

3.3.5. Pullback along $\pi_{\mathrm{Ran}_{x_0}}$ is a functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0}} \to \mathrm{D}((\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}}),$$

which naturally factors via a functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0}} \to (\mathrm{D}((\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}}))^{\mathrm{Hecke}_{x_0}}. \tag{3.16}$$

Interpreting $\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$ as (3.11), the functor (3.16) provides a right adjoint to (3.15).
□[Proposition 3.2.3]

Remark 3.3.6. Unwinding the construction, one can describe the functor right adjoint (which is also the inverse) of (3.15) as follows:

By (3.7), a datum of a functor

$$\mathbf{Res}_{\pi_!}(\mathrm{Vect}^{\mathrm{fact}_{x_0}}) \to \mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \tag{3.17}$$

is equivalent to that of a functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0}} \to \mathbf{Res}^{\mathrm{untl}}_{\pi^!}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}),$$

while the latter is equivalent to that of a functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0}} \to \mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \tag{3.18}$$

of factorization module categories over Vect and $\mathrm{D}(\mathrm{Gr}_G)$, respectively, compatible with $\pi^!$.

The functor (3.18) is given by the natural factorization of

$$\pi^!_{x_0} : \mathrm{Vect}^{\mathrm{fact}_{x_0}} \to \mathrm{D}((\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}})$$

as

$$\mathrm{Vect}^{\mathrm{fact}_{x_0}} \to (\mathrm{D}(\mathrm{Gr}_G)^{\mathrm{fact}_{x_0}})^{\mathrm{Hecke}_{x_0}} \simeq \mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}\,.$$

3.4. **Example: the case of** Vect**.** In this section we run a plausibility check for Theorem 3.1.7 when $\mathbf{C} = \text{Vect}$ and the group G is semi-simple and simply-connected. We calculate explicitly both sides and show that they are abstractly isomorphic.

3.4.1. Let us apply Theorem 3.1.7 to $\mathbf{C} := \text{Vect}$, viewed as an object of $\mathfrak{L}(G)_{x_0}$-**mod**, equipped with the trivial action. We obtain:

Corollary 3.4.2. *The functor*

$$\text{Vect}^{\mathfrak{L}(G)_{x_0}} \to \omega_{\text{Gr}_G}\text{-mod}^{\text{fact}}(\text{Vect}^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)})_{x_0} \tag{3.19}$$

of (3.6) *is an equivalence.*

3.4.3. In Sect. 4.7.7 we will show that the assertion of Corollary 3.4.2 is equivalent to a key calculation involved in the proof of Theorem 3.1.7.

In the rest of this subsection, we will explain that the existence of *an* equivalence

$$\text{Vect}^{\mathfrak{L}(G)_{x_0}} \simeq \omega_{\text{Gr}_G}\text{-mod}^{\text{fact}}(\text{Vect}^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)})_{x_0} \tag{3.20}$$

is a priori known, at least when G is semi-simple and simply-connected.

3.4.4. Recall that according to Proposition 3.2.3, we have

$$\text{Vect}^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)} \simeq \mathbf{Res}_{\pi_!}(\text{Vect}^{\text{fact}_{x_0}}).$$

Combining with (1.9), we obtain

$$\omega_{\text{Gr}_G}\text{-mod}^{\text{fact}}(\text{Vect}^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)})_{x_0} \simeq \pi_!(\omega_{\text{Gr}_G})\text{-mod}^{\text{fact}}(\text{Vect}^{\text{fact}_{x_0}})_{x_0} =: \pi_!(\omega_{\text{Gr}_G})\text{-mod}^{\text{fact}}_{x_0},$$

where in the right-hand side, $\pi_!(\omega_{\text{Gr}_G})$ is viewed as a plain unital factorization algebra, and $\text{-mod}^{\text{fact}}_{x_0}$ refers to the plain category of unital factorization modules at x_0.

3.4.5. Assume now that G is semi-simple and simply-connected. Let $\mathfrak{a}_{BG}$ be the Lie algebra[4] that controls the rational homotopy type of the classifying stack BG of G, which is characterized by a canonical isomorphism between cocommutative coalgebras

$$\text{C}_\cdot(\mathfrak{a}_{BG}) \simeq \text{C}_\cdot(BG),$$

where the first $\text{C}_\cdot(-)$ denoted the homological Chevalley complex of a Lie algebra.

Let $\mathfrak{a}_{BG,X} := \mathfrak{a}_{BG} \otimes k_X$ be the corresponding constant Lie*-algebra and $U^{\text{ch}}(\mathfrak{a}_{BG,X})$ be its chiral universal enveloping algebra. We have the following result:

Lemma 3.4.6. *Let*

$$\pi_!(\omega_{\text{Gr}_G})^{\text{ch}} := \pi_!(\omega_{\text{Gr}_G})|_X[-1]$$

be the unital chiral algebra corresponding to the unital factorization algebra $\pi_!(\omega_{\text{Gr}_G})$ *(see* [GLC2, Sect. D.1]*). We have a canonical isomorphism*

$$\pi_!(\omega_{\text{Gr}_G})^{\text{ch}} \simeq U^{\text{ch}}(\mathfrak{a}_{BG,X}).$$

Sketch. By [Ga1, Theorem 15.3.3], the *augmented cocommutative factorization algebra*[5] $\mathcal{A} := \pi_!(\omega_{\text{Gr}_G})$ and the *augmented commutative factorization algebra*[6] $\mathcal{B} := \text{C}^\cdot(BG)$ are "Verdier dual" to each other

[4]The Lie algebra $\mathfrak{a}_{BG}$ is actually abelian. Indeed, $\text{C}^\cdot(\mathfrak{a}_{BG}) \simeq \text{C}^\cdot(BG)$, which is isomorphic to a polynomial algebra.

[5]A unital cocommutative factorization algebra is a unital factorization algebra $\mathcal{A}$ whose structural isomorphism

$$\text{union}^!(\mathcal{A})|_{(\text{Ran}\times\text{Ran})_{\text{disj}}} \xrightarrow{\simeq} (\mathcal{A} \boxtimes \mathcal{A})|_{(\text{Ran}\times\text{Ran})_{\text{disj}}}$$

is extended to a not necessarily invertible map $\text{union}^!(\mathcal{A}) \to \mathcal{A} \boxtimes \mathcal{A}$ (and is equipped with a homotopy-coherent data of commutativity and associativity). Being augmented means it is equipped with a homomorphism $\mathcal{A} \to \text{unit}_{\text{Vect}}$ compatible with the cocommutative factorization structures. Informally speaking, this means the cocommutative coalgebra structure on the cochain complex $\text{C}_\cdot(\text{Gr}_G)$ is naturally compatible with its factorization structure induced from the factorization space Gr_G.

[6]A unital commutative factorization algebra is a unital factorization algebra $\mathcal{B}$ whose structural isomorphism is extended to a not necessarily invertible map $\mathcal{B} \boxtimes \mathcal{B} \to \text{union}^!(\mathcal{B})$ (and is equipped with a homotopy-coherent data). By [BD1, Sect. 3.4.20-3.4.22], knowing a unital commutative factorization algebra is equivalent to knowing a unital commutative algebra in the symmetric monoidal category $\text{D}(X)$. Via this correspondence, $\mathcal{B}$ is given by $\text{C}^\cdot(BG) \otimes \omega_X$.

after removing the augmented units. Here the notion of Verdier duality is developed in [Ga1] and is subtler than the usual one because the D-modules $\mathcal{A}_{\mathrm{Ran}}$ and $\mathcal{B}_{\mathrm{Ran}}$ are not compact. Nevertheless, [Ho, Theorem 1.5.9] shows that after passing to the *chiral Koszul duals* (see [FG] for what this means), this Verdier duality becomes the usual one. In other words,

$$\mathcal{A}^{\vee,\mathrm{KD}} \simeq \mathbb{D}_X^{\mathrm{Ver}}(\mathcal{B}^{\vee,\mathrm{KD}}),$$

where $\mathcal{A}^{\vee,\mathrm{KD}}$ is the Lie*-algebra that is Koszul dual to $\mathcal{A}$, while $\mathcal{B}^{\vee,\mathrm{KD}}$ is the Lie*-coalgebra that is Koszul dual to $\mathcal{B}$. Since $\mathcal{B}|_X$ is constant with !-fibers equal to $\mathrm{C}^\cdot(BG)$, we see $\mathcal{B}^{\vee,\mathrm{KD}}$ is constant with !-fibers equal to the Koszul dual of $\mathrm{C}^\cdot(BG)$, which is just the Lie coalgebra $(\mathfrak{a}_{BG})^*$. It follows that $\mathcal{A}^{\vee,\mathrm{KD}}$ is constant with $*$-fibers equal to the Lie algebra $\mathfrak{a}_{BG}$. In other words,

$$\mathcal{A}^{\vee,\mathrm{KD}} \simeq \mathfrak{a}_{BG,X}.$$

By [FG, Proposition 6.1.2], this implies

$$\mathcal{A} \simeq U^{\mathrm{ch}}(\mathfrak{a}_{BG,X}).$$

□

3.4.7. As a corollary, we have

$$\pi_!(\omega_{\mathrm{Gr}_G})\text{-mod}^{\mathrm{fact}}_{x_0} \simeq \pi_!(\omega_{\mathrm{Gr}_G})^{\mathrm{ch}}\text{-mod}^{\mathrm{ch}}_{x_0} \simeq \mathfrak{a}_{BG,X}\text{-mod}^{\mathrm{Lie}^\star}_{x_0}.$$

Since $\mathfrak{a}_{BG,X}$ is a constant Lie*-algebra, the above category is equivalent to the category of modules over the Lie algebra

$$\mathfrak{a}_{BG} \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0}), \tag{3.21}$$

or which is the same, to the category of modules over

$$U(\mathfrak{a}_{BG} \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0})). \tag{3.22}$$

3.4.8. The category $\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}}$ is equivalent to the category of modules over

$$\mathrm{C}_\cdot(\mathfrak{L}(G)_{x_0}), \tag{3.23}$$

viewed as an associative algebra via the product operation on $\mathfrak{L}(G)_{x_0}$.

Let $\mathfrak{a}_G$ be the group-object in the category of Lie algebras that controls the homotopy of G. The assumption that G is semi-simple and simply-connected implies that $\mathfrak{L}(G)_{x_0}$ is connected and simply-connected. According to [GL, Theorem 1.4.4], the group-object in the category of Lie algebras that controls the rational homotopy of $\mathfrak{L}(G)_{x_0}$ is canonically isomorphic to

$$\mathfrak{a}_G \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0}).$$

Hence,

$$\mathrm{C}_\cdot(\mathfrak{L}(G)_{x_0}) \simeq \mathrm{C}_\cdot(\mathfrak{a}_G \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0})). \tag{3.24}$$

The structure of an associative algebra on $\mathrm{C}_\cdot(\mathfrak{a}_G \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0}))$ is induced by the group structure on $\mathfrak{a}_G \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0})$ as a Lie algebra, where the latter results from the group structure on G.

3.4.9. Note that we have

$$\mathfrak{a}_G = \Omega(\mathfrak{a}_{BG}),$$

as group-objects in the category of Lie algebras, where $\Omega(-)$ is the loop functor on the category of Lie algebras.

Hence, we also have

$$\mathfrak{a}_G \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0}) \simeq \Omega(\mathfrak{a}_{BG} \otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0})).$$

Finally, according to [GR2, Chapter 5, Theorem 6.1.2], we have

$$\mathrm{C}_\cdot(\Omega(-)) \simeq U(-),$$

and hence the associative algebras (3.22) and (3.23) are canonically isomorphic.

3.4.10. Summarizing, we obtain

$$\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \simeq \mathrm{C}_\cdot(\mathfrak{L}(G)_{x_0})\text{-mod} \simeq \mathrm{C}_\cdot(\Omega(\mathfrak{a}_{BG}\otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0})))\text{-mod} \simeq$$
$$\simeq U(\mathfrak{a}_{BG}\otimes \mathrm{C}^\cdot(\mathcal{D}^\times_{x_0}))\text{-mod} \simeq \pi_!(\omega_{\mathrm{Gr}_G})\text{-mod}^{\mathrm{fact}}_{x_0} \simeq \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}.$$

4. Proof of Theorem 3.1.7: reduction to the case of a trivial action

In this subsection we will reduce the assertion of Theorem 3.1.7 to the case when $\mathbf{C} = \mathrm{Vect}$. The main tool will be the notion of *almost trivial* action of a group on a category.

4.1. **Almost constant sheaves.** In this subsection we discuss the notion of *almost constant* sheaf on a scheme of finite type. This notion will be relevant for defining the notion of *almost trivial* action in the subsequent subsections.

4.1.1. Let Y be a scheme of finite type. Let

$$\mathrm{D}(Y)^{\text{alm-const}} \subset \mathrm{D}(Y) \tag{4.1}$$

be the full subcategory generated by the constant sheaf $\underline{k}_Y$.

We have a canonical identification

$$\mathrm{D}(Y)^{\text{alm-const}} \simeq \mathrm{C}^\cdot(Y)\text{-mod}, \tag{4.2}$$

given by the action of $\mathrm{C}^\cdot(Y)$ on $\underline{k}_Y$.

Remark 4.1.2. We warn the reader that the assignment $Y \mapsto \mathrm{D}(Y)^{\text{alm-const}}$ (unlike its close relative $\mathrm{D}(Y)^{\text{q-const}}$, see Sect. 4.1.4) is *not* a sheaf even for the Zariski topology. See, however, Sect. 5.2.3 for a descent-type statement.

4.1.3. The embedding (4.1) admits a right adjoint. In terms of the identification (4.2), this right adjoint is given by

$$\mathcal{F} \in \mathrm{D}(Y) \mapsto \mathrm{C}^\cdot(Y,\mathcal{F}) \in \mathrm{C}^\cdot(Y)\text{-mod}.$$

Thus, we can view $\mathrm{D}(Y)^{\text{alm-const}}$ as a quotient of $\mathrm{D}(Y)$ by the full subcategory

$$\{\mathcal{F} \in \mathrm{D}(Y) \mid \mathrm{C}^\cdot(Y,\mathcal{F}) = 0\}.$$

4.1.4. The category $\mathrm{D}(Y)^{\text{alm-const}}$ carries a natural t-structure, in which $\underline{k}_Y$ is in the heart.

Let

$$\mathrm{D}(Y)^{\text{q-const}}$$

be the left-completion of $\mathrm{D}(Y)^{\text{alm-const}}$ with respect to this t-structure. It is easy to see that the embedding (4.1) extends to a fully faithful embedding

$$\mathrm{D}(Y)^{\text{q-const}} \subset \mathrm{D}(Y). \tag{4.3}$$

The essential image of (4.3) is the full subcategory of $\mathrm{D}^{\mathrm{hol}}(Y)$ consisting of objects, whose cohomologies with respect to either perverse or the *usual* t-structure[7] admit a filtration with subquotients isomorphic to $\underline{k}_Y$.

Remark 4.1.5. The (fully faithful) embedding

$$\mathrm{D}(Y)^{\text{alm-const}} \hookrightarrow \mathrm{D}(Y)^{\text{q-const}} \tag{4.4}$$

is not always an equivalence. E.g., it fails to be an equivalence for $Y = \mathbb{P}^1$ (and, which is more, relevant for us, for Y being a semi-simple group).

[7] One can mimic the definition of the usual t-structure on constructible sheaves and define its counterpart on $\mathrm{D}^{\mathrm{hol}}(Y)$.

4.1.6. Assume that Y is connected and simply-connected. Choose a base point $y \in Y$, and let L_Y be the Lie algebra that controls the homotopy type of Y (since Y is simply-connected, L_Y sits in cohomological degrees ≤ -1). In particular, we have

$$\mathrm{C}^{\cdot}(Y) \simeq \mathrm{C}^{\cdot}(L_Y) \text{ and } \mathrm{C}_{\cdot}(Y) \simeq \mathrm{C}_{\cdot}(L_Y).$$

The functor of *-fiber at y defines an equivalence

$$\mathrm{D}(Y)^{\text{q-const}} \simeq L_Y\text{-mod}.$$

In particular, we obtain that (4.4) is an equivalence if and only if the universal enveloping algebra of L_Y is eventually coconnetive (equivalently, if L_Y sits only in *odd* degrees).

Note also that if $U(L_Y)$ is *not* eventually coconnetive, the embedding (4.3) does not preserve compactness; in particular, its right adjoint is discontinuous.

4.1.7. Let

$$\mathrm{D}(Y)^{\omega,\text{alm-const}} \hookrightarrow \mathrm{D}(Y) \tag{4.5}$$

be the full subcategory of $\mathrm{D}(Y)$ obtained from ones in (4.4) by $\overset{*}{\otimes}$-tensoring with ω_Y.

We still have an equivalence

$$\mathrm{C}^{\cdot}(Y)\text{-mod} \simeq \mathrm{D}(Y)^{\omega,\text{alm-const}}, \tag{4.6}$$

given by sending

$$\mathrm{C}^{\cdot}(Y) \in \mathrm{C}^{\cdot}(Y)\text{-mod} \mapsto \omega_Y \in \mathrm{D}(Y)^{\omega,\text{alm-const}}.$$

4.1.8. Note that Verdier duality identifies

$$\mathrm{D}(Y)^{\omega,\text{alm-const}} \simeq (\mathrm{D}(Y)^{\text{alm-const}})^{\vee}.$$

Under this identification, the functor (4.5) is the dual of the right adjoint of (4.1).

4.1.9. Let now $\mathcal{Y}$ be an ind-scheme of ind-finite type:

$$\mathcal{Y} \simeq \text{“}\underset{i}{\operatorname{colim}}\text{”}\, Y_i,$$

where Y_i's are schemes of finite type, and the transition maps $Y_{i_1} \to Y_{i_2}$ are closed embeddings.

Recall that the category $\mathrm{D}(\mathcal{Y})$ is defined as

$$\lim_{i}\, \mathrm{D}(Y_i),$$

where the limit is taken with respect to the !-pullbacks.

Let

$$\mathrm{D}(\mathcal{Y})^{\omega,\text{alm-const}} \subset \mathrm{D}(\mathcal{Y})$$

be the full subcategory equal to

$$\lim_{i}\, \mathrm{D}(Y_i)^{\omega,\text{alm-const}} \subset \lim_{i}\, \mathrm{D}(Y_i)$$

4.1.10. Recall that

$$\mathrm{C}^{\cdot}(\mathcal{Y}) \simeq \lim_{i}\, \mathrm{C}^{\cdot}(Y_i).$$

Hence, $\mathrm{C}^{\cdot}(\mathcal{Y})$ acts on $\omega_{\mathcal{Y}} \in \mathrm{D}(\mathcal{Y})^{\omega,\text{alm-const}}$, and we obtain a functor

$$\mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod} \to \mathrm{D}(\mathcal{Y})^{\omega,\text{alm-const}}. \tag{4.7}$$

However, the functor (4.7) is in general not even fully faithful (e.g., it fails to be such for $\mathcal{Y} = \mathbb{P}^{\infty}$).

4.1.11. Note that we can also write

$$\mathrm{D}(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{D}(Y_i), \tag{4.8}$$

with transition functors being given by *-pushforward.

We let $\mathrm{D}(\mathcal{Y})^{\text{alm-const}}$ be the *quotient* category of $\mathrm{D}(\mathcal{Y})$, defined in terms of (4.8) as

$$\underset{i}{\mathrm{colim}}\, \mathrm{D}(Y_i)^{\text{alm-const}}.$$

We do not know whether the category $\mathrm{D}(\mathcal{Y})^{\text{alm-const}}$ is dualizable in general. However, we have:

$$\mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}(\mathcal{Y})^{\text{alm-const}}, \mathrm{Vect}) \simeq \mathrm{D}(\mathcal{Y})^{\omega,\text{alm-const}}. \tag{4.9}$$

4.1.12. In terms of the equivalence (4.2), we have:

$$\mathrm{D}(\mathcal{Y})^{\text{alm-const}} \simeq \underset{i}{\mathrm{colim}}\, \mathrm{C}^{\cdot}(Y_i)\text{-mod}, \tag{4.10}$$

where the transition functors are given by restriction along the maps

$$\mathrm{C}^{\cdot}(Y_{i_2}) \to \mathrm{C}^{\cdot}(Y_{i_1}).$$

We have a naturally defined functor

$$\underset{i}{\mathrm{colim}}\, \mathrm{C}^{\cdot}(Y_i)\text{-mod} \to \mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod}, \tag{4.11}$$

given by restriction.

In terms of (4.9), the dual of the functor (4.11) is the functor (4.7).

4.2. **Almost trivial actions: the case of algebraic groups.** In this subsection we develop the notion of *almost trivial* action for groups of finite type.

4.2.1. Let H be an algebraic group of finite type.

Consider the embedding

$$\mathrm{D}(H)^{\text{alm-const}} \hookrightarrow \mathrm{D}(H). \tag{4.12}$$

By (4.2), we can identify

$$\mathrm{D}(H)^{\text{alm-const}} \simeq \mathrm{C}^{\cdot}(H)\text{-mod}. \tag{4.13}$$

4.2.2. The subcategory (4.12) is preserved by the monoidal operation, and hence inherits a monoidal structure. The *right* adjoint to (4.12) is (strictly) compatible with monoidal structures.

In terms of the identification (4.13), the monoidal structure on $\mathrm{D}(H)^{\text{alm-const}}$ corresponds to the Hopf algebra structure on $\mathrm{C}^{\cdot}(H)$, induced by the group-structure on H.

The latter description implies that the monoidal category $\mathrm{D}(H)^{\text{alm-const}}$ is *semi-rigid* (see [AGKRRV, Appendix C] for what this means).

4.2.3. The monoidal functor

$$\mathrm{C}^{\cdot}(H, -) : \mathrm{D}(H) \to \mathrm{Vect}$$

induces a monoidal functor

$$\mathrm{D}(H)^{\text{alm-const}} \to \mathrm{Vect}, \tag{4.14}$$

which admits a left adjoint. The functor (4.14) is conservative.

In terms of the identification (4.13), the functor (4.14) corresponds to the forgetful functor

$$\mathrm{C}^{\cdot}(H)\text{-mod} \to \mathrm{Vect}.$$

4.2.4. Let $\mathbf{C}$ be a category acted on by H. Set

$$\text{alm-inv}_H(\mathbf{C}) := \mathrm{D}(H)^{\text{alm-const}} \underset{\mathrm{D}(H)}{\otimes} \mathbf{C}.$$

The embedding (4.12) and its right adjoint give rise to a pair of adjoint functors

$$\text{alm-inv}_H(\mathbf{C}) \rightleftarrows \mathbf{C}, \tag{4.15}$$

with the left adjoint being fully faithful.

4.2.5. We shall say that action of $\mathbf{C}$ is *almost trivial* if the functors (4.15) are mutually inverse equivalences.

4.2.6. *Example.* An example to keep in mind of an action that is *not* almost trivial is

$$\mathbf{C} := \mathrm{D}(G)^{\text{q-const}}.$$

In fact, for

$$\mathbf{C}' := \mathrm{D}(G)^{\text{q-const}} / \mathrm{D}(G)^{\text{alm-const}},$$

we have $(\mathbf{C}')^G = 0$.

4.2.7. Let $(H\text{-}\mathbf{mod})_{\text{alm-trivial}} \subset H\text{-}\mathbf{mod}$ be the full subcategory that consists of H-module categories equipped with an almost trivial action.

The embedding

$$(H\text{-}\mathbf{mod})_{\text{alm-trivial}} \hookrightarrow H\text{-}\mathbf{mod} \tag{4.16}$$

admits a right adjoint, given by

$$\mathbf{C} \mapsto \text{alm-inv}_H(\mathbf{C}). \tag{4.17}$$

The counit of this adjunction is the left adjoint in (4.15). Since (4.15) admits a right adjoint, we can identify (4.17) also with the *left* adjoint of (4.16).

4.2.8. Consider the category

$$\text{inv}_H(\mathbf{C}) := \mathbf{C}^H \simeq \text{Vect} \underset{\mathrm{D}(H)}{\otimes} \mathbf{C}.$$

Consider the corresponding pair of adjoint functors

$$\text{oblv}_H : \mathbf{C}^H \rightleftarrows \mathbf{C} : \text{Av}^H_* .$$

It is clear that the functor oblv_H has essential image contains in $\text{alm-inv}_H(\mathbf{C})$. Hrnce, the functor Av^H_* factors as

$$\mathbf{C} \to \text{alm-inv}_H(\mathbf{C}) \to \mathbf{C}^H,$$

where the first arrow is the right adjoint in (4.15), and the second arrow is the right adjoint to

$$\text{oblv}_H : \mathbf{C}^H \to \text{alm-inv}_H(\mathbf{C}). \tag{4.18}$$

One can view the adjunction

$$\mathbf{C}^H \rightleftarrows \text{alm-inv}_H(\mathbf{C})$$

as obtained by tensoring $- \underset{\mathrm{D}(H)}{\otimes} \mathbf{C}$ from the adjunction

$$\text{Vect} \rightleftarrows \mathrm{D}(H)^{\text{alm-const}}. \tag{4.19}$$

From the above it follows that the essential image of oblv_H generates $\text{alm-inv}_H(\mathbf{C})$ under colimits. Moreover, we have:

Lemma 4.2.9. *The kernel of the right adjoint in* (4.15) *equals* $\ker(\text{Av}^H_*)$.

4.2.10. Note that the left adjoint in (4.19) itself admits a left adjoint[8]. Since the monoidal category $\mathrm{D}(H)^{\text{alm-const}}$ is semi-rigid, this left adjoint is automatically $\mathrm{D}(H)^{\text{alm-const}}$-linear.

This implies that the forgetful functor $\mathbf{C}^H \to \text{alm-inv}_H(\mathbf{C})$ also admits a left adjoint, functorially in $\mathbf{C}$. We denote this left adjoint by $\mathrm{Av}_!^H$.

4.2.11. Consider the functor

$$\mathrm{inv}_H : H\text{-}\mathbf{mod} \to \mathrm{DGCat}\,.$$

It naturally enhances to a functor

$$\mathrm{inv}_H^{\mathrm{enh}} : H\text{-}\mathbf{mod} \to \mathrm{Vect}^H\text{-}\mathbf{mod},$$

where we identify

$$\mathrm{Vect}^H \simeq \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Vect})$$

as a monoidal category.

The functor $\mathrm{inv}_H^{\mathrm{enh}}$ admits a left adjoint, given by

$$\widetilde{\mathbf{C}} \mapsto \mathrm{Vect} \underset{\mathrm{Vect}^H}{\otimes} \widetilde{\mathbf{C}}.$$

It is clear, however, that the adjunction $((\mathrm{inv}_H^{\mathrm{enh}})^L, \mathrm{inv}_H^{\mathrm{enh}})$ factors as

$$\mathrm{Vect}^H\text{-}\mathbf{mod} \rightleftarrows (H\text{-}\mathbf{mod})_{\text{alm-triv}} \rightleftarrows H\text{-}\mathbf{mod}.$$

We claim:

Proposition 4.2.12. *The adjoint functors*

$$\mathrm{Vect}^H\text{-}\mathbf{mod} \rightleftarrows (H\text{-}\mathbf{mod})_{\text{alm-triv}} \tag{4.20}$$

are mutually inverse equivalences.

The proposition will be proved in Sect. 5.1.

Corollary 4.2.13. *For* $\mathbf{C} \in H\text{-}\mathbf{mod}$*, the counit of the adjunction*

$$\mathrm{Vect} \underset{\mathrm{Vect}^H}{\otimes} \mathbf{C}^H \to \mathbf{C}$$

is fully faithful with essential image $\text{alm-inv}_H(\mathbf{C})$*.*

4.3. **Almost constant sheaves on the affine Grassmannian.** In this subsection we give a Koszul-dual description of the category of almost constant sheaves on the affine Grassmannian and related geometries.

4.3.1. *Notational change.* From now and until Sect. 4.7 we will adopt the following notational change[9]

$$\mathfrak{L}(G)_{x_0} \rightsquigarrow \mathfrak{L}(G),\ \mathfrak{L}^+(G)_{x_0} \rightsquigarrow \mathfrak{L}^+(G),\ \mathrm{Gr}_{G,x_0} \rightsquigarrow \mathrm{Gr}_G.$$

4.3.2. We return to the setting of Sect. 4.1.12. We take $\mathcal{Y}$ to be the *neutral connected component* of $\mathfrak{L}(G)/K$, where $K = K_i$ for $i \geq 0$.

[8]This follows, e.g., from the fact that $\mathrm{C}^{\cdot}(H)$ is finite-dimensional.

[9]We do it since the geometry of the curve will not be involved, unlike other places in this paper.

4.3.3. Let $\mathcal{Y}_0 := \mathfrak{L}^+(G)/K$. Consider the restriction map

$$\mathrm{C}^\cdot(\mathcal{Y}) \to \mathrm{C}^\cdot(\mathcal{Y}_0).$$

From (3.24)[10], we obtain that the restriction map

$$\mathrm{C}^\cdot(\mathcal{Y}_0)\text{-mod} \to \mathrm{C}^\cdot(\mathcal{Y})\text{-mod} \tag{4.21}$$

preserves compactness.

We let $\mathrm{C}^\cdot(\mathcal{Y})\text{-mod}_0$ be the full subcategory of $\mathrm{C}^\cdot(\mathcal{Y})\text{-mod}$, generated by objects in the essential image of (4.21). The embedding

$$\mathrm{C}^\cdot(\mathcal{Y})\text{-mod}_0 \hookrightarrow \mathrm{C}^\cdot(\mathcal{Y})\text{-mod}$$

admits a continuous right adjoint.

4.3.4. *Example.* Let $i = 0$, so $\mathcal{Y}$ is the neutral component of Gr_{G,x_0}. We have

$$\mathrm{C}^\cdot(\mathcal{Y}) \simeq \mathrm{Sym}(V),$$

where V is a finite-dimensional, cohomologically graded vector space, concentrated in positive even degrees.

In this case

$$\mathrm{C}^\cdot(\mathcal{Y})\text{-mod}_0 \simeq \mathrm{Sym}(V)\text{-mod}_0,$$

where the category in the right-hand side is the full subcategory of $\mathrm{Sym}(V)$-mod, generated by the augmentation module.

4.3.5. Recall the functor (4.11). We claim:

Proposition 4.3.6. *The functor* (4.11)*, i.e.,*

$$\mathrm{D}(\mathcal{Y})^{\text{alm-const}} \to \mathrm{C}^\cdot(\mathcal{Y})\text{-mod},$$

is an equivalence onto $\mathrm{C}^\cdot(\mathcal{Y})\text{-mod}_0 \subset \mathrm{C}^\cdot(\mathcal{Y})\text{-mod}$*.*

The proposition will be proved in Sect. 5.3.

Corollary 4.3.7. *The category* $\mathrm{D}(\mathcal{Y})^{\text{alm-const}}$ *is dualizable.*

Corollary 4.3.8. *The category* $\mathrm{D}(\mathfrak{L}(G)/K)^{\text{alm-const}}$ *is dualizable.*

Corollary 4.3.9. *For a category* $\mathbf{C}$*, the functor*

$$\mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}} \otimes \mathbf{C} \to \mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}(\mathcal{Y})^{\text{alm-const}}, \mathbf{C})$$

is an equivalence.

Corollary 4.3.10. *For a category* $\mathbf{C}$*, the functor*

$$\mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}} \otimes \mathbf{C} \to \mathrm{D}(\mathfrak{L}(G)/K) \otimes \mathbf{C}$$

is fully faithful.

Proof. Follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}} \otimes \mathbf{C} \arrow[r] \arrow[d, "\sim"'] & \mathrm{D}(\mathfrak{L}(G)/K) \otimes \mathbf{C} \arrow[d, "\sim"] \\
\mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}(\mathfrak{L}(G)/K)^{\text{alm-const}}, \mathbf{C}) \arrow[r] & \mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}(\mathfrak{L}(G)/K), \mathbf{C}),
\end{tikzcd}$$

in which the right vertical arrow is given by Verdier duality on $\mathrm{D}(\mathfrak{L}(G)/K)$, and the bottom horizontal arrow is fully faithful. □

[10]Indeed, the cochain algebras involved are the same as in the semi-simple simply-connected case.

Remark 4.3.11. As another consequence of Proposition 4.3.6, we obtain that

$$\mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}} \subset \mathrm{D}(\mathfrak{L}(G)/K)$$

is the full subcategory generated under colimits by the dualizing sheaf.

4.4. **Almost constant sheaves on the loop group.** In this subsection we define what we mean by the category of almost constant sheaves on $\mathfrak{L}(G)$. As usual, some extra care is needed here, since $\mathfrak{L}(G)$ is of infinite type.

4.4.1. Let us return to the setting of Sect. 4.1.7. Let $f : Y' \to Y$ be a map of schemes of finite type that is a universal homological equivalence.

In this case the functor

$$f^! : \mathrm{D}(Y) \to \mathrm{D}(Y')$$

gives rise to an *equivalence*

$$\mathrm{D}(Y)^{\omega,\text{alm-const}} \overset{\sim}{\to} \mathrm{D}(Y')^{\omega,\text{alm-const}}.$$

A similar observation applies to a map between indschemes $f : \mathcal{Y} \to \mathcal{Y}'$.

4.4.2. Let $K' \subset K$ be subgroups as in Sect. 4.3.2, but we assume that K is pro-unipotent. We obtain that the pullback functor

$$\mathrm{D}(\mathfrak{L}(G)/K) \to \mathrm{D}(\mathfrak{L}(G)/K')$$

induces an equivalence

$$\mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}} \overset{\sim}{\to} \mathrm{D}(\mathfrak{L}(G)/K')^{\omega,\text{alm-const}}.$$

Denote by

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const,right}} \subset \mathrm{D}(\mathfrak{L}(G))$$

the full subcategory equal to the essential image of

$$\mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}} \hookrightarrow \mathrm{D}(\mathfrak{L}(G)/K) \overset{!\text{-pullback}}{\hookrightarrow} \mathrm{D}(\mathfrak{L}(G))$$

for some/any K as above.

Define

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const,left}} \subset \mathrm{D}(\mathfrak{L}(G))$$

similarly, by swapping the roles of left and right.

4.4.3. We claim:

Proposition 4.4.4. *The subcategories*

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const,right}} \subset \mathrm{D}(\mathfrak{L}(G)) \supset \mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const,left}}$$

coincide.

Proof. Let Y be a $\mathfrak{L}^+(G) \times \mathfrak{L}^+(G)$-invariant subscheme of $\mathfrak{L}(G)$. Let K be as above. Note we can find K' sufficiently small so that the projection

$$Y \to Y/K$$

factors $K'\backslash Y \to Y/K$; moreover, the latter map is smooth with contractible fibers, and hence is a universal homological equivalence.

This implies that the subcategory

$$\mathrm{D}(Y)^{\omega,\text{alm-const,right}} \subset \mathrm{D}(Y)$$

defined to be the essential image of

$$\mathrm{D}(Y/K_1)^{\omega,\text{alm-const}} \hookrightarrow \mathrm{D}(Y/K_1) \overset{!\text{-pullback}}{\hookrightarrow} \mathrm{D}(Y) \tag{4.22}$$

for some/any K_1, coincides with the subcategory

$$\mathrm{D}(Y)^{\omega,\text{alm-const,left}} \subset \mathrm{D}(Y)$$

defined to be the essential image of

$$\mathrm{D}(K_2\backslash Y)^{\omega,\text{alm-const}} \hookrightarrow \mathrm{D}(K_2\backslash Y) \overset{!\text{-pullback}}{\hookrightarrow} \mathrm{D}(Y) \tag{4.23}$$

for some/any K_2.

This implies the statement of the proposition, since

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const},\text{right}} \subset \mathrm{D}(\mathfrak{L}(G))$$

equals

$$\lim_{Y} \mathrm{D}(Y)^{\omega,\text{alm-const},\text{right}} \subset \lim_{Y} \mathrm{D}(Y)$$

and

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const},\text{left}} \subset \mathrm{D}(\mathfrak{L}(G))$$

equals

$$\lim_{Y} \mathrm{D}(Y)^{\omega,\text{alm-const},\text{left}} \subset \lim_{Y} \mathrm{D}(Y).$$

□

4.4.5. Denote the subcategory

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const},\text{right}} = \mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const},\text{left}}$$

of $\mathrm{D}(\mathfrak{L}(G))$ by $\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}}$.

4.4.6. From Proposition 4.4.4 and Corollary 4.3.9, we obtain:

Corollary 4.4.7. *The coproduct functor*

$$\mathrm{D}(\mathfrak{L}(G)) \to \mathrm{D}(\mathfrak{L}(G)) \otimes \mathrm{D}(\mathfrak{L}(G))$$

sends $\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}} \subset \mathrm{D}(\mathfrak{L}(G))$ *to the full subcategory*

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}} \otimes \mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}} \subset \mathrm{D}(\mathfrak{L}(G)) \otimes \mathrm{D}(\mathfrak{L}(G)).$$

4.4.8. It follows from Corollary 4.3.8 that the category $\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}}$ is dualizable. Denote its dual by $\mathrm{D}(\mathfrak{L}(G))^{\text{alm-const}}$; we can view it as a quotient category of $\mathrm{D}(\mathfrak{L}(G))$.

From Corollary 4.4.7 we obtain that the *monoidal* structure on $\mathrm{D}(\mathfrak{L}(G))$ gives rise to a monoidal structure on $\mathrm{D}(\mathfrak{L}(G))^{\text{alm-const}}$.

Remark 4.4.9. It follows from Remark 4.3.11 that

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}} \subset \mathrm{D}(\mathfrak{L}(G))$$

is the full subcategory, generated under colimits by the dualizing sheaf.

4.5. **Almost invariants for the loop group.**

4.5.1. Let $\mathbf{C}$ be a category equipped with an action of $\mathfrak{L}^+(G)$. We let $\text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C})$ be the full subcategory of $\mathbf{C}$ equal to

$$\text{alm-inv}_G(\mathbf{C}^{K_1}).$$

The contents of Sect. 4.2 apply equally well to this situation.

4.5.2. Let now $\mathbf{C}$ be equipped with an action of $\mathfrak{L}(G)$. We let $\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C})$ to be the full subcategory of $\mathbf{C}$ consisting of objects that are sent by the co-action functor

$$\mathbf{C} \to \mathrm{D}(\mathfrak{L}(G)) \otimes \mathbf{C}$$

to the full subcategory

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}} \otimes \mathbf{C} \subset \mathrm{D}(\mathfrak{L}(G)) \otimes \mathbf{C}.$$

We can identify

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}) = \mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}} \underset{\mathrm{D}(\mathfrak{L}(G))}{\otimes} \mathbf{C} \simeq \mathrm{Funct}_{\mathrm{D}(\mathfrak{L}(G))}(\mathrm{D}(\mathfrak{L}(G))^{\text{alm-const}}, \mathbf{C}).$$

4.5.3. It is easy to see that $\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C})$, viewed as a full subcategory of $\mathbf{C}$, is preserved by the action of $\mathfrak{L}(G)$, i.e., it can itself be viewed as a category acted on by $\mathfrak{L}(G)$.

4.5.4. *Example.* Let $G = T$ be a torus. Then it is easy to see that the inclusion

$$\text{alm-inv}_{\mathfrak{L}(T)}(\mathbf{C}) \subset \text{alm-inv}_{\mathfrak{L}^+(T)}(\mathbf{C})$$

is an equality.

4.5.5. Unwinding the definitions, we obtain:

Lemma 4.5.6.
(i) *The full subcategories*

$$\text{alm-inv}_{\mathfrak{L}(G)\text{-right}}(\mathrm{D}(\mathfrak{L}(G))),\ \text{alm-inv}_{\mathfrak{L}(G)\text{-left}}(\mathrm{D}(\mathfrak{L}(G))),\ \text{alm-inv}_{\mathfrak{L}(G)\times\mathfrak{L}(G)}(\mathrm{D}(\mathfrak{L}(G)))$$

and

$$\mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}}$$

of $\mathrm{D}(\mathfrak{L}(G))$*, coincide.*
(ii) *For* $K = K_i$*,* $i \geq 1$*, the full subcategories*

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathrm{D}(\mathfrak{L}(G)/K)) \text{ and } \mathrm{D}(\mathfrak{L}(G)/K)^{\omega,\text{alm-const}}$$

of $\mathrm{D}(\mathfrak{L}(G)/K)$*, coincide.*

4.5.7. Consider

$$\mathrm{D}(\mathrm{Gr}_G) \in \mathfrak{L}(G)\text{-}\mathbf{mod}.$$

It follows from the definitions that we have an inclusion of subcategories

$$\mathrm{D}(\mathrm{Gr}_G)^{\omega,\text{alm-const}} \subset \text{alm-inv}_{\mathfrak{L}(G)}(\mathrm{D}(\mathrm{Gr}_G)). \tag{4.24}$$

We claim:

Proposition 4.5.8. *The inclusion* (4.24) *is an equality.*

The proof will be given in Sect. 5.4.

4.5.9. Let $\mathbf{C}$ be an object of $\mathfrak{L}(G)$-**mod**. Recall that $\mathbf{C}$ is said to be *spherically generated* if the (a priori fully faithful) functor

$$\mathrm{D}(\mathrm{Gr}_G) \underset{\mathrm{Sph}_G}{\otimes} \mathbf{C}^{\mathfrak{L}^+(G)} \to \mathbf{C} \tag{4.25}$$

is an equivalence, where

$$\mathrm{Sph}_G := \mathrm{D}(\mathfrak{L}(G))^{\mathfrak{L}^+(G)\times\mathfrak{L}^+(G)}.$$

We claim:

Proposition 4.5.10. *Suppose that* $\mathbf{C}$ *is spherically generated. Then the embedding*

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}) \hookrightarrow \mathbf{C}$$

admits a left adjoint.

Proof. It suffices to consider the universal case, i.e., $\mathbf{C} = \mathrm{D}(\mathrm{Gr}_G)$. I.e., we need to show that the embedding

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathrm{D}(\mathrm{Gr}_G)) \hookrightarrow \mathrm{D}(\mathrm{Gr}_G)$$

admits a left adjoint.

By Proposition 4.5.8, this is equivalent to showing that the embedding

$$\mathrm{D}(\mathrm{Gr}_G)^{\omega,\text{alm-const}} \hookrightarrow \mathrm{D}(\mathrm{Gr}_G)$$

admits a left adjoint.

Dually, we need to show that the projection functor

$$\mathrm{D}(\mathrm{Gr}_G) \to \mathrm{D}(\mathrm{Gr}_G)^{\text{alm-const}}$$

admits a continuous right adjoint, i.e., that it preserves compactness.

We can work at one connected component of Gr_G at a time, and it is enough to consider the neutral connected component; denote it by $\mathcal{Y}$. Thus, by Proposition 4.3.6, we have to show that the functor

$$\mathcal{F} \mapsto \mathrm{C}^{\cdot}(\mathcal{Y}, \mathcal{F}), \quad \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod}_0$$

preserves compactness.

Note, however, that since $\mathrm{C}^{\cdot}(\mathcal{Y})$ is isomorphic to a polynomial algebra on generators in *even* degrees, an object of $\mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod}_0$ is compact if and if the underlying vector space is finite-dimensional.

The required assertion follows now from the fact that $\mathcal{Y}$ is ind-proper, and so the functor

$$\mathrm{C}^{\cdot}(\mathcal{Y}, -) : \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{Vect}$$

preserves compactness.

□

Corollary 4.5.11. *The embedding*

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}) \hookrightarrow \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C})$$

admits a left adjoint.

Proof. Note that $\text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C})$ is contained in the spherically generated subcategory of $\mathbf{C}$, i.e., the essential image of (4.25). Hence, we can assume that $\mathbf{C}$ is spherically generated.

Now the assertion follows from Proposition 4.5.10.

□

Remark 4.5.12. It follows from the proof of Proposition 4.5.10 that the formation of the left adjoint in Proposition 4.5.10 (resp., Corollary 4.5.11) is functorial in $\mathbf{C}$, i.e., the corresponding Beck-Chevalley natural transformation is an isomorophism.

4.6. **Almost trivial actions of the loop group.** In this subsection we define what it means for a $\mathfrak{L}(G)_{x_0}$-action on a group to be almost trivial, and we give a Koszul-dual description of the totality of such categories.

4.6.1. We shall say that an action of $\mathfrak{L}(G)$ on $\mathbf{C}$ is *almost trivial* if the embedding

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}) \hookrightarrow \mathbf{C} \tag{4.26}$$

is an equivalence.

4.6.2. Note that we can also characterize almost trivial actions as follows: an action of $\mathfrak{L}(G)$ on $\mathbf{C}$ is almost trivial if and only if the monoidal action of $\mathrm{D}(\mathfrak{L}(G))$ on $\mathbf{C}$ factors through the quotient

$$\mathrm{D}(\mathfrak{L}(G)) \twoheadrightarrow \mathrm{D}(\mathfrak{L}(G))^{\text{alm-const}}.$$

4.6.3. For future use, we notice:

Lemma 4.6.4. *Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a 1-morphism in $\mathfrak{L}(G)$-**mod**, which is* conservative *as a functor on the underlying categories. Then if the action on $\mathbf{C}_2$ is almost trivial, then so it is on $\mathbf{C}_1$.*

Proof. Since the action on $\mathbf{C}_2$ is almost trivial, both arrows in

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}_2) \hookrightarrow \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_2) \hookrightarrow \mathbf{C}_2$$

are equivalences. We need to show that the same is true for

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}_1) \hookrightarrow \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_1) \hookrightarrow \mathbf{C}_1.$$

We have a commutative diagram[11]

$$\begin{tikzcd} \mathbf{C}_1 \arrow[r] \arrow[d, "F"'] & \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_1) \arrow[r] \arrow[d] & \mathbf{C}_1 \arrow[d, "F"] \\ \mathbf{C}_2 \arrow[r] & \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_2) \arrow[r] & \mathbf{C}_2. \end{tikzcd}$$

Hence, the fact that the counit of the adjunction

$$\text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_2) \rightleftarrows \mathbf{C}_2$$

is an isomorphism and the conservativity of F imply that the unit of the adjunction

$$\text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_1) \rightleftarrows \mathbf{C}_1$$

is an isomorphism.

The proof for

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}_i) \hookrightarrow \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}_i)$$

is similar using left adjoints and Corollary 4.5.11.

□

4.6.5. Let

$$(\mathfrak{L}(G)\text{-}\mathbf{mod})_{\text{alm-triv}} \subset \mathfrak{L}(G)\text{-}\mathbf{mod} \tag{4.27}$$

be the full subcategory, consisting of $\mathfrak{L}(G)$-module categories, on which the action is almost trivial.

Note that we can identify

$$(\mathfrak{L}(G)\text{-}\mathbf{mod})_{\text{alm-triv}} \simeq \mathrm{D}(\mathfrak{L}(G))^{\text{alm-const}}\text{-}\mathbf{mod},$$

viewed as a full subcategory of

$$\mathfrak{L}(G)\text{-}\mathbf{mod} \simeq \mathrm{D}(\mathfrak{L}(G))\text{-}\mathbf{mod}.$$

The assignment

$$\mathbf{C} \mapsto \text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C})$$

is a right adjoint to the embedding (4.27).

4.6.6. Consider the (symmetric) monoidal category $\mathrm{Vect}^{\mathfrak{L}(G)}$.

The functor

$$\mathrm{inv}_{\mathfrak{L}(G)} : \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

naturally upgrades to a functor

$$\mathrm{inv}^{\mathrm{enh}}_{\mathfrak{L}(G)} : \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{Vect}^{\mathfrak{L}(G)}\text{-}\mathbf{mod},$$

which admits a left adjoint, given by

$$\widetilde{\mathbf{C}} \mapsto \mathrm{Vect} \underset{\mathrm{Vect}^{\mathfrak{L}(G)}}{\otimes} \widetilde{\mathbf{C}}. \tag{4.28}$$

It is clear that the above adjoint pair factors as

$$\mathrm{Vect}^{\mathfrak{L}(G)}\text{-}\mathbf{mod} \rightleftarrows (\mathfrak{L}(G)\text{-}\mathbf{mod})_{\text{alm-triv}} \rightleftarrows \mathfrak{L}(G)\text{-}\mathbf{mod}.$$

4.6.7. We will prove:

Theorem 4.6.8. *The adjoint functors*

$$\mathrm{Vect}^{\mathfrak{L}(G)}\text{-}\mathbf{mod} \rightleftarrows (\mathfrak{L}(G)\text{-}\mathbf{mod})_{\text{alm-triv}}$$

are mutually inverse equivalences.

The theorem will be proved in Sect. 5.5.

[11] The horizontal arrows are the adjoint functors from (4.15), both of which are functorial in $\mathbf{C}$.

4.6.9. From Theorem 4.6.8 we obtain:

Corollary 4.6.10. *The naturally defined functor*

$$\mathrm{Vect} \underset{\mathrm{Vect}^{\mathfrak{L}(G)}}{\otimes} \mathrm{Vect} \to \mathrm{D}(\mathfrak{L}(G))^{\omega,\text{alm-const}}$$

is an equivalence.

4.6.11. For future use, we record the following consequence of Theorem 4.6.8.

Let $\mathbf{C}$ be an object of $\mathfrak{L}(G)$-**mod**. Note that since Gr_G is proper, the forgetful functor

$$\mathrm{oblv}_{\mathfrak{L}(G)\to\mathfrak{L}^+(G)} : \mathbf{C}^{\mathfrak{L}(G)} \to \mathbf{C}^{\mathfrak{L}^+(G)}$$

admits a *left* adjoint, to be denoted

$$\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)},$$

see Sect. B.4.1.

We claim:

Proposition 4.6.12. *Let $\mathbf{C}$ be spherically generated. Then the action of $\mathfrak{L}(G)$ is almost trivial if the functor $\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}$ is conservative.*

Proof. Since both $\mathbf{C}$ and $\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C})$ are spherically generated, the inclusion

$$\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}) \hookrightarrow \mathbf{C}$$

is an equivalence if and only if the functor

$$((\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}))^{\mathfrak{L}^+(G)} \to \mathbf{C}^{\mathfrak{L}^+(G)},$$

which is a priori also a fully faithful inclusion, is an equivalence.

By Theorem 4.6.8, we can identify the above functor with

$$\mathrm{Vect}^{\mathfrak{L}^+(G)} \underset{\mathrm{Vect}^{\mathfrak{L}(G)}}{\otimes} \mathbf{C}^{\mathfrak{L}(G)} \to \mathbf{C}^{\mathfrak{L}^+(G)}. \tag{4.29}$$

Thus, the action of $\mathfrak{L}(G)$ is almost trivial if and only if the left adjoint to (4.29) is conservative.

The precomposition of (4.29) with

$$\mathbf{C}^{\mathfrak{L}(G)} \simeq \mathrm{Vect}^{\mathfrak{L}(G)} \underset{\mathrm{Vect}^{\mathfrak{L}(G)}}{\otimes} \mathbf{C}^{\mathfrak{L}(G)} \to \mathrm{Vect}^{\mathfrak{L}^+(G)} \underset{\mathrm{Vect}^{\mathfrak{L}(G)}}{\otimes} \mathbf{C}^{\mathfrak{L}(G)} \tag{4.30}$$

is the forgetful functor $\mathrm{oblv}_{\mathfrak{L}(G)\to\mathfrak{L}^+(G)}$.

Thus, if the left adjoint to $\mathrm{oblv}_{\mathfrak{L}(G)\to\mathfrak{L}^+(G)}$, i.e., $\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}$, is conservative, then so is the left adjoint to (4.29).

□

Remark 4.6.13. In fact, one can show that the assertion of Proposition 4.6.12 is "if and only if", but we will not need this.

4.6.14. Note that we an identify

$$\mathrm{Vect}^{\mathfrak{L}(G)} \simeq \mathrm{C}_\cdot(\mathfrak{L}(G))\text{-mod},$$

as monoidal categories.

Hence, being equivalent to the category of modules over a Hopf algebra, $\mathrm{Vect}^{\mathfrak{L}(G)}$ is a Frobenius algebra in DGCat. In particular, we have a canonical equivalence

$$(\mathrm{Vect}^{\mathfrak{L}(G)})^\vee \simeq \mathrm{Vect}^{\mathfrak{L}(G)}$$

as $\mathrm{Vect}^{\mathfrak{L}(G)}$-module categories.

4.7. **The reduction step.** After all the preparations, in this subsection we will finally finally formulate a reduction step in the proof of Theorem 3.1.7: the claim is that it is sufficient to prove it for $\mathbf{C} = \mathrm{Vect}$.

4.7.1. It is clear that the embedding (4.26) induces an equivalence

$$(\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}))^{\mathfrak{L}(G)} \overset{\sim}{\to} \mathbf{C}^{\mathfrak{L}(G)}.$$

4.7.2. We will prove:

Theorem 4.7.3. *The embedding* (4.26) *induces an equivalence*

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\text{fact}}((\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}))^{\text{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \overset{\sim}{\to} \omega_{\mathrm{Gr}_G}\text{-mod}^{\text{fact}}(\mathbf{C}^{\text{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}.$$

Theorem 4.7.3 will be proved in Sects. 7-9. In the remainder of this section and Sect. 6, we will show how Theorem 4.7.3 implies Theorem 3.1.7.

4.7.4. Note that by Theorem 4.7.3 and Sect. 4.7.1 we obtain that it suffices to prove Theorem 3.1.7 for $\mathbf{C} \in (\mathfrak{L}(G)\text{-}\mathbf{mod})_{\text{alm-triv}}$.

By Theorem 4.6.8, we can assume that $\mathbf{C}$ is of the form

$$\mathrm{Vect} \underset{\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}}}{\otimes} \widetilde{\mathbf{C}}, \quad \widetilde{\mathbf{C}} \in \mathrm{Vect}^{\mathfrak{L}(G)_{x_0}}\text{-}\mathbf{mod}.$$

4.7.5. It follows from Theorem 4.6.8 that the operation

$$\widetilde{\mathbf{C}} \mapsto \mathrm{Vect} \underset{\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}}}{\otimes} \widetilde{\mathbf{C}}$$

preserves limits.

Moreover, by Sect. 4.6.14, any object of $\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}}\text{-}\mathbf{mod}$ can be written as a totalization of a cosimplicial object with terms of the form

$$\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \otimes \widetilde{\mathbf{C}}_0, \quad \widetilde{\mathbf{C}}_0 \in \mathrm{DGCat}\,.$$

4.7.6. This reduces the assertion of Theorem 3.1.7 to the case when the action

$$\mathbf{C} = \mathrm{Vect} \otimes \widetilde{\mathbf{C}}_0 \simeq \widetilde{\mathbf{C}}_0, \quad \widetilde{\mathbf{C}}_0 \in \mathrm{DGCat},$$

equipped with the trivial action.

In Sect. 6, we will prove the assertion of Theorem 3.1.7 in this case by an explicit calculation.

4.7.7. To simplify the exposition we will consider the case when $\widetilde{\mathbf{C}}_0 = \mathrm{Vect}$. The case of a general category is completely analogous.

Note that the assertion of Theorem 3.1.7 for $\mathbf{C} = \mathrm{Vect}$ coincides with that of Corollary 3.4.2.

5. Proofs of Propositions 4.2.12, 4.3.6, 4.5.8 and Theorem 4.6.8

The goal of this subsection is to supply proofs of Proposition 4.2.12, 4.3.6, 4.5.8 and Theorem 4.6.8. These all are Koszul duality-type statements, and we essentially need to take care of convergence issues.

For the duration of this section we keep the notational change

$$\mathfrak{L}(G)_{x_0} \rightsquigarrow \mathfrak{L}(G),\ \mathfrak{L}^+(G)_{x_0} \rightsquigarrow \mathfrak{L}^+(G),\ \mathrm{Gr}_{G,x_0} \rightsquigarrow \mathrm{Gr}_G.$$

5.1. **Proof of Proposition 4.2.12.**

5.1.1. It is easy to see that the statement of Proposition 4.2.12 for H follows from the corresponding statement for its neutral connected component. Hence, for the duration of this subsection, we will assume that H is connected.

5.1.2. Let $\mathbf{A}$ be a monoidal category equipped with a monoidal ϕ functor to Vect. Denote

$$\mathbf{B} := \mathrm{Funct}_{\mathbf{A}\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Vect})^{\mathrm{rev}},$$

where $\mathbf{A}$ acts on Vect via ϕ.

The functor

$$\mathbf{A}\text{-}\mathbf{mod} \to \mathrm{Vect}, \quad \mathbf{C} \mapsto \mathrm{inv}_{\mathbf{A}} = \mathrm{Funct}_{\mathbf{A}\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C})$$

upgrades to a functor

$$\mathrm{inv}_{\mathbf{A}}^{\mathrm{enh}} : \mathbf{A}\text{-}\mathbf{mod} \to \mathbf{B}\text{-}\mathbf{mod},$$

and the latter admits a left adjoint given by

$$\widetilde{\mathbf{C}} \mapsto \mathrm{Vect} \underset{\mathbf{B}}{\otimes} \widetilde{\mathbf{C}},$$

where the augmentation on $\mathbf{B}$ is given by the forgetful functor

$$\mathrm{Funct}_{\mathbf{A}\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathrm{Vect}) \to \mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{Vect}, \mathrm{Vect}) \simeq \mathrm{Vect}\,.$$

We shall say that the pair $(\mathbf{A}, \phi)$ satisfies Koszul duality if the adjoint functors $((\mathrm{inv}_{\mathbf{A}}^{\mathrm{enh}})^L, \mathrm{inv}_{\mathbf{A}}^{\mathrm{enh}})$ are mutually inverse equivalences.

5.1.3. Let A be a finite-dimensional Hopf algebra, and let $\mathbf{A} := A$-mod. Let ϕ be the tautological forgetful functor A-mod $\to$ Vect.

It is easy to see that in this case

$$\mathbf{B} \simeq B\text{-mod},$$

where B is the linear dual of A.

5.1.4. *Example.* Let W be a compact object of Vect, concentrated in odd degrees. Set

$$\mathbf{A} := \mathrm{Sym}(W)\text{-mod},$$

which we regard as a monoidal category with respect to *convolution*. Let ϕ be the tautological forgetful functor $\mathrm{Sym}(W)$-mod $\to$ Vect.

According to Sect. 5.1.3, $\mathbf{B} \simeq \mathrm{Sym}(W^*)$-mod, viewed as a monoidal category also with respect to *convolution*.

Then it is easy to see that this pair $(\mathbf{A}, \phi)$ satisfies Koszul duality.

5.1.5. The statement of Proposition 4.2.12 is equivalent to the fact that the monoidal category $\mathrm{C}^{\cdot}(H)$-mod equipped with the tautological forgetful functor to Vect satisfies Koszul duality.

By construction, this example fits the pattern of Sect. 5.1.3. Hence, it suffices to show that it fits in fact the pattern of Sect. 5.1.4.

Let $\mathfrak{a}$ be the Lie algebra that controls the rational homotopy type of BH. Note that $\mathrm{C}^{\cdot}(H) \simeq \mathrm{C}^{\cdot}(\Omega(\mathfrak{a}))$. Recall also that $\mathfrak{a}$ is abelian and is concentrated in odd degrees. Hence,

$$\mathrm{C}^{\cdot}(\Omega(\mathfrak{a})) \simeq \mathrm{Sym}(\underline{\mathfrak{a}}^*),$$

as Hopf algebras, where $\underline{\mathfrak{a}}$ is the vector space underlying $\mathfrak{a}$.

$\square$[Proposition 4.2.12]

Remark 5.1.6. Note that by Sect. 5.1.3, we obtain that

$$\mathrm{Vect}^H \simeq \mathrm{C}_{\cdot}(H)\text{-mod},$$

as is supposed to be the case.

5.2. **A descent result for almost constant sheaves.**

5.2.1. Let H be a connected algebraic group, and let

$$f:\widetilde{Y}\to Y$$

be an H-torsor and Y is a scheme of finite type.

We consider the category $\mathrm{D}(\widetilde{Y})$ as equipped with an action of H. Its full subcategory $\mathrm{D}(\widetilde{Y})^{\text{alm-const}}$ is stable under this action; moreover the H-action on $\mathrm{D}(\widetilde{Y})^{\text{alm-const}}$ is almost trivial.

5.2.2. The functor of *-pullback identifies

$$\mathrm{D}(Y)\overset{\sim}{\to}(\mathrm{D}(\widetilde{Y}))^H.$$

Since f^* sends

$$\mathrm{D}(Y)^{\text{alm-const}}\to\mathrm{D}(\widetilde{Y})^{\text{alm-const}}$$

and

$$(\mathrm{D}(\widetilde{Y})^{\text{alm-const}})^H=\mathrm{D}(\widetilde{Y})^{\text{alm-const}}\underset{\mathrm{D}(\widetilde{Y})}{\times}(\mathrm{D}(\widetilde{Y}))^H,$$

we obtain a commutative diagram

$$\begin{tikzcd}
(\mathrm{D}(\widetilde{Y})^{\text{alm-const}})^H \arrow[r] & \mathrm{D}(\widetilde{Y})^H \\
\mathrm{D}(Y)^{\text{alm-const}} \arrow[r] \arrow[u] & \mathrm{D}(Y), \arrow[u, "\sim"']
\end{tikzcd} \tag{5.1}$$

where the horizontal arrows are fully faithful.

We claim:

Lemma 5.2.3. *The above functor*

$$\mathrm{D}(Y)^{\text{alm-const}}\to(\mathrm{D}(\widetilde{Y})^{\text{alm-const}})^H$$

is an equivalence.

Proof. The functor in question is fully faithful and preserves compactness (since the other three arrows in (5.1) have this property).

Hence, it is enough to show that it sends compact generators of $\mathrm{D}(Y)^{\text{alm-const}}$ to generators of $(\mathrm{D}(\widetilde{Y})^{\text{alm-const}})^H$.

This functor sends

$$\underline{k}_Y\mapsto\underline{k}_{\widetilde{Y}}.$$

Hence, it remains to show that the latter is a generator of $(\mathrm{D}(\widetilde{Y})^{\text{alm-const}})^H$. (It is here that the assumption that H is connected will be used.)

Indeed, for a category $\mathbf{C}$ with an action of H and $\mathbf{c},\mathbf{c}'\in\mathbf{C}^H$, the object

$$\mathcal{H}om_{\mathbf{C}}(\mathbf{c},\mathbf{c}')\in\mathrm{Vect}$$

naturally upgrades to an object of Vect^H, while

$$\mathcal{H}om_{\mathbf{C}^H}(\mathbf{c},\mathbf{c}')\simeq\mathrm{inv}_H(\mathcal{H}om_{\mathbf{C}}(\mathbf{c},\mathbf{c}')).$$

Now, if $\mathbf{c}$ is a generator of $\mathbf{C}$, we have

$$\mathbf{c}'\neq 0\Rightarrow\mathcal{H}om_{\mathbf{C}}(\mathbf{c},\mathbf{c}')\neq 0.$$

We now use the fact that for a connected H, the functor

$$\mathrm{inv}_H:\mathrm{Vect}^H\to\mathrm{Vect}$$

is conservative.[12]

$\square$

[12]Indeed, in the notations of Sect. 5.1.5, we have $\mathrm{Vect}^H\simeq\mathrm{Sym}(\underline{\mathfrak{a}}^*)$-mod, and since $\underline{\mathfrak{a}}^*$ is concentrated in *odd* degrees, this category is generated by the augmentation module.

Corollary 5.2.4. *The functor $f^!$ induces an equivalence*

$$\mathrm{D}(Y)^{\omega,\text{alm-const}} \to (\mathrm{D}(\widetilde{Y})^{\omega,\text{alm-const}})^H.$$

5.3. **Proof of Proposition 4.3.6.**

5.3.1. *Reduction to the case of the affine Grassmannian.* We change the notations slightly and denote by $\mathcal{Y}$ the neutral connected component of Gr_G, and by $\widetilde{\mathcal{Y}}$ its preimage in $\mathfrak{L}(G)/K$.

Write

$$\mathcal{Y}_i = \text{“}\underset{i}{\mathrm{colim}}\text{”}\, Y_i,$$

and set

$$\widetilde{Y}_i := Y_i \underset{\mathcal{Y}}{\times} \widetilde{\mathcal{Y}},$$

so that

$$\widetilde{\mathcal{Y}} \simeq \text{“}\underset{i}{\mathrm{colim}}\text{”}\, \widetilde{Y}_i.$$

We view both sides of

$$\underset{i}{\mathrm{colim}}\, \mathrm{C}^{\cdot}(\widetilde{Y}_i)\text{-mod} \to \mathrm{C}^{\cdot}(\widetilde{\mathcal{Y}})\text{-mod},$$

as acted on (almost trivially) by $\mathfrak{L}^+(G)/K$. Since operation $\mathrm{inv}_{\mathfrak{L}^+(G)/K}(-)$ is conservative on the subcategory of almost trivial modules, it suffices to show that the functor

$$\underset{i}{\mathrm{colim}}\, (\mathrm{C}^{\cdot}(\widetilde{Y}_i)\text{-mod})^{\mathfrak{L}^+(G)/K} \overset{\sim}{\to} (\underset{i}{\mathrm{colim}}\, \mathrm{C}^{\cdot}(\widetilde{Y}_i)\text{-mod})^{\mathfrak{L}^+(G)/K} \to (\mathrm{C}^{\cdot}(\widetilde{\mathcal{Y}})\text{-mod})^{\mathfrak{L}^+(G)/K} \tag{5.2}$$

is an equivalence onto

$$(\mathrm{C}^{\cdot}(\widetilde{\mathcal{Y}})\text{-mod}_0)^{\mathfrak{L}^+(G)/K} \subset (\mathrm{C}^{\cdot}(\widetilde{\mathcal{Y}})\text{-mod})^{\mathfrak{L}^+(G)/K}.$$

By Lemma 5.2.3, we can identify the terms

$$(\mathrm{C}^{\cdot}(\widetilde{Y}_i)\text{-mod})^{\mathfrak{L}^+(G)/K} \simeq (\mathrm{D}(\widetilde{Y}_i)^{\text{alm-const}})^{\mathfrak{L}^+(G)/K}$$

with

$$\mathrm{D}(Y_i)^{\text{alm-const}} \simeq \mathrm{C}^{\cdot}(Y_i)\text{-mod}.$$

Similarly, it is easy to see (e.g., using (3.24)) that the right-hand side in (5.2) identifies with $\mathrm{C}^{\cdot}(\mathcal{Y})$-mod, which contains $\mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod}_0$ as a full subcategory.

Thus, we obtain that it suffices to show that the resulting map

$$\underset{i}{\mathrm{colim}}\, \mathrm{C}^{\cdot}(Y_i)\text{-mod} \to \mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod}_0.$$

is an equivalence.

5.3.2. Note that for a *coconnective* algebra A and $n \geq 0$, the truncation $A^{\leq n}$ has a natural structure of algebra, equipped with a map from A.

Consider the commutative diagram

$$\begin{array}{ccc} \underset{i,n}{\mathrm{colim}}\, (\mathrm{C}^{\cdot}(Y_i))^{\leq n}\text{-mod} & \longrightarrow & \underset{i}{\mathrm{colim}}\, \mathrm{C}^{\cdot}(Y_i)\text{-mod} \\ \downarrow & & \downarrow \\ \underset{n}{\mathrm{colim}}\, (\mathrm{C}^{\cdot}(\mathcal{Y}))^{\leq n}\text{-mod} & \longrightarrow & \mathrm{C}^{\cdot}(\mathcal{Y})\text{-mod}_0. \end{array} \tag{5.3}$$

We need to show that the right vertical arrow is an equivalence. We will achieve this by showing that the other three arrows in (5.3) are equivalences.

5.3.3. The equivalence is immediate for the top horizontal arrow: indeed for a fixed i, the family

$$n \rightsquigarrow (\mathrm{C}^{\cdot}(Y_i))^{\leq n}$$

stabilizes to $\mathrm{C}^{\cdot}(Y_i)$, since Y_i is finite-dimensional.

5.3.4. We claim that for a fixed n, the family

$$i \rightsquigarrow (\mathrm{C}^\cdot(Y_i))^{\leq n}$$

also stabilizes to $(\mathrm{C}^\cdot(\mathcal{Y}))^{\leq n}$. Indeed, this follows from the cellular decomposition of the affine Grassmannian.

Hence, the left vertical arrow in (5.3) is an equivalence.

5.3.5. Thus, it remains to show that the functor

$$\operatorname*{colim}_n (\mathrm{C}^\cdot(\mathcal{Y}))^{\leq n}\text{-mod} \to \mathrm{C}^\cdot(\mathcal{Y})\text{-mod}_0$$

is an equivalence.

We identify $\mathrm{C}^\cdot(\mathcal{Y})$ with $\mathrm{Sym}(V)$, see Sect. 4.3.4. So we need to show that the functor

$$\operatorname*{colim}_n (\mathrm{Sym}(V)/\operatorname{Sym}^{>n}(V))\text{-mod} \to \mathrm{Sym}(V)\text{-mod}_0$$

is an equivalence.

5.3.6. It is easy to reduce the assertion to the case when V is one dimensional. In this case, we can use the grading-shearing trick (see [AG, Sect. A.2]), and assume that V is a finite-dimensional vector space in cohomological degree 0.

Hence, the assertion becomes that

$$\operatorname*{colim}_n (\mathrm{Sym}(V)/\operatorname{Sym}^{>n}(V))\text{-mod} \to \mathrm{Sym}(V)\text{-mod}_0$$

is an equivalence. I.e., we have to show that

$$\operatorname*{colim}_n \mathrm{QCoh}(S_n) \to \mathrm{QCoh}(\mathbb{A}^1)_0 \tag{5.4}$$

is an equivalence, where $S_n = \mathrm{Spec}(k[t]/t^n)$.

Note, however, that the composition

$$\operatorname*{colim}_n \mathrm{IndCoh}(S_n) \to \operatorname*{colim}_n \mathrm{QCoh}(S_n) \overset{(5.4)}{\longrightarrow} \mathrm{QCoh}(\mathbb{A}^1)_0 \tag{5.5}$$

is an equivalence (e.g., by [GR0, Proposition 7.4.5]). This implies that (5.4) is an equivalence, since the first arrow in (5.5) is a Verdier quotient.

□[Proposition 4.3.6]

5.4. **Proof of Proposition 4.5.8.**

5.4.1. Denote $\widetilde{\mathrm{Gr}}_G := \mathfrak{L}(G)/K_1$, so that

$$\mathrm{Gr}_G \simeq \widetilde{\mathrm{Gr}}_G/G.$$

Unwinding the definitions and using Lemma 4.5.6, we obtain that the category $\text{alm-inv}_{\mathfrak{L}(G)}(\mathrm{D}(\mathrm{Gr}_G))$ identifies with

$$(\mathrm{D}(\widetilde{\mathrm{Gr}}_G)^{\omega,\text{alm-const}})^G.$$

Thus, we need to show that the inclusion

$$\mathrm{D}(\mathrm{Gr}_G)^{\omega,\text{alm-const}} \hookrightarrow (\mathrm{D}(\widetilde{\mathrm{Gr}}_G)^{\omega,\text{alm-const}})^G$$

is an equality.

5.4.2. Write

$$\mathrm{Gr}_G = \text{``}\underset{i}{\mathrm{colim}}\text{''}\, Y_i,$$

and set

$$\widetilde{Y}_i := Y_i \underset{\mathrm{Gr}_G}{\times} \widetilde{\mathrm{Gr}}_G,$$

so that

$$\widetilde{\mathrm{Gr}}_G \simeq \text{``}\underset{i}{\mathrm{colim}}\text{''}\, \widetilde{Y}_i.$$

We have:

$$\mathrm{D}(\mathrm{Gr}_G)^{\omega,\text{alm-const}} \simeq \lim_i \mathrm{D}(Y_i)^{\omega,\text{alm-const}}$$

and

$$\mathrm{D}(\widetilde{\mathrm{Gr}}_G)^{\omega,\text{alm-const}} \simeq \lim_i \mathrm{D}(\widetilde{Y}_i)^{\omega,\text{alm-const}},$$

and hence

$$(\mathrm{D}(\widetilde{\mathrm{Gr}}_G)^{\omega,\text{alm-const}})^G \simeq \lim_i (\mathrm{D}(\widetilde{Y}_i)^{\omega,\text{alm-const}})^G.$$

Now the required isomorphism follows from Corollary 5.2.4.

□[Proposition 4.5.8]

5.5. **Proof of Theorem 4.6.8.**

5.5.1. We consider the adjunction

$$\mathrm{Vect}^{\mathfrak{L}(G)}\text{-}\mathbf{mod} \rightleftarrows (\mathfrak{L}(G)\text{-}\mathbf{mod})_{\text{alm-triv}}. \tag{5.6}$$

Since the functor $\mathrm{inv}_{\mathfrak{L}(G)}$ (and hence $\mathrm{inv}^{\mathrm{enh}}_{\mathfrak{L}(G)}$) commutes with tensor products, the left adjoint in (5.6) is fully faithful.

Hence, the two functors are mutually inverse equivalences if and only if $\mathrm{inv}^{\mathrm{enh}}_{\mathfrak{L}(G)}$ is conservative.

5.5.2. We first consider the case when G is semi-simple and simply-connected.

We view $\mathbf{A} := \mathrm{D}(\mathfrak{L}(G))^{\text{alm-const}}$ as a monoidal category under convolution and a natural monoidal functor ϕ to Vect. We claim that this pair $(\mathbf{A}, \phi)$ satisfies Koszul duality (see Sect. 5.1.2).

By Proposition 4.3.6, we can identify

$$\mathbf{A} \simeq \mathrm{C}^\cdot(\mathfrak{L}(G))\text{-mod}_0,$$

equipped with the tautological forgetful functor to Vect.

By (3.24) and Sect. 3.4.9, we can identify $\mathrm{C}^\cdot(\mathfrak{L}(G))$ as a Hopf algebra with

$$\mathrm{Sym}(V) \otimes \mathrm{Sym}(W),$$

where V is a cohomologically graded vector space concentrated in positive even degrees, and W is a cohomologically graded vector space concentrated in positive odd degrees.

Under this identification $\mathrm{C}^\cdot(\mathfrak{L}(G))\text{-mod}_0$ corresponds to

$$\mathrm{Sym}(V)\text{-mod}_0 \otimes \mathrm{Sym}(W)\text{-mod}.$$

5.5.3. It is enough to show that both monoidal categories

$$\mathbf{A}_1 := \mathrm{Sym}(V)\text{-mod}_0 \text{ and } \mathbf{A}_2 := \mathrm{Sym}(W)\text{-mod},$$

equipped with the forgetful functors to Vect, satisfy Koszul duality.

The case of $\mathbf{A}_2$ is immediate, see Sect. 5.1.4.

5.5.4. In the case of $\mathbf{A}_1$, Using the grading-shearing trick (see [AG, Sect. A.2]), we can assume that V is a finite-dimensional vector space in cohomological degree 0. In this case, we identify

$$\mathrm{Sym}(V)\text{-mod}_0 \simeq \mathrm{Rep}(V^*),$$

where V^* is regarded as an (additive) algebraic group. In this case, the Koszul duality statement is well-known.

5.5.5. Next we consider the case when $G = T$ is a torus. We will show directly that the counit of the adjunction in (5.6) is an equivalence. Let $\mathbf{C}$ be an object of $\mathfrak{L}(T)$-$\mathbf{mod}_{\text{alm-triv}}$. The counit is the functor

$$\text{Vect} \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)} \to \mathbf{C}. \tag{5.7}$$

Consider the short exact sequence

$$1 \to \mathfrak{L}^+(T) \to \mathfrak{L}(T) \to \Lambda \to 1.$$

By Proposition 4.2.12, in order to show that (5.7) is an equivalence, it suffices to show that

$$\text{Vect}^{\mathfrak{L}^+(T)} \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)} \overset{\sim}{\to} (\text{Vect} \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)})^{\mathfrak{L}^+(T)} \to \mathbf{C}^{\mathfrak{L}^+(T)} \tag{5.8}$$

is an equivalence.

We regard both sides of (5.8) as acted on by $\mathfrak{L}(T)/\mathfrak{L}^+(T) \simeq \Lambda$. In order to show that (5.8) is an equivalences, it is sufficient that it becomes so after taking Λ-invariants:

$$(\text{Vect}^{\mathfrak{L}^+(T)})^\Lambda \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)} \overset{\sim}{\to} \left(\text{Vect}^{\mathfrak{L}^+(T)} \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)}\right)^\Lambda \to (\mathbf{C}^{\mathfrak{L}^+(T)})^\Lambda \simeq \mathbf{C}^{\mathfrak{L}(T)}. \tag{5.9}$$

However, the latter composition is the identity functor

$$\mathbf{C}^{\mathfrak{L}(T)} \simeq \text{Vect}^{\mathfrak{L}(T)} \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)} \simeq (\text{Vect}^{\mathfrak{L}^+(T)})^\Lambda \underset{\text{Vect}^{\mathfrak{L}(T)}}{\otimes} \mathbf{C}^{\mathfrak{L}(T)} \to \mathbf{C}^{\mathfrak{L}(T)}.$$

5.5.6. We now consider the case when the derived group G' of G is simply connected. We have a short exact sequence

$$1 \to G' \to G \to T_0 \to 1,$$

where T_0 is a torus.

In this case, the fact that the counit of the adjunction in (5.6) is an equivalence follows from the validity of Theorem 4.6.8 for G' and T_0 by the argument in Sect. 5.5.5 above.

5.5.7. Finally, let G be arbitrary. We wish to show that the functor $\text{inv}_{\mathfrak{L}(G)}(-)$ is conservative.

Choose a short exact sequence

$$1 \to T_0 \to \widetilde{G} \to G \to 1,$$

where T_0 is a torus and $\widetilde{G}$ is such that its derived group is simply-connected. By what we proved above, the the operation $\text{inv}_{\mathfrak{L}(\widetilde{G})_{x_0}}(-)$ is conservative. Hence, it suffices to show that for a functor $\mathbf{C}_1 \to \mathbf{C}_2$ if

$$\text{inv}_{\mathfrak{L}(G)}(\mathbf{C}_1) \to \text{inv}_{\mathfrak{L}(G)}(\mathbf{C}_2) \tag{5.10}$$

is an equivalence, then so is

$$\text{inv}_{\mathfrak{L}(\widetilde{G})}(\mathbf{C}_1) \to \text{inv}_{\mathfrak{L}(\widetilde{G})}(\mathbf{C}_2). \tag{5.11}$$

For a category $\mathbf{C}$ with an action of $\mathfrak{L}(G)$, we have

$$\text{inv}_{\mathfrak{L}(\widetilde{G})}(\mathbf{C}) \simeq \text{inv}_{\mathfrak{L}(G)}(\mathbf{C} \otimes \text{Vect}^{\mathfrak{L}(T_0)}).$$

We have a monadic adjunction

$$\text{Vect}^{\mathfrak{L}(T_0)} \rightleftarrows \text{Vect}\,.$$

From here we obtain a monadic adjunction

$$\text{inv}_{\mathfrak{L}(\widetilde{G})}(\mathbf{C}) \rightleftarrows \text{inv}_{\mathfrak{L}(G)}(\mathbf{C}).$$

Moreover, for a functor $\mathbf{C}_1 \to \mathbf{C}_2$, the functor (5.10) intertwines the two monads. Hence, if (5.10) is an equivalence, so is (5.11).

□[Theorem 4.6.8]

6. Proof of Theorem 3.1.7: the case of a trivial action

In this section we will prove Theorem 3.1.7 for $\mathbf{C} = \mathrm{Vect}$. The proof will amount to calculating a certain monad, and this calculation will turn out to be equivalent to the *contractibility*[13] statement from [Ga1].

This calculation is the crux of the proof, and expresses the intuitive idea (alluded to in the Introduction) that

$$\int_{\text{punctured disc}} \mathrm{Gr}_G \simeq \mathfrak{L}(G).$$

6.1. **Setting up the monad.** In this subsection we will reduce the assertion of Theorem 3.1.7 for $\mathbf{C} = \mathrm{Vect}$ to a calculation that says that some particular map (in Vect) is an isomorphism.

6.1.1. We consider the functor

$$\Phi : \mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(Gr_G)})_{x_0} \tag{6.1}$$

of (3.19) (which is a particular case of (3.6)).

It makes the diagram

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\mathrm{Id}"] & \mathrm{Vect} \\ \mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \arrow[u, "\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}}"] \arrow[r, "\Phi"] & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \arrow[u, "\mathrm{oblv}_{\omega_{\mathrm{Gr}_G}}"'] \end{tikzcd} \tag{6.2}$$

commute, where the vertical arrows are the tautological forgetful functors.

6.1.2. Note that the left vertical arrow in (6.2) is conservative and admits a left adjoint, to be denoted $\mathrm{Av}_!^{\mathfrak{L}(G)_{x_0}}$.

By the Barr-Beck-Lurie theorem, we can identify $\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}}$ with the category of modules in Vect over the resulting monad.

The right vertical arrow in (6.2) is also conservative. We will show (shortly) that it also admits a left adjoint.

Thus, in order to prove that Φ is an equivalence, it suffices to show that Φ induces an isomorphism between the two monads.

6.1.3. Denote $\widetilde{\mathrm{Gr}}_{G,x_0} := \mathfrak{L}(G)_{x_0}/K$, where $K = K_i$ for some/any $i \geq 1$. We consider it as an ind-scheme, equipped with an action of $\mathfrak{L}(G)_{x_0}$. Consider the corresponding factorization module category at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$:

$$\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}. \tag{6.3}$$

Denote:

$$\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}_{x_0}} := \mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}/K_i.$$

We regard $\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}_{x_0}}$ as a factorization module space at x_0 with respect to the factorization space Gr_G. The factorization module category (6.3) is given by considering D-modules on $\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}_{x_0}}$, viewed as a crystal of categories over Ran_{x_0} equipped with a natural factorization structure against $\mathrm{D}(\mathrm{Gr}_G)$.

---

[13] A.k.a., *non-abelian Poincaré duality.*

6.1.4. Let $\widetilde{\pi}_{x_0}$ denote the projection $\widetilde{\mathrm{Gr}}_{G,x_0} \to \mathrm{pt}$, and let $\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G}$ denote the projection

$$\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}_{x_0}} \to \mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}/\mathfrak{L}(G)_{x_0} \simeq \mathrm{Gr}_{G,\mathrm{Ran}_{x_0}}/\,\mathrm{Hecke}_{x_0},$$

viewed as a map between factorization module spaces at x_0 over Gr_G.

Pullback with respect to $\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G}$ can be viewed as a functor

$$\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \to \mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$$

as factorization module categories at x_0 with respect to $\mathrm{D}(\mathrm{Gr}_G)$.

6.1.5. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\mathrm{Id}"] & \mathrm{Vect} \\
 & \mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0}) \arrow[u, "\iota_1^!"'] \\
\mathrm{Vect} \arrow[uu, "\mathrm{Id}"'] \arrow[r, "\widetilde{\Phi}"] & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \arrow[u, "\mathrm{oblv}_{\omega_{\mathrm{Gr}_G}}"'] \\
\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \arrow[u, "\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}}"] \arrow[r, "\Phi"] & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}, \arrow[u, "(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})^!"']
\end{tikzcd} \tag{6.4}$$

where:

- ι_1 denotes the embedding of the unit point into $\widetilde{\mathrm{Gr}}_{G,x_0}$;
- $\widetilde{\Phi}$ denotes the functor that sends the generator $k \in \mathrm{Vect}$ to $\omega_{\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}_{x_0}}}$, equipped with its natural factorization structure against ω_{Gr_G}.

It is easy to see that the outer diagram in (6.4) identifies with (6.2).

6.1.6. We will show that:

(1) The functor $\iota_1^! \circ \mathrm{oblv}_{\omega_{\mathrm{Gr}_G}}$ admits a left adjoint;

(2) The partially defined functor $(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})_!$, left adjoint to the lower-right vertical functor in (6.4), is defined on the essential images of $\widetilde{\Phi}$ and $(\iota_1^! \circ \mathrm{oblv}_{\omega_{\mathrm{Gr}_G}})^L$;

(3) The Beck-Chevalley natural transformation

$$(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})_! \circ \widetilde{\Phi} \to \Phi \circ \mathrm{Av}_!^{\mathfrak{L}(G)_{x_0}}$$

(arising from the lower portion of (6.4)) becomes an isomorphism after applying the functor

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \overset{\mathrm{oblv}_{\omega_{\mathrm{Gr}_G}}}{\longrightarrow} \mathrm{Vect}\,.$$

(4) The Beck-Chevalley natural transformation

$$(\iota_1^! \circ \mathrm{oblv}_{\omega_{\mathrm{Gr}_G}})^L \to \widetilde{\Phi}$$

(arising from the upper portion of (6.4)) becomes an isomorphism after applying the functor

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \overset{(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})_!}{\longrightarrow} \\ \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \overset{\mathrm{oblv}_{\omega_{\mathrm{Gr}_G}}}{\longrightarrow} \mathrm{Vect}\,. \tag{6.5}$$

It is clear that the above properties (1)-(4) imply the required property of the monad from Sect. 6.1.2.

The rest of this section is devoted to the verification of properties (1)-(4).

Remark 6.1.7. We will give a purely geometric proof of Properties (2) and (3). However, if we allow ourselves to use Theorem 4.7.3 (which will be proved independently), the proof of both properties can be significantly simplified.

Indeed, by Theorem 4.7.3, we can replace

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \rightsquigarrow \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\text{alm-inv}_{\mathfrak{L}(G)}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0}))^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}.$$

Then, the left adjoint in point (2) is induced by the functor

$$\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\omega,\text{alm-const}} \simeq \text{alm-inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})) \to \mathrm{Vect} \tag{6.6}$$

as objects of $\mathfrak{L}(G)_{x_0}$-**mod**, where (6.6) is the left adjoint to

$$\mathrm{Vect} \to \mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\omega,\text{alm-const}}, \quad k \mapsto \omega_{\widetilde{\mathrm{Gr}}_{G,x_0}},$$

which exists, e.g., since the objects of $\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\omega,\text{alm-const}}$ are ind-holonomic[14].

Property (3) follows from the fact that the Beck-Chevalley natural transformation arising from the commutative diagram

$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "k\mapsto \omega_{\widetilde{\mathrm{Gr}}_{G,x_0}}"] & \mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\omega,\text{alm-const}} \\
\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \arrow[u, "\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}}"] \arrow[r, "\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}}"] & \mathrm{Vect} \arrow[u, "\widetilde{\pi}^!_{x_0}"']
\end{tikzcd}$$

is an isomorphism. This follows by identifying the above diagram with

$$\begin{tikzcd}
\mathbf{C}_1^{\mathfrak{L}(G)_{x_0}} \arrow[r] & \mathbf{C}_1 \\
\mathbf{C}_2^{\mathfrak{L}(G)_{x_0}} \arrow[u] \arrow[r] & \mathbf{C}_2 \arrow[u]
\end{tikzcd}$$

with $\mathbf{C}_1 := \mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\omega,\text{alm-const}}$, $\mathbf{C}_2 = \mathrm{Vect}$ and the functor being $\widetilde{\pi}^!_{x_0}$, and the above fact that $\widetilde{\pi}^!_{x_0}$ admits a left adjoint in $\mathfrak{L}(G)_{x_0}$-**mod**.

6.2. **Left adjoint for factorization modules.** In this subsection we establish Property (1) in Sect. 6.1.6.

6.2.1. We will apply Proposition C.12.11 to

$$\mathbf{A} := \mathrm{D}(\mathrm{Gr}_G),\ \mathbf{C} = \mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)},\ \mathcal{A} := \omega_{\mathrm{Gr}_G}.$$

Lemma 6.2.2. *In the above setting, any object $\mathcal{F} \in \mathrm{D}_{\mathrm{hol}}(\widetilde{\mathrm{Gr}}_{G,x_0})$ is adapted to ω_{Gr_G}-induction (see Sect. C.12.8).*

6.2.3. To prove the lemma, we need some notations.

Unwinding the definitions, the functor (C.51) is given by

$$\jmath_I^! : \mathrm{D}^{\mathrm{lax}}(\mathcal{Z}_I) \to \mathrm{D}^{\mathrm{lax}}(\mathcal{Y}_I),$$

where

- $\mathcal{Z}_I$ is the categorical prestack (see Sect. C.1.1)

$$\Big(\big(\prod_{i\in I^\circ} \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}}\big) \times \widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}\Big)_{\mathrm{disj}},$$

  where
  - $\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}}$ is the space encoding the unital factorization structure of Gr_G (see Sect. C.1.7);

[14]Alternatively for any $\mathbf{C} \in \mathfrak{L}(G)$-**mod**, the functor $\mathbf{C}^{\mathfrak{L}(G)} \to \text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C})$ admits a left adjoint, which is the composition $\text{alm-inv}_{\mathfrak{L}(G)}(\mathbf{C}) \hookrightarrow \text{alm-inv}_{\mathfrak{L}^+(G)}(\mathbf{C}) \xrightarrow{\mathrm{Av}_!^{\mathfrak{L}^+(G)}} \mathbf{C}^{\mathfrak{L}^+(G)} \xrightarrow{\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}} \mathbf{C}^{\mathfrak{L}(G)}$.

– $\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}$ is the space encoding the unital factorization Gr_G-module structure of $\widetilde{\mathrm{Gr}}_{G,x_0}$
– $(-)_{\mathrm{disj}}$ means we apply base-change along the open subspace

$$\Big(\big(\prod_{i\in I^\circ}\mathrm{Ran}^{\mathrm{untl}}\big)\times\mathrm{Ran}^{\mathrm{untl}}_{x_0}\Big)_{\mathrm{disj}}\subseteq\big(\prod_{i\in I^\circ}\mathrm{Ran}^{\mathrm{untl}}\big)\times\mathrm{Ran}^{\mathrm{untl}}_{x_0},$$

see Sect. C.1.13.

- $\mathcal{Y}_I$ is the categorical prestack

$$\big(\prod_{i\in I}\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_\circ}\big)_{\mathrm{disj}}\times\widetilde{\mathrm{Gr}}_{G,x_0},$$

where $\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_\circ}$ is the space encoding the unital factorization structure of Gr_G, but for the punctured curve $X_\circ:=X\setminus x_0$. See Sect. C.12.4.

- The morphism

$$\jmath_I:\mathcal{Y}_I\to\mathcal{Z}_I$$

is the base-change of the morphism (C.50). The functor

$$\jmath_I^!:\mathrm{D}^{\mathrm{lax}}(\mathcal{Z}_I)\to\mathrm{D}^{\mathrm{lax}}(\mathcal{Y}_I)$$

is the !-pullback functors for lax D-modules (see Sect. C.3.4 and Sect. C.3.7).

6.2.4. Let $\mathcal{F}\in\mathrm{D}_{\mathrm{hol}}(\widetilde{\mathrm{Gr}}_{G,x_0})$ and write

$$\omega\in\mathrm{D}^{\mathrm{lax}}(\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_\circ})$$

for the dualizing D-module. To prove Lemma 6.2.2, we need to check:

- For any marked finite set I, the partially defined left adjoint $\jmath_{I,!}$ of $\jmath_I^!$ is defined on the object

$$(\underset{i\in I}{\boxtimes}\omega)|_{\mathrm{disj}}\boxtimes\mathcal{F}\in\mathrm{D}^{\mathrm{lax}}(\mathcal{Z}_I). \tag{6.7}$$

- The canonical morphism

$$\jmath_{I,!}((\underset{i\in I}{\boxtimes}\omega)|_{\mathrm{disj}}\boxtimes\mathcal{F})\to\big((\underset{i\in I}{\boxtimes}\omega)\boxtimes\jmath_!(\omega\boxtimes\mathcal{F})\big)|_{\mathrm{disj}}$$

is invertible.

6.2.5. To verify the claims in Sect. 6.2.4, we need some preparations.

Let $\mathcal{Y}$ be any categorical prestack. We say a lax D-module $\mathcal{F}\in\mathrm{D}^{\mathrm{lax}}(\mathcal{Y})$ is ind-holonomic if its !-pullback along any affine point $S\to\mathcal{Y}$ is contained in $\mathrm{D}_{\mathrm{hol}}(S)$. Note that (6.7) is an ind-holonomic object.

By definition, !-pullback functors preserve ind-holonomic lax D-modules.

Let $\mathcal{Y}$ be a categorical prestack. We say a collection of (finite type) indschemes $(f_\alpha:Y_\alpha\to\mathcal{Y})_{\alpha\in A}$ over $\mathcal{Y}$ is *adapted to !-direct images* if

- The functors

$$f_\alpha^!:\mathrm{D}^{\mathrm{lax}}(\mathcal{Y})\to\mathrm{D}(Y_\alpha)$$

are jointly conservative.
- The left adjoint of $f_\alpha^!$ exists, i.e., we have the !-direct image functor

$$f_{\alpha,!}:\mathrm{D}(Y_\alpha)\to\mathrm{D}^{\mathrm{lax}}(\mathcal{Y}).$$

- The functor $f_{\alpha,!}$ preserves ind-holonomic lax D-modules, i.e., we have a functor

$$f_{\alpha,!}:\mathrm{D}_{\mathrm{hol}}(Y_\alpha)\to\mathrm{D}^{\mathrm{lax}}_{\mathrm{hol}}(\mathcal{Y}). \tag{6.8}$$

It is clear these conditions imply that $\mathrm{D}^{\mathrm{lax}}_{\mathrm{hol}}(\mathcal{Y})$ is generated by the images of (6.8).

6.2.6. Let $\jmath : \mathcal{Y} \to \mathcal{Z}$ be a morphism between categorical prestacks, and $(f_\alpha : Y_\alpha \to \mathcal{Y})_{\alpha\in A}$ and $(g_\beta : Z_\beta \to \mathcal{Z})_{\beta\in B}$ be collections of indschemes that are adapted to !-direct images. We say these two collections are *compatible with* $\jmath$ if for any α, there exists β such that the composition $Y_\alpha \to \mathcal{Y} \to \mathcal{Z}$ factors through Z_β.

We claim the above condition implies the partially defined left adjoint $\jmath_!$ of

$$\jmath^! : \mathrm{D}^{\mathrm{lax}}(\mathcal{Z}) \to \mathrm{D}^{\mathrm{lax}}(\mathcal{Y})$$

is defined on $\mathrm{D}^{\mathrm{lax}}_{\mathrm{hol}}(\mathcal{Y})$. Namely, we only need to show $(\jmath\circ f_\alpha)_!$ is defined on $\mathrm{D}_{\mathrm{hol}}(Y_\alpha)$ for any $\alpha\in A$. Note that $\jmath\circ f_\alpha$ factors as $\jmath\circ f_\alpha$ as

$$Y_\alpha \xrightarrow{\jmath_{\alpha\beta}} Z_\beta \xrightarrow{g_\beta} \mathcal{Z}$$

for some β. Now the claim follows from the following two facts:

- The partially defined left adjoint $\jmath_{\alpha\beta,!}$ of
$$\jmath^!_{\alpha\beta} : \mathrm{D}(Z_\beta) \to \mathrm{D}(Y_\alpha)$$
is defined on $\mathrm{D}_{\mathrm{hol}}(Y_\alpha)$, because Y_α and β are (ind-finite type) indschemes;
- The functor $g_{\beta,!} : \mathrm{D}(Z_\beta) \to \mathrm{D}^{\mathrm{lax}}(Z)$ left adjoint to $g^!_\beta$ exists by assumption.

6.2.7. Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism between categorical prestacks and $\mathcal{W}$ be another categorical prestack. Let $(f_\alpha : Y_\alpha \to \mathcal{Y})_{\alpha\in A}$, $(g_\beta : Z_\beta \to \mathcal{Z})_{\beta\in B}$ and $(h_\gamma : W_\gamma \to \mathcal{W})_{\gamma\in C}$ be collections of indschemes that are adapted to !-direct images. We say these collections are *compatble with f and $(f, \mathrm{id}_{\mathcal{W}})$* if

- $(f_\alpha, h_\gamma)_{(\alpha,\gamma)\in A\times C}$ is a collection of schemes over $\mathcal{Y}\times\mathcal{W}$ that is adapted to !-direct images;
- $(g_\beta, h_\gamma)_{(\beta,\gamma)\in B\times C}$ is a collection of schemes over $\mathcal{Z}\times\mathcal{W}$ that is adapted to !-direct images;
- $(f_\alpha)_{\alpha\in A}$ and $(g_\beta)_{\beta\in B}$ are compatible with f.

As in Sect. 6.2.6, one can show these conditions imply for $\mathcal{M}\in\mathcal{D}^{\mathrm{lax}}_{\mathrm{hol}}(\mathcal{Y})$ and $\mathcal{N}\in\mathcal{D}^{\mathrm{lax}}_{\mathrm{hol}}(\mathcal{W})$, we have

$$(\jmath, \mathrm{id}_{\mathcal{W}})_!(\mathcal{M}\boxtimes\mathcal{N}) \xrightarrow{\simeq} \jmath_!(\mathcal{M})\boxtimes\mathcal{N}.$$

6.2.8. Finally, let us apply the above paradigm to the claims in Sect. 6.2.4. We only need to find collections of indschemes over

$$\mathcal{W} := \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_\circ},\ \mathcal{Y} := \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_\circ}\times\widetilde{\mathrm{Gr}_{G,x_0}},\ \mathcal{Z} := \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}$$

satisfying the conditions in Sect. 6.2.7. Note that here we can ignore the functor $(-)|_{\mathrm{disj}}$ because it commutes with arbitrary !-direct images.

Note that $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ are defined over $\mathrm{Ran}^{\mathrm{untl}}_{x_0}$ (resp. $\mathrm{Ran}^{\mathrm{untl}}_\circ$). Now the desired collections of indschemes over them can be given by applying base-change to the schemes

$$\begin{aligned} X^I\times x_0 \to &\quad \mathrm{Ran}^{\mathrm{untl}}_{x_0},\ |I|<\infty \\ (X-x)^I \to &\quad \mathrm{Ran}^{\mathrm{untl}}_\circ,\ |I|<\infty \end{aligned}$$

respectively. Namely, the conditions in Sect. 6.2.7 can be verified by combining the following two arguments:

- The maps $X^I\times x_0 \to \mathrm{Ran}_{x_0}$ into the *non-unital* marked Ran space are pseudo-proper, hence there is a !-direct image functor along
$$\mathcal{T}|_{X^I\times x_0} \to \mathcal{T}|_{\mathrm{Ran}_{x_0}}$$
for $\mathcal{T} = \mathcal{Y}$ or $\mathcal{Z}$. Moreover, these functors preserve ind-holonomic objects and commute with external tensor products.

- $\mathcal{T} = \mathcal{Y}$ or $\mathcal{Z}$ (resp. $\mathcal{T} = \mathcal{W}$) is a *coCartesian space* over $\mathrm{Ran}^{\mathrm{untl}}_{x_0}$ (resp. $\mathrm{Ran}^{\mathrm{untl}}_{\circ}$), see Sect. C.1.5. Also, for any morphism $\underline{x} \subseteq \underline{x}'$ in $\mathrm{Ran}^{\mathrm{untl}}_{x_0}$ (resp. in $\mathrm{Ran}^{\mathrm{untl}}_{\circ}$), the structural morphism
$$\mathcal{T}_{\underline{x}} \to \mathcal{T}_{\underline{x}'}$$
is ind-proper. Then we can mimic the construction in [Ga1, Sect. 4.3] to obtain a !-direct image functor along
$$\mathcal{T}|_{\mathrm{Ran}_{\underline{x}_0}} \to \mathcal{T},$$
which preserves ind-holonomic objects and commute with external tensor products.

$\square$[Lemma 6.2.2]

6.2.9. As a consequence of Lemma 6.2.2, we can apply Proposition C.12.11 to $V := \delta_1$, where $1 \in \widetilde{\mathrm{Gr}}_{G,x_0}$ is a unit point. This gives an object
$$\mathrm{ind}_{\omega_{\mathrm{Gr}_G}}(\delta_1) \simeq \jmath_!(\omega_{\mathrm{Gr}_{G,\circ}} \boxtimes \delta_1) \in \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}).$$
Note that the underlying lax D-module of this object is
$$\jmath_!(\omega_{\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}}} \boxtimes \delta_1) \in \mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}),$$
where we recall that $\jmath$ is the map
$$\jmath : \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}} \times \widetilde{\mathrm{Gr}}_{G,x_0} \to \widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}.$$

6.3. **Verification of Properties (2) and (3).**

6.3.1. Using the method in Sect. 6.2.6 - Sect. 6.2.8, one can show the partially defined functor $(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})_!$, left adjoint to
$$(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})^! : \mathrm{D}^{\mathrm{lax}}(\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}} / \mathrm{Hecke}_{x_0}) \to \mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}),$$
is defined on ind-holonomic objects, and it sends objects in
$$\omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0},$$
whose underlying object of $\mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}_{x_0}})$ is ind-holonomic to objects in
$$\omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0},$$
thereby providing a left adjoint to the lower-right vertical functor in (6.4).

This establishes Property (2) in Sect. 6.1.6.

6.3.2. We now proceed to establishing Property (3). We need to establish that the Beck-Chevalley natural transformation corresponding to the diagram

$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\widetilde{\Phi}"] & \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \\
\mathrm{Vect}^{\mathfrak{L}(G)_{x_0}} \arrow[u, "\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}}"] \arrow[r, "\Phi"] & \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \arrow[u, "(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})^!"']
\end{tikzcd} \tag{6.9}$$

is an isomorphism.

Note that we can view (6.9) as a commutative diagram involving the functors (3.6):

$$\begin{tikzcd}
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^K) \arrow[r, "\widetilde{\Phi}"] & \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^K)^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \\
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}}) \arrow[u] \arrow[r, "\Phi"] & \omega_{\mathrm{Gr}_G}\text{-}\mathrm{mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}, \arrow[u]
\end{tikzcd} \tag{6.10}$$

where $\mathrm{D}(\mathfrak{L}(G)_{x_0})^K$ and $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}}$ are considered as objects of $\mathfrak{L}(G)_{x_0}$-**mod** with respect to the *left* action of $\mathfrak{L}(G)_{x_0}$, and $(-)^K$ and $(-)^{\mathfrak{L}(G)_{x_0}}$ are taken with respect to the *right* action.

The vertical arrows in (6.10) are induced by the 1-morphism

$$\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}} \to \mathrm{D}(\mathfrak{L}(G)_{x_0})^K$$

in $\mathfrak{L}(G)_{x_0}$-**mod**, given by $\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}\to K}$ with respect to the *right* action.

6.3.3. Let Φ^+ be the functor

$$\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}}) \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0},$$

which is a particular case of (3.6) for $\mathbf{C} := (\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}} = \mathrm{D}(\mathrm{Gr}_{G,x_0})$.

We expand diagram (6.10) as

$$\begin{array}{ccc}
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^K) & \xrightarrow{\widetilde{\Phi}} & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^K)^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \\
{\scriptstyle \mathrm{oblv}_{\mathfrak{L}^+(G)_{x_0}\to K}}\uparrow & & \uparrow{\scriptstyle \mathrm{oblv}_{\mathfrak{L}^+(G)_{x_0}\to K}} \\
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}}) & \xrightarrow{\Phi^+} & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \\
{\scriptstyle \mathrm{oblv}_{\mathfrak{L}(G)_{x_0}\to\mathfrak{L}^+(G)_{x_0}}}\uparrow & & \uparrow{\scriptstyle \mathrm{oblv}_{\mathfrak{L}(G)_{x_0}\to\mathfrak{L}^+(G)_{x_0}}} \\
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}}) & \xrightarrow{\Phi} & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}.
\end{array} \tag{6.11}$$

It is enough to show that the Beck-Chevalley natural transformations

$$(\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}\to\mathfrak{L}^+(G)_{x_0}})^L \circ \Phi^+_{\mathrm{oblv}} \to \Phi \circ (\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}\to\mathfrak{L}^+(G)_{x_0}})^L \tag{6.12}$$

and

$$(\mathrm{oblv}_{\mathfrak{L}^+(G)_{x_0}\to K})^L \circ \widetilde{\Phi} \to \Phi^+ \circ (\mathrm{oblv}_{\mathfrak{L}^+(G)_{x_0}\to K})^L \tag{6.13}$$

are both isomorphisms.

6.3.4. The assertion for (6.12) follows from the fact that the 1-morphism

$$\mathrm{oblv}_{\mathfrak{L}(G)_{x_0}\to\mathfrak{L}^+(G)_{x_0}} : (\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}(G)_{x_0}} \to (\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}}$$

admits a left adjoint already in $\mathfrak{L}(G)_{x_0}$-**mod**, see Sect. 4.6.11.

6.3.5. To prove the assertion for (6.13), we expand the upper portion of (6.11):

$$\begin{array}{ccc}
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^K) & \xrightarrow{\widetilde{\Phi}} & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^K)^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \\
\uparrow & & \uparrow \\
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{alm}\text{-}\,\mathfrak{L}^+(G)_{x_0}}) & \xrightarrow{\Phi^{\mathrm{alm}\text{-}\,+}} & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{alm}\text{-}\,\mathfrak{L}^+(G)_{x_0}})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \\
\uparrow & & \uparrow \\
\mathrm{inv}_{\mathfrak{L}(G)_{x_0}}(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}}) & \xrightarrow{\Phi^+} & \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0},
\end{array} \tag{6.14}$$

where the symbol alm- $\mathfrak{L}^+(G)_{x_0}$ refers to *almost invariants* with respect to $\mathfrak{L}^+(G)_{x_0}$.

It suffices to show that the Beck-Chevalley natural transformations corresponding to both subdiagrams are isomorphisms.

For the lower square this follows from the fact that the 1-morphism

$$(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathfrak{L}^+(G)_{x_0}} \to (\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{alm}\text{-}\,\mathfrak{L}^+(G)_{x_0}}$$

admits a left adjoint already in $\mathfrak{L}(G)_{x_0}$-**mod**, see Sect. 4.2.10.

6.3.6. For the upper square in (6.14) we note:

- The left vertical arrow is an equivalence;
- The right vertical arrow is fully faithful. Indeed, the 1-morphism
$$(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\text{alm-}\mathfrak{L}^+(G)_{x_0}} \to \mathrm{D}(\mathfrak{L}(G)_{x_0})^K$$
in $\mathfrak{L}(G)_{x_0}$-**mod** is fully faithful and admits a *right* adjoint.

Now, the fact that the Beck-Chevalley natural transformation is an isomorphism follows from the next general assertion:

Lemma 6.3.7. *Let*
$$\begin{tikzcd}
\mathbf{C}_1 \arrow[r, "\Phi"] & \mathbf{C}_2 \\
\mathbf{C}'_1 \arrow[u, "\iota_1"] \arrow[r, "\Phi'"] & \mathbf{C}'_2 \arrow[u, "\iota_2"]
\end{tikzcd}$$
be a commutative diagram, in which ι_1 is an equivalence and ι_2 is fully faithful. Then the Beck-Chevalley natural transformation
$$(\iota_2)^L \circ \Phi \to \Phi' \circ (\iota_1)^L$$
is an isomorphism.

Proof. It suffices to show that the natural transformation
$$(\iota_2)^L \circ \Phi \circ \iota_1 \to \Phi' \circ (\iota_1)^L \circ \iota_1$$
is an isomorphism.

We have a commutative diagram of functors
$$\begin{tikzcd}
(\iota_2)^L \circ \Phi \circ \iota_1 \arrow[r] \arrow[d, "\sim"'] & \Phi' \circ (\iota_1)^L \circ \iota_1 \arrow[d] \\
(\iota_2)^L \circ \iota_2 \circ \Phi' \arrow[r] & \Phi'.
\end{tikzcd}$$

We have to show that the top horizontal arrow is an isomorphism. However, the conditions of the lemma imply that all three other arrows are isomorphisms.

□

6.4. **Verification of Property (4).** In this subsection we perform the key calculation involved in the proof of Theorem 3.1.7.

6.4.1. We rewrite the functor (6.5) as

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\widetilde{\mathrm{Gr}}_{G,x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \overset{\mathrm{oblv}}{\to} \mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}) \overset{(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})_!}{\longrightarrow}$$
$$\to \mathrm{D}^{\mathrm{lax}}(\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}/\,\mathrm{Hecke}_{x_0}) \overset{\iota^!_{x_0}}{\longrightarrow} \mathrm{Vect}, \tag{6.15}$$

where
$$\iota_{x_0} : \mathrm{pt} \to \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}/\,\mathrm{Hecke}_{x_0}$$
is the base change of the map $x_0 \to \mathrm{Ran}^{\mathrm{untl}}_{x_0}$.

Note that by Sect. 6.2.9, the image of the natural transformation
$$(\iota^!_1 \circ \mathrm{oblv}_{\omega_{\mathrm{Gr}_G}})^L \to \widetilde{\Phi}$$
evaluated on $k \in \mathrm{Vect}$ under the first arrow in (6.15) is the map

$$\jmath_!(\omega_{\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}}} \boxtimes \delta_1) \to \jmath_!(\omega_{\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}}} \boxtimes \omega_{\widetilde{\mathrm{Gr}}_{G,x_0}}) \simeq \jmath_! \circ \jmath^!(\omega_{\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}}) \to \omega_{\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}}. \tag{6.16}$$

6.4.2. Denote by $\pi_{\mathrm{Ran}^{\mathrm{untl}}}/\operatorname{Hecke}_{x_0}$ the projection

$$\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}/\operatorname{Hecke}_{x_0} \to \mathrm{Ran}^{\mathrm{untl}}_{x_0}.$$

Note that this morphism is pseudo-proper (see [Ga1, Sect. 1.5.3]); hence, the functor

$$(\pi_{\mathrm{Ran}^{\mathrm{untl}}}/\operatorname{Hecke}_{x_0})_! : \mathrm{D}^{\mathrm{lax}}(\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}/\operatorname{Hecke}_{x_0}) \to \mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}),$$

left adjoint to $(\pi_{\mathrm{Ran}^{\mathrm{untl}}}/\operatorname{Hecke}_{x_0})^!$, is well-defined and satisfies base change.

Hence, we obtain that the functor

$$\mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}) \overset{(\widetilde{\pi}^{\mathrm{fact}_{x_0},\mathrm{Gr}_G})_!}{\longrightarrow} \mathrm{D}^{\mathrm{lax}}(\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}/\operatorname{Hecke}_{x_0}) \overset{(\iota_{x_0})^!}{\longrightarrow} \mathrm{Vect},$$

appearing in (6.15) can be rewritten as

$$\mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}) \overset{(\widetilde{\pi}_{\mathrm{Ran}^{\mathrm{untl}}})_!}{\longrightarrow} \mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}) \overset{(-)_{x_0}}{\longrightarrow} \mathrm{Vect}, \tag{6.17}$$

where:

- $\widetilde{\pi}_{\mathrm{Ran}^{\mathrm{untl}}}$ denotes the projection $\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}} \to \mathrm{Ran}^{\mathrm{untl}}_{x_0}$;
- $(\widetilde{\pi}_{\mathrm{Ran}^{\mathrm{untl}}})_!$ denotes the partially defined left adjoint to $(\widetilde{\pi}_{\mathrm{Ran}^{\mathrm{untl}}})^!$.

Thus, we obtain that we need to show that the morphism (6.16) becomes an isomorphism after applying the functor (6.17).

6.4.3. The question is local; hence we can assume that $(X, x_0) = (\mathbb{A}^1, 0)$. Consider the objects of $\mathrm{D}(\mathrm{Ran}_{x_0})$ obtained by applying the functor

$$\mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}) \overset{(\widetilde{\pi}_{\mathrm{Ran}^{\mathrm{untl}}})_!}{\longrightarrow} \mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}) \overset{\imath^!}{\longrightarrow} \mathrm{D}(\mathrm{Ran}_{x_0})$$

to the two sides of (6.16), where Ran_{x_0} is the *non-unital* marked Ran space (which is a non-categorical prestack) and

$$\imath : \mathrm{Ran}_{x_0} \to \mathrm{Ran}^{\mathrm{untl}}_{x_0}$$

is the obvious map. Note that both these objects are equivariant with respect to the action of $\mathbb{G}_m$ on Ran_{x_0} by dilations.

6.4.4. Now we apply the contraction principle to the $\mathbb{G}_m$-action on Ran_{x_0}, which says the functors

$$x_0^! : \mathrm{D}(\mathrm{Ran}_{x_0}) \to \mathrm{Vect}$$

and

$$\mathrm{C}^\cdot_c(\mathrm{Ran}_{x_0}, -) : \mathrm{D}(\mathrm{Ran}_{x_0}) \to \mathrm{Vect}$$

are canonically equivalent when restricted to $\mathbb{G}_m$-equivariant D-modules on Ran_{x_0}[15]. This implies the functors

$$\mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}) \overset{(-)_{x_0}}{\longrightarrow} \mathrm{Vect}$$

and

$$\mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}) \overset{\imath^!}{\longrightarrow} \mathrm{D}(\mathrm{Ran}_{x_0}) \xrightarrow{\mathrm{C}^\cdot_c(\mathrm{Ran}_{x_0},-)} \mathrm{Vect},$$

are canonically equivalent when restricted to $\mathbb{G}_m$-equivariant D-modules on $\mathrm{Ran}_{x_0^{\mathrm{untl}}}$. Note that by [GLC2, Lemma C.5.12], the latter functor is canonically equivalent to

$$\mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}) \xrightarrow{\mathrm{C}^\cdot_c(\mathrm{Ran}^{\mathrm{untl}}_{x_0},-)} \mathrm{Vect}\,.$$

[15]Proof: the map $x_0 : \mathrm{pt} \to \mathrm{Ran}_{x_0}$ is right inverse to $p : \mathrm{Ran}_{x_0} \to \mathrm{pt}$. Hence $x_0^! \circ p^! \simeq \mathrm{Id}$. We only need to show this natural isomorphism exhibits $x_0^!$ and the left adjoint of $p^!$, when restricted to $\mathbb{G}_m$-equivariant objects. Note that we have

$$\mathrm{Ran}_{x_0} \simeq \operatorname{colim}_{I\in \mathrm{Fin}}(\mathrm{Ran}_{x_0} \times_{\mathrm{Ran}} X^I)$$

where Fin is the category of finite sets. Moreover, this isomorphism is compatible with the $\mathbb{G}_m$-actions on both sides. Hence the desired claim follows from the contraction principle for schemes ([DG, Appendix A]).

6.4.5. Thus, it remains to show that the morphism (6.16) becomes an isomorphism after applying the functor

$$\mathrm{D}^{\mathrm{lax}}(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}) \overset{(\widetilde{\pi}_{\mathrm{Ran}^{\mathrm{untl}}})_!}{\longrightarrow} \mathrm{D}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}) \overset{\mathrm{C}^{\cdot}_c(\mathrm{Ran}^{\mathrm{untl}}_{x_0},-)}{\longrightarrow} \mathrm{Vect}, \tag{6.18}$$

6.4.6. Let us again assume that (X, x_0) is arbitrary. We obtain that it suffices to show that the morphism (6.16) induces an isomorphism after applying the functor

$$\mathrm{C}^{\cdot}_c(\widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}}, -).$$

I.e., we need to show that the locally closed embedding

$$\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}} \times \{1\} \to \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}} \times \widetilde{\mathrm{Gr}}_{G,x_0} \to \widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}} \tag{6.19}$$

induces an isomorphism on homology.

As we will show in the next subsection, the latter assertion follows from the *homological contractibility* statement from [Ga1].

6.5. **The contractibility statement.**

6.5.1. Write

$$X = \overline{X} - \underline{x},$$

where $\overline{X}$ is a complete (and smooth) curve and $\underline{x} = \{x_1, ..., x_n\}$ is a finite collection of points on $\overline{X}$.

Let Bun_G denote the moduli stack of G-bundles on $\overline{X}$. Let $\mathrm{Bun}_G^{\mathrm{level}^1_{\underline{x}}}$ (resp., $\widetilde{\mathrm{Bun}}_G^{\mathrm{level}^1_{\underline{x}}}$) be the moduli stack of G-bundles with structure of level 1 at $\{x_1, ..., x_n\}$ (resp., additional structure of level K at x_0).

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{\circ}} & \xrightarrow{(6.19)} & \widetilde{\mathrm{Gr}}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}} \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Bun}}_G^{\mathrm{level}^1_{\underline{x}}} & \xrightarrow{\mathrm{id}} & \widetilde{\mathrm{Bun}}_G^{\mathrm{level}^1_{\underline{x}}}. \end{array} \tag{6.20}$$

We will show that both vertical maps in (6.20) induce isomorphisms at the level of $\mathrm{C}_{\cdot}(-)$. This will establish the corresponding fact for the morphism (6.19).

We will show that both vertical maps in (6.20) are in fact universal homological equivalences.

6.5.2. The fact that the left vertical arrow in (6.20) is a universal homological equivalence is the statement of the contractibility theorem from [Ga1], applied to the complete curve $\overline{X}$ and $\{x_0, x_1, ..., x_n\}$ as marked points.

6.5.3. Note that the right vertical arrow in (6.20) is the base change of the map

$$\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}_{x_0}} \to \mathrm{Bun}_G^{\mathrm{level}^1_{\underline{x}}}. \tag{6.21}$$

Hence, it suffices to show that (6.21) is a universal homological equivalence.

6.5.4. Consider the forgetful map $\mathrm{Ran}^{\mathrm{untl}}_{x_0} \to \mathrm{Ran}^{\mathrm{untl}}$ and the map $\mathrm{add}_{x_0} : \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}_{x_0}$. They realize Ran_{x_0} as a retract of Ran. Similarly, the map (6.21) is a retract of the map

$$\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{Bun}_G^{\mathrm{level}^1_{\underline{x}}}.$$

Hence, it suffices to show that the latter map is a universal homological equivalence. However, this is again an instance of the contractibility theorem, applied to the complete curve $\overline{X}$ and $\{x_1, ..., x_n\}$ as marked points.

□

7. Proof of Theorem 4.7.3 for a torus

For the next three sections we will be concerned with the proof of Theorem 4.7.3.

In this section we take $G = T$ to be a torus, and we will prove Theorem 4.7.3 using local geometric class field theory.

This special case will also be used in the proof of the general case of Theorem 4.7.3.

7.1. **Reduction to the case of characters.**

7.1.1. Write

$$\mathfrak{L}^+(T)_{x_0} \simeq T \times \ker(\mathfrak{L}^+(\mathfrak{t}) \to \mathfrak{t}).$$

Fourier-Laumon transforms identifies the monoidal category $\mathrm{D}(\mathfrak{L}^+(T)_{x_0})$ with

$$\mathrm{QCoh}(\check{\mathfrak{t}}/\check{\Lambda}) \otimes \mathrm{D}(\ker(\mathfrak{L}^+(\mathfrak{t}) \to \mathfrak{t})^*),$$

equipped with the *pointwise* tensor product.

7.1.2. Note that for $\mathbf{C} \in \mathfrak{L}^+(T)_{x_0}$-**mod**, the inclusion

$$\text{alm-inv}_{\mathfrak{L}^+(T)_{x_0}}(\mathbf{C}) \to \mathbf{C}$$

is an equivalence if and only if for every *non-zero* geometric point

$$\chi^+ \in (\check{\mathfrak{t}}/\check{\Lambda}) \times (\ker(\mathfrak{L}^+(\mathfrak{t}) \to \mathfrak{t})^*),$$

the fiber $\mathbf{C}_{\chi^+}$ is zero, see [AGKRRV, Lemma 21.4.6].

7.1.3. Let $\mathbf{C}$ be an object of $\mathfrak{L}(T)_{x_0}$-**mod**. Thus, we obtain that in order to prove Theorem 4.7.3, it suffices to show that for any such χ^+,

$$\omega_{\mathrm{Gr}_T}\text{-mod}^{\mathrm{fact}}(\mathbf{C}_{\chi^+}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} = 0.$$

Up to changing the ground field, we can assume that χ^+ is a rational point.

7.1.4. Note that k-rational points of $(\check{\mathfrak{t}}/\check{\Lambda}) \times (\ker(\mathfrak{L}^+(\mathfrak{t}) \to \mathfrak{t})^*)$ can be thought of character sheaves on $\mathfrak{L}^+(T)_{x_0}$.

For $\mathbf{C} \in \mathfrak{L}(T)_{x_0}$-**mod** and a k-rational point χ^+ as above, we have

$$\mathbf{C}_{\chi^+} \simeq \mathrm{D}(\mathfrak{L}(T)_{x_0})^{(\mathfrak{L}^+(T)_{x_0},\chi^+)} \otimes \mathbf{C}_0,$$

where $\mathbf{C}_0 \in \mathrm{DGCat}$ and the action of $\mathfrak{L}(T)_{x_0}$ is via the first factor.

Thus, we need to show that for a non-trivial χ^+ and any $\mathbf{C}_0$, the category

$$\omega_{\mathrm{Gr}_T}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(T)_{x_0})^{(\mathfrak{L}^+(T)_{x_0},\chi^+)})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)} \otimes \mathbf{C}_0)_{x_0}$$

is zero.

7.1.5. We will prove:

Theorem 7.1.6. *For a non-trivial χ^+, the category*

$$\omega_{\mathrm{Gr}_T}\text{-mod}^{\mathrm{fact}}((\mathrm{D}(\mathfrak{L}(T)_{x_0})^{(\mathfrak{L}^+(T)_{x_0},\chi^+)})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)})_{x_0}$$

is zero.

The proof in the presence of $\mathbf{C}_0$ is the same. The rest of this section is devoted to the proof of Theorem 7.1.6.

7.2. **Character sheaves on $\mathfrak{L}(T)_{x_0}$.**

7.2.1. Let χ be an extension of χ^+ to a character sheaf on all of $\mathfrak{L}(T)_{x_0}$. (Note that the space of such extensions is a torsor over

$$\mathrm{Hom}(\Lambda, B(k^\times)) \simeq B(\check{T}(k)),$$

where $\check{T}$ is the Langlands-dual torus.)

The choice of χ equips the category $\mathrm{D}(\mathfrak{L}(T)_{x_0})^{(\mathfrak{L}^+(T)_{x_0},\chi^+)}$ with an action of Λ, so that

$$(\mathrm{D}(\mathfrak{L}(T)_{x_0})^{(\mathfrak{L}^+(T)_{x_0},\chi^+)})^\Lambda \simeq \mathrm{Vect}\,.$$

Let us denote the above copy of Vect, viewed as an object of $\mathfrak{L}(T)_{x_0}$-**mod**, by Vect_χ.

7.2.2. Using the equivalence[16]

$$\Lambda\text{-}\mathbf{mod} \simeq \mathrm{Rep}(\Lambda)\text{-}\mathbf{mod}$$

of [Ga3], we can recover $\mathrm{D}(\mathfrak{L}(T)_{x_0})^{(\mathfrak{L}^+(T)_{x_0},\chi^+)}$ as

$$\mathrm{Funct}_{\mathrm{Rep}(\Lambda)}(\mathrm{Vect}, \mathrm{Vect}_\chi).$$

Hence, we can reformulate Theorem 7.1.6 as:

Theorem 7.2.3. *The category*

$$\omega_{\mathrm{Gr}_T}\text{-mod}^{\mathrm{fact}}(\mathrm{Vect}_\chi^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)})_{x_0}$$

is zero.

7.2.4. The rest of this section is devoted to the proof of Theorem 7.2.3. The proof will amount to a simple computation, once we input the assertion of geometric class field theory (gCFT), reviewed in the next subsection.

7.3. **Geometric class field theory.**

7.3.1. Consider the map

$$X \to \mathrm{Ran}_{x_0}, \quad x \mapsto \{x, x_0\}.$$

Let $\mathrm{Gr}_{T,X}^{\mathrm{level}_{x_0}^\infty}$ denote the pullback of $\mathrm{Gr}_{G,\mathrm{Ran}_{x_0}}^{\mathrm{level}_{x_0}^\infty}$ along this map. Denote by j and i the open and closed embeddings

$$\mathrm{Gr}_{T,X-x_0}^{\mathrm{level}_{x_0}^\infty} \hookrightarrow \mathrm{Gr}_{T,X}^{\mathrm{level}_{x_0}^\infty} \hookleftarrow \mathrm{Gr}_{T,x_0}^{\mathrm{level}_{x_0}^\infty},$$

respectively.

Note that we have the identifications

$$\mathrm{Gr}_{T,X-x_0}^{\mathrm{level}_{x_0}^\infty} \simeq \mathrm{Gr}_{T,X-x_0} \times \mathfrak{L}(T)_{x_0} \text{ and } \mathrm{Gr}_{T,x_0}^{\mathrm{level}_{x_0}^\infty} \simeq \mathfrak{L}(T)_{x_0}.$$

7.3.2. For $\lambda \in \Lambda$ consider the connected component

$$(\mathrm{Gr}_{T,X-x_0})^\lambda \times \mathfrak{L}^+(T)_{x_0} = (\mathrm{Gr}_{T,X-x_0})^\lambda \times (\mathfrak{L}(T)_{x_0})^0 \subset (\mathrm{Gr}_{T,X-x_0})^\lambda \times \mathfrak{L}(T)_{x_0}.$$

Its closure in $\mathrm{Gr}_{T,X}^{\mathrm{level}_{x_0}^\infty}$, denoted $\overline{(\mathrm{Gr}_{T,X-x_0})^\lambda \times \mathfrak{L}^+(T)_{x_0}}$, is a $\mathfrak{L}^+(T)_{x_0}$-torsor over X, and its special fiber identifies with

$$(\mathfrak{L}(T)_{x_0})^\lambda \subset \mathfrak{L}(T)_{x_0}.$$

[16] Note that the category $\mathrm{Rep}(\Lambda)$-**mod**, which appears in the formula below, identifies with $\mathrm{QCoh}(\check{T})$, viewed as a monoidal category under *convolution*.

7.3.3. We will need the following statement from gCFT:

Theorem 7.3.4. *Up to replacing X by an open subset containing x_0, there exists a $\check{T}$-local system σ on $X - x$ such that for every $\lambda \in \Lambda$, the local system*

$$\lambda(\sigma) \boxtimes \chi^+$$

on

$$X \times \mathfrak{L}^+(T)_{x_0} \simeq (\mathrm{Gr}_{T,X-x_0})^\lambda \times \mathfrak{L}^+(T)_{x_0}$$

extends to a local system on $\overline{(\mathrm{Gr}_{T,X-x_0})^\lambda \times \mathfrak{L}^+(T)_{x_0}}$.

Moreover, σ is unique, up to tensoring with $\check{T}$-local systems unramified near x.

7.3.5. Note that a choice of σ as in the theorem determines an extension $\chi^+ \rightsquigarrow \chi$. Indeed, the restriction of χ to $(\mathfrak{L}(T)_{x_0})^\lambda$ equals the restriction of the extended local system to

$$\overline{(\mathrm{Gr}_{T,X-x_0})^\lambda \times \mathfrak{L}^+(T)_{x_0}} \hookleftarrow (\mathfrak{L}(T)_{x_0})^\lambda.$$

7.3.6. Note that the assignment

$$\lambda \rightsquigarrow \lambda(\sigma)$$

extends to a local system on $\mathrm{Gr}_{T,\mathrm{Ran}_\circ}$, to be denoted $\Lambda(\sigma)$, which is equipped with a natural factorization structure.

It follows formally from Theorem 7.3.4 that the local system

$$\Lambda(\sigma) \boxtimes \chi$$

on

$$\mathrm{Gr}_{T,\mathrm{Ran}_\circ} \times \mathfrak{L}(T)_{x_0} \simeq (\mathrm{Ran}_\circ \times \{x_0\}) \underset{\mathrm{Ran}_{x_0}}{\times} \mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$$

extends (uniquely) to a local system, to be denoted $\chi_{\mathrm{Ran}_{x_0}}$, on $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\mathrm{Ran}_{x_0}}$.

Moreover, $\chi_{\mathrm{Ran}_{x_0}}$ has a natural factorization structure with respect to $\Lambda(\sigma)$.

7.4. **The module** $\mathrm{Vect}_\chi^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)}$**.** Recall the object

$$\mathrm{Vect}_\chi^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)} \in \mathrm{D}(\mathrm{Gr}_T)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}. \tag{7.1}$$

In this subsection we will describe it as the factorization restriction of the tautological (i.e., vacuum) object $\mathrm{Vect}^{\mathrm{fact}_{x_0}} \in \mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}$.

7.4.1. Denote by $\pi_!^\sigma$ the factorization functor $\mathrm{D}(\mathrm{Gr}_T) \to \mathrm{Vect}$, given by the precomposition of $\pi_!$ (see Sect. 3.2.1) with the operation of tensoring by the inverse of $\Lambda(\sigma)$.

We claim:

Proposition 7.4.2. *The object* (7.1) *identifies canonically with* $\mathbf{Res}_{\pi_!^\sigma}(\mathrm{Vect}\text{-}\mathbf{mod}^{\mathrm{fact}_{x_0}})$.

Proof. Note that the operation of tensoring by $\Lambda(\sigma)$ is a factorization automorphism of $\mathrm{D}(\mathrm{Gr}_T)$.

Moreover, tensoring by $\chi_{\mathrm{Ran}_{x_0}}$ defines an *isomorphism*

$$(\mathrm{D}(\mathfrak{L}(T)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)})^{\mathfrak{L}(T)_{x_0}} \to (\mathrm{D}(\mathfrak{L}(T)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)})^{(\mathfrak{L}(T)_{x_0},\chi)},$$

i.e.,

$$\mathrm{Vect}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)} \to \mathrm{Vect}_\chi^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_T)},$$

compatible with the above automorphism of $\mathrm{D}(\mathrm{Gr}_T)$.

Now, the assertion of the proposition follows from that of Proposition 3.2.3.

□

7.4.3. Denote

$$R_{\check{T},\sigma} := \pi_!^\sigma(\omega_{\mathrm{Gr}_T}) \in \mathrm{FactAlg}(\mathrm{Vect}).$$

Thus, by (1.9), we can reformulate Theorem 7.1.6 as follows:

Theorem 7.4.4. *The category* $R_{\check{T},\sigma}\text{-mod}_{x_0}^{\mathrm{fact}}$ *is zero.*

Remark 7.4.5. The assertion of Theorem 7.4.4 follows easily from that of [Bogd, Equation (4.10) and/or Theorem 3.8]. Below will give an alternative (in a sense, more elementary) proof.

7.5. **Proof of Theorem 7.4.4 via chiral algebras.**

7.5.1. Let $R^{\mathrm{ch}}_{\check{T},\sigma}$ be the chiral algebra (on $X - x_0$) corresponding to $R_{\check{T},\sigma}$, so that

$$R_{\check{T},\sigma}\text{-mod}_{x_0}^{\mathrm{fact}} \simeq R^{\mathrm{ch}}_{\check{T},\sigma}\text{-mod}_{x_0}^{\mathrm{ch}},$$

see [GLC2, Sect. D.1].

Thus, we need to show that $R^{\mathrm{ch}}_{\check{T},\sigma}\text{-mod}_{x_0}^{\mathrm{ch}} = 0$.

7.5.2. Note that the D-module on $X - x_0$ underlying $R^{\mathrm{ch}}_{\check{T},\sigma}$ identifies canonically with

$$\underset{\lambda}{\oplus} R^{\mathrm{ch},\lambda}_{\check{T},\sigma}, \quad R^{\mathrm{ch},\lambda}_{\check{T},\sigma} = \lambda(\sigma) \otimes \omega_{X-x_0}[-1].$$

The chiral operation is given by

$$(j_{x_1\neq x_2})_*((\lambda(\sigma)\otimes\omega_{X-x_0}[-1]) \boxtimes (\mu(\sigma)\otimes\omega_{X-x_0}[-1])) \to$$
$$\to \Delta_*((\lambda(\sigma)\otimes\omega_{X-x_0}[-1]) \overset{!}{\otimes} (\mu(\sigma)\otimes\omega_{X-x_0}[-1]))[1] \simeq \Delta_*((\lambda+\mu)(\sigma)\otimes\omega_{X-x_0}[-1]).$$

7.5.3. Let $\mathcal{M}$ be an object of $R^{\mathrm{ch}}_{\check{T},\sigma}\text{-mod}_{x_0}^{\mathrm{ch}}$; let M denote the underlying the vector space, so that the D-module underlying $\mathcal{M}$ is $i_*(M)$.

The chiral action is given by

$$\mathrm{act} : (j_{x\neq x_0})_*(R^{\mathrm{ch}}_{\check{T},\sigma}) \otimes M \to i_*(M).$$

The axiom of chiral action implies that the (signed) sum of the following three morphisms

$$(j_{x_1\neq x_2, x_1\neq x_0, x_2\neq x_0})_*(R^{\mathrm{ch}}_{\check{T},\sigma} \boxtimes R^{\mathrm{ch}}_{\check{T},\sigma}) \otimes M \to \Delta_* \circ i_*(M) \tag{7.2}$$

(as D-modules on X^2) is zero:

- $(j_{x_1\neq x_2, x_1\neq x_0, x_2\neq x_0})_*(R^{\mathrm{ch}}_{\check{T},\sigma} \boxtimes R^{\mathrm{ch}}_{\check{T},\sigma}) \otimes M \overset{\{-,-\}\otimes\mathrm{Id}}{\to} \Delta_* \circ (j_{x\neq x_0})_*(R^{\mathrm{ch}}_{\check{T},\sigma}) \otimes M \overset{\mathrm{act}}{\to} \Delta_* \circ i_*(M)$;
-

$$(j_{x_1\neq x_2, x_1\neq x_0, x_2\neq x_0})_*(R^{\mathrm{ch}}_{\check{T},\sigma} \boxtimes R^{\mathrm{ch}}_{\check{T},\sigma}) \otimes M \overset{\mathrm{Id}\otimes\mathrm{act}}{\longrightarrow}$$
$$\to (\mathrm{id}\times i)_* \circ (j_{x\neq x_0})_*(R^{\mathrm{ch}}_{\check{T},\sigma}) \otimes M \overset{\mathrm{act}}{\to} (\mathrm{id}\times i)_* \circ i_*(M) = (i\times i)_*(M) = \Delta_* \circ i_*(M);$$

- The map, obtained from the previous one by interchanging the roles of x_1 and x_2.

7.5.4. Since σ is non-trivial at x, we can choose $\lambda \in \Lambda$, so that the 1-dimensional local system $\lambda(\sigma)$ is non-trivially ramified at x.

Let us restrict the three maps in (7.2) to

$$(j_{x_1\neq x_2, x_1\neq x_0, x_2\neq x_0})_*(R^{\mathrm{ch},\lambda}_{\check{T},\sigma} \boxtimes R^{\mathrm{ch},-\lambda}_{\check{T},\sigma}) \otimes M. \tag{7.3}$$

Note that the space of maps

$$(j_{x\neq x_0})_*(\lambda(\sigma)\otimes\omega_{X-x_0}) \to i_*(M)$$

is zero, and similarly for $-\lambda$.

Hence, the restriction of the 2nd and 3rd maps to (7.3) are zero. Hence, so is the restriction of the 1st map.

7.5.5. By definition, the first map identifies with

$$(j_{x_1\neq x_2,x_1\neq x_0,x_2\neq x_0})_*\left((\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\boxtimes(-\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\right)\otimes M\to \\ \to\Delta_*\circ(j_{x\neq x_0})_*(\omega_{X-x_0}[-1])\otimes M\overset{\text{act}}{\to}\Delta_*\circ i_*(M). \tag{7.4}$$

Note however, that the above copy of

$$\omega_{X-x_0}[-1]=R^{\mathrm{ch},0}_{\check{T},\sigma}$$

is the chiral unit in $R^{\mathrm{ch}}_{\check{T},\sigma}$, and since $\mathcal{M}$ is a unital chiral module, the last arrow in (7.4) comes from the canonical map

$$(j_{x\neq x_0})_*(\omega_{X-x_0}[-1])\to i_*(k).$$

Since (7.4) is zero, this means that the first arrow in (7.4) factors via a map

$$(j_{x_1\neq x_2,x_1\neq x_0,x_2\neq x_0})_*\left((\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\boxtimes(-\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\right)\otimes M\to\Delta_*(\omega_X[-1])\otimes M.$$

If $M\neq 0$, this would mean that the canonical map

$$(j_{x_1\neq x_2,x_1\neq x_0,x_2\neq x_0})_*\left((\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\boxtimes(-\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\right)\to\Delta_*\circ(j_{x\neq x_0})_*(\omega_{X-x_0}[-1]))$$

factors as

$$(j_{x_1\neq x_2,x_1\neq x_0,x_2\neq x_0})_*\left((\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\boxtimes(-\lambda(\sigma)\otimes\omega_{X-x_0}[-1])\right)\dashrightarrow \\ \dashrightarrow\Delta_*(\omega_X[-1])\to\Delta_*\circ(j_{x\neq x_0})_*(\omega_{X-x_0}[-1])),$$

which is false.

□[Theorem 7.4.4]

8. Towards the proof of Theorem 4.7.3: the key geometric lemma

In this subsection we supply a key geometric input for the proof of Theorem 4.7.3, incarnated by Lemma 8.2.7.

8.1. **Extending the loop group action.**

8.1.1. Consider the (corr-unital) factorization group ind-scheme $\mathfrak{L}(G)$ and its (co-unital) factorization group subscheme $\mathfrak{L}^+(G)$, so that their fibers at x_0 are $\mathfrak{L}(G)_{x_0}$ and $\mathfrak{L}^+(G)_{x_0}$, respectively and

$$\mathrm{Gr}_G\simeq\mathfrak{L}(G)/\mathfrak{L}^+(G).$$

We consider also the corresponding tautological (i.e., vacuum) factorization module spaces at x_0, denoted $\mathfrak{L}(G)^{\mathrm{fact}_{x_0}}$ and $\mathfrak{L}^+(G)^{\mathrm{fact}_{x_0}}$ over $\mathfrak{L}(G)$ and $\mathfrak{L}^+(G)$, respectively.

8.1.2. Unwinding the definitions, the !-pullback along the multiplication map of $\mathfrak{L}(G)$ defines a *lax-unital* factorization functor

$$\mathrm{D}(\mathfrak{L}(G))\to\mathrm{D}(\mathfrak{L}(G))\otimes\mathrm{D}(\mathfrak{L}(G)),$$

while the similar functor for $\mathfrak{L}^+(G)$ is a strictly unital factorization functor (see Sect. C.2.6 and Sect. C.5). In other words, we obtain (associative) coalgebra objects

$$\mathrm{D}(\mathfrak{L}(G))\in\mathbf{coAlg}(\mathbf{UntlFactCat}^{\text{lax-untl}}) \tag{8.1}$$

while

$$\mathrm{D}(\mathfrak{L}^+(G))\in\mathbf{coAlg}(\mathbf{UntlFactCat}). \tag{8.2}$$

Note that we have a homomorphism between these coalgebra objects

$$i^!:\mathrm{D}(\mathfrak{L}(G))\to\mathrm{D}(\mathfrak{L}^+(G))$$

given by !-pullback along the factorization map $i:\mathfrak{L}^+(G)\to\mathfrak{L}(G)$.

8.1.3. We define a unital factorization action of $\mathfrak{L}(G)$ on a unital factorization category $\mathbf{A}$ to be a comodule structure of $\mathbf{A}$ with respect to the coalgebra (8.1), such that the composition

$$\mathbf{A} \to \mathrm{D}(\mathfrak{L}(G)) \otimes \mathbf{A} \xrightarrow{i^! \otimes \mathrm{Id}_{\mathbf{A}}} \mathrm{D}(\mathfrak{L}^+(G)) \otimes \mathbf{A}$$

is *strictly* unital.

If $\mathfrak{L}(G)$ acts on a unital factorization category $\mathbf{A}$, and $\mathbf{D}$ is a unital factorization $\mathbf{A}$-module category at x_0, we can similarly define the notion of unital factorization actions of $\mathfrak{L}(G)^{\mathrm{fact}_{x_0}}$ on $\mathbf{D}$ that are compatible with the given $\mathfrak{L}(G)$-action on $\mathbf{A}$.

Note that in this case, the fiber $\mathbf{D}_{x_0}$ at x_0 acquires an action of $\mathfrak{L}(G)_{x_0}$ as a DG category.

8.1.4. *Example.* An example of a unital factorization category equipped with an action of $\mathfrak{L}(G)$ is $\mathbf{A} := \mathrm{D}(\mathrm{Gr}_G)$.

For this choice of $\mathbf{A}$, an example of a unital factorization module category $\mathbf{D}$ equipped with a compatible $\mathfrak{L}(G)^{\mathrm{fact}_{x_0}}$-action is $\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$.

Note that the resulting $\mathfrak{L}(G)_{x_0}$-action on

$$(\mathrm{D}(\mathfrak{L}(G)_{x_0})^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \simeq \mathfrak{L}(G)_{x_0}$$

is given by *left* translations.

8.1.5. For $\mathbf{A}$ as above, denote

$$\mathbf{A}^0 := \mathbf{A}^{\mathfrak{L}^+(G)},$$

which is defined as the cosimplicial limit of

$$\mathbf{A} \rightrightarrows \mathrm{D}(\mathfrak{L}^+(G)) \otimes \mathbf{A} \cdots$$

in **UntlFactCat**. Note that the forgetful functor

$$\iota : \mathbf{A}^0 \to \mathbf{A}$$

is a *strictly* unital factorization functor.

For $\mathbf{D}$ as in Sect. 8.1.3, consider

$$\mathbf{D}^0 := \mathbf{Res}_\iota(\mathbf{D}) \in \mathbf{A}^0\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}.$$

We claim:

Proposition-Construction 8.1.6. *Under the above circumstances, we have a natural action of $\mathfrak{L}(G)_{x_0}$ on $\mathbf{D}^0$* as an object of $\mathbf{A}^0$-$\mathbf{mod}^{\mathrm{fact}}_{x_0}$, *so that the action on*

$$\mathbf{D}^0_{x_0} \simeq \mathbf{D}_{x_0}$$

is the action from Sect. 8.1.3.

Proof. Let

$$\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\mathrm{Ran}_{x_0}} \subset \mathfrak{L}(G)_{\mathrm{Ran}_{x_0}} \tag{8.3}$$

be the group ind-scheme over Ran_{x_0} defined in [GLC2, Sect. C.10.10].

By construction, $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\mathrm{Ran}_{x_0}}$ comes equipped with a projection to the constant group ind-scheme with fiber $\mathfrak{L}(G)_{x_0}$.

By the construction of the operation of factorization restriction (see [GLC2, Secrt. B.9.28]), the action of $\mathfrak{L}(G)_{\mathrm{Ran}_{x_0}}$ on the crystal of categories underlying $\mathbf{D}$ restricts to an action of $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\mathrm{Ran}_{x_0}}$ on $\mathbf{Res}_\iota(\mathbf{D})$. Moreover, this action factors via

$$\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\mathrm{Ran}_{x_0}} \to \mathfrak{L}(G)_{x_0} \times \mathrm{Ran}_{x_0}.$$

□

Remark 8.1.7. Informally, Proposition 8.1.6 reads as follows: the action of $\mathfrak{L}(G)_{x_0}$ on $\mathbf{D}_{x_0}$ commutes with the factorization module structure with respect to $\mathbf{A}^0$.

Corollary 8.1.8. *For a factorization algebra $\mathcal{A} \in \mathbf{A}^0$, the category*

$$\iota(\mathcal{A})\text{-mod}^{\text{fact}}(\mathbf{D})_{x_0} \overset{\text{[GLC2, Lemma B.12.12]}}{\simeq} \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{D}^0)_{x_0}$$

carries an action of $\mathfrak{L}(G)_{x_0}$ compatible with the forgetful functor

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{D}^0)_{x_0} \to \mathbf{D}_{x_0}.$$

8.1.9. *Example.* Let us consider what may be a familiar situation in which Proposition 8.1.6 is applicable.

Consider the factorization category $\mathrm{KM}(\mathfrak{g}, \kappa)$ of Kac-Moody representations (at a given level). It carries an action of $\mathfrak{L}(G)_{\text{Ran}}$ (twisted by the level), defined as in [GLC2, Sect. B.14.22].

The corresponding category $(\mathrm{KM}(\mathfrak{g}, \kappa))^{\mathfrak{L}^+(G)}$ is by the definition the factorization version of the Kazhdan-Lusztig category, denoted $\mathrm{KL}(G, \kappa)$, see [GLC2, Sect. B.14.28].

Consider

$$\mathrm{KM}(\mathfrak{g}, \kappa)^{\text{fact}_{x_0}} \in \mathrm{KM}(\mathfrak{g}, \kappa)\text{-}\mathbf{mod}^{\text{fact}}_{x_0}.$$

Then Proposition 8.1.6 says that the action of $\mathfrak{L}(G)_{x_0}$ on $\mathrm{KM}(\mathfrak{g}, \kappa)_{x_0}$ commutes with fusion against objects of $\mathrm{KL}(G, \kappa)$.

In particular, Corollary 8.1.8 says that for a factorization algebra $\mathcal{A} \in \mathrm{KL}(G, \kappa)_{\text{Ran}}$, the category

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathrm{KM}(\mathfrak{g}, \kappa))_{x_0},$$

carries a natural action of $\mathfrak{L}(G)_{x_0}$.

8.1.10. Let $\mathbf{C}$ be an object of $\mathfrak{L}(G)_{x_0}$-**mod**. By the construction of the functor (2.1), the example in Sect. 8.1.4 shows that the resulting object

$$\mathbf{C}^{\text{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\text{fact}}_{x_0}$$

carries a compatible action of $\mathfrak{L}(G)^{\text{fact}_{x_0}}$.

Applying Corollary 8.1.8 to

$$\omega_{\mathrm{Gr}_G} \in \mathrm{Alg}^{\text{fact}}(\mathrm{D}(\mathrm{Gr}_G)),$$

viewed as a $\mathfrak{L}(G)$- (and hence $\mathfrak{L}^+(G)$)-equivariant object, we obtain that the category

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\text{fact}}(\mathbf{C}^{\text{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)})_{x_0}$$

carries an action of $\mathfrak{L}(G)_{x_0}$, which commutes with the forgetful functor

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\text{fact}}(\mathbf{C}^{\text{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)})_{x_0} \to (\mathbf{C}^{\text{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)})_{x_0} \simeq \mathbf{C}.$$

8.2. The key geometric lemma.

8.2.1. We continue to be in the context of Sect. 8.1.3. Consider the restrictions

$$\mathbf{A}_X := \mathbf{A}|_X \text{ and } \mathbf{D}_{X \times x_0} := \mathbf{D}|_{X \times x_0},$$

along $X \to \text{Ran}$ and $X \times x_0 \to \text{Ran}_{x_0}$, respctively.

Denote by

$$(X - x_0) \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} \{x_0\}$$

the corresponding morphisms, and by

$$\mathbf{A}_{X - x_0} \otimes \mathbf{D}_{x_0} \overset{j_*}{\hookrightarrow} \mathbf{D}_{X \times x_0} \overset{i_*}{\hookleftarrow} \mathbf{D}_{x_0}$$

the corresponding functors.

8.2.2. For a triple of objects $\mathbf{a} \in \mathbf{A}_{X-x_0}$ and $\mathbf{d}, \mathbf{d}' \in \mathbf{D}_{x_0}$, consider the space

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\mathbf{a} \boxtimes \mathbf{d}), i_*(\mathbf{d}')).$$

For future reference, we note that if $\mathbf{a} \in \mathbf{A}^0_{X-x_0}$, we have a canonical identification

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\iota(\mathbf{a}) \boxtimes \mathbf{d}), i_*(\mathbf{d}')) \simeq \mathcal{H}om_{\mathbf{D}^0_{X \times x_0}}(j_*(\mathbf{a} \boxtimes \mathbf{d}), i_*(\mathbf{d}')), \tag{8.4}$$

see Sect. 8.1.5 for the notation. This follows from the construction of the operation of factorization restriction (see [GLC2, Secrt. B.9.28]).

8.2.3. More generally, for an affine test-scheme S and an S-point g of $\mathfrak{L}(G)_{x_0}$, consider the spaces

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0} \otimes \mathrm{D}^!(S)}\left(\mathrm{pr}^!(j_*(\mathbf{a} \boxtimes \mathbf{d})), \mathrm{pr}^!(i_*(\mathbf{d}'))\right) \tag{8.5}$$

and

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0} \otimes \mathrm{D}^!(S)}\left(\mathrm{pr}^!(j_*(\mathbf{a} \boxtimes \mathbf{d})), \mathrm{act}^!(i_*(\mathbf{d}'))\right), \tag{8.6}$$

where $\mathrm{pr}^!$ and $\mathrm{act}^!$ are the two functors

$$\mathbf{D}_{X \times x_0} \to \mathbf{D}_{X \times x_0} \otimes \mathrm{D}^!(S).$$

8.2.4. In what follows, we will abuse the notation slightly and omit S. So we will simply write $j_*(\mathbf{a} \boxtimes \mathbf{d})$ instead of $\mathrm{pr}^!(j_*(\mathbf{a} \boxtimes \mathbf{d}))$, and we will write

$$g \cdot i_*(\mathbf{d}') := \mathrm{act}^!(i_*(\mathbf{d}')).$$

Note also that

$$g \cdot i_*(\mathbf{d}') \simeq i_*(g \cdot \mathbf{d}'),$$

where in the right-hand side $g \cdot -$ refers to the $\mathfrak{L}(G)_{x_0}$-action on $\mathbf{D}_{x_0}$.

Thus, instead of (8.5) we will write

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\mathbf{a} \boxtimes \mathbf{d}), i_*(\mathbf{d}')), \tag{8.7}$$

and instead of (8.6) we will write

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\mathbf{a} \boxtimes \mathbf{d}), i_*(g \cdot \mathbf{d}')). \tag{8.8}$$

8.2.5. Let us come back for a moment to the statement of Proposition 8.1.6. It implies that for $\mathbf{a}, \mathbf{d}, \mathbf{d}'$ and g as above, we have a canonical isomorphism

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\mathbf{a} \boxtimes \mathbf{d}), i_*(\mathbf{d}')) \simeq \mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\mathbf{a} \boxtimes g \cdot \mathbf{d}), i_*(g \cdot \mathbf{d}')). \tag{8.9}$$

Hence, by Proposition 8.1.6, the expression in (8.8) is canonically isomorphic to

$$\mathcal{H}om_{\mathbf{D}_{X \times x_0}}(j_*(\mathbf{a} \boxtimes (g^{-1} \cdot \mathbf{d})), i_*(\mathbf{d}')). \tag{8.10}$$

8.2.6. The key assertion behind the proof of Theorem 4.7.3 is the following:

Main Lemma 8.2.7. *Suppose that the object* $\mathbf{a}$ *belongs to* $(\mathbf{A}_{X-x_0})^{\mathfrak{L}(G)_{X-x_0}}$.

(a) *If* g *is a point of*[17] $\mathfrak{L}(N)_{x_0}$, *there exists a canonical isomorphism between the spaces* (8.7) *and* (8.8).

[17] In the formula below, N is the maximal unipotent subgroup of G.

(b) *If* $\mathbf{d} \in (\mathbf{D}_{x_0})^{\mathfrak{L}^+(G)_{x_0}}$*, then for any* $g \in \mathfrak{L}(G)_{x_0}$*, there exists a natural isomorphism, to be denoted* $\alpha_{g,\mathbf{d},\mathbf{d}'}$*, between the spaces (8.7) and (8.8), such that for* $g_1 \in \mathfrak{L}^+(G)_{x_0}$ *the diagram*

$$\begin{tikzcd}
\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathbf{d}')) \arrow[r, "\alpha_{g_1\cdot g,\mathbf{d},\mathbf{d}'}"] \arrow[dd, "\mathbf{d}\ \textit{is spherical}"', "\sim"] & \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*((g_1\cdot g)\cdot\mathbf{d}')) \arrow[d, "="'] \\
& \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(g_1\cdot(g\cdot\mathbf{d}'))) \arrow[d, "\sim"', "(8.9)"] \\
\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes g_1^{-1}\cdot\mathbf{d}), i_*(\mathbf{d}')) \arrow[r, "\alpha_{g,g_1^{-1}\cdot\mathbf{d},\mathbf{d}'}"] & \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes g_1^{-1}\cdot\mathbf{d}), i_*(g\cdot\mathbf{d}'))
\end{tikzcd}$$

commutes.

In what follows, adopting the conventions of Remark Sect. 8.2.4, we will write the sought-for isomorphism in Main Lemma 8.2.7 as

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathbf{d}')) \overset{\alpha_{g,\mathbf{d},\mathbf{d}'}}{\simeq} \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(g\cdot\mathbf{d}')). \tag{8.11}$$

Remark 8.2.8. Let us emphasize the difference between the assertion of Main Lemma 8.2.7 and that of Proposition 8.1.6:

In (8.9), both $\mathbf{d}$ and $\mathbf{d}'$ are moved by g (while $\mathbf{a}$ is only required to be $\mathfrak{L}^+(G)$-equivariant).

By contrast, in Main Lemma 8.2.7 only $\mathbf{d}$ (or $\mathbf{d}'$) is moved by g, but $\mathbf{a}$ is required to be $\mathfrak{L}(G)$-equivariant.

Remark 8.2.9. One can show that when G is semi-simple, one can get rid of the condition that $\mathbf{d}$ be $\mathfrak{L}^+(G)_{x_0}$-equivariant. However, when G has a non-trivial connected center, the $\mathfrak{L}^+(G)_{x_0}$-equivariance condition on $\mathbf{d}$ is necessary.

8.2.10. Here is how Main Lemma 8.2.7 will be used:

Corollary 8.2.11. *Let* $\mathbf{a}$*,* $\mathbf{d}$ *and* $\mathbf{d}'$ *be as in Lemma 8.2.7.*

(a) *We have an isomorphism*

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}\Big(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathbf{d}')\Big) \simeq \mathcal{H}om_{\mathbf{D}_{X\times x_0}}\left(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathrm{Av}_*^{\mathfrak{L}^+(N)_{x_0}}(\mathbf{d}'))\right).$$

(b) *For G reductive, under the additional assumptions of Lemma 8.2.7(b), we have*

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}\Big(j_*\left(\mathbf{a}\boxtimes\mathbf{d}\right)\otimes \mathrm{C}.(\mathrm{Gr}_{G,x_{x_0}}), i_*(\mathbf{d}')\Big) \simeq$$
$$\simeq \mathcal{H}om_{\mathbf{D}_{X\times x_0}}\left(j_*\Big(\mathbf{a}\boxtimes \mathrm{Av}_!^{\mathfrak{L}^+(G)_{x_0}\to\mathfrak{L}(G)_{x_0}}(\mathbf{d})\Big), i_*(\mathbf{d}')\right),$$

where $\mathrm{Av}_!^{\mathfrak{L}^+(G)_{x_0}\to\mathfrak{L}(G)_{x_0}}$ *is as in Sect. 4.6.11.*

Proof. We prove point (a), as point (b) is similar. By adjunction,

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}\left(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathrm{Av}_*^{\mathfrak{L}^+(N)_{x_0}}(\mathbf{d}'))\right) \simeq \tag{8.12}$$
$$\simeq \mathcal{H}om_{\mathbf{D}_{X\times x_0}\otimes \mathrm{D}^!(\mathfrak{L}^+(N))}\left(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(g_{\mathrm{univ}}\cdot\mathbf{d}')\right),$$

where g_{univ} is the tautological $\mathfrak{L}^+(N)$-point of $\mathfrak{L}^+(N)$.

Now, Main Lemma 8.2.7(a) implies that the left-hand side in (8.12) identifies with

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}\otimes \mathrm{D}^!(\mathfrak{L}^+(N))}\left(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathbf{d}')\right). \tag{8.13}$$

Both objects

$$j_*(\mathbf{a}\boxtimes\mathbf{d}) \text{ and } i_*(\mathbf{d}') \in \mathbf{D}_{X\times x_0}\otimes \mathrm{D}^!(\mathfrak{L}^+(N))$$

are the same-named objects in $\mathbf{D}_{X\times x_0}$, tensored with $\omega_{\mathfrak{L}^+(N)} \in \mathrm{D}^!(\mathfrak{L}^+(N))$. Since the latter object is compact, the expression in (8.13) is isomorphic to

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}\left(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathbf{d}')\right)\otimes \mathrm{End}_{\mathrm{D}^!(\mathfrak{L}^+(N))}(\omega_{\mathfrak{L}^+(N)}) \simeq \mathcal{H}om_{\mathbf{D}_{X\times x_0}}\left(j_*(\mathbf{a}\boxtimes\mathbf{d}), i_*(\mathbf{d}')\right)\otimes \mathrm{C}^\cdot(\mathfrak{L}^+(N)).$$

Now, since N is unipotent, $\mathfrak{L}^+(N))$ is contractible, and hence $\mathrm{C}^\cdot(\mathfrak{L}^+(N)) \simeq k$.

□

8.3. **Proof of Main Lemma 8.2.7: reduction to a particular congruence level.**

In this subsection we will show that we can assume that both $\mathbf{d}$ and $\mathbf{d}'$ are invariant with respect to some congruence subgroup $K_n \subset \mathfrak{L}^+(G)_{x_0}$.

8.3.1. First, as the category $\mathbf{D}_{x_0}$ is acted on by $\mathfrak{L}(G)_{x_0} \supset \mathfrak{L}^+(G)_{x_0}$, every object can be written as a colimit of objects invariant with respect to congruence subgroups.

Thus, we can assume that $\mathbf{d}$ is invariant with respect to some K_n, $n \geq 1$.

8.3.2. We now use the fact that $\mathbf{a}$ is $\mathfrak{L}^+(G)_{X-x_0}$-equivariant, and we apply isomorphism (8.4):

By Proposition 8.1.6, the category $\mathbf{D}^0_{X\times x_0}$ is acted on by $\mathfrak{L}(G)_{x_0}$, and in particular by K_n. By assumption, with respect to this action, the object

$$j_*(\mathbf{a}\boxtimes\mathbf{d}) \in \mathbf{D}^0_{X\times x_0}$$

is K_n-invariant.

Hence, the left-hand side in (8.11) remains unchanged if we replace $\mathbf{d}'$ by $\mathrm{Av}^{K_n}_*(\mathbf{d}')$.

Similarly, the right-hand side in (8.11) remains unchanged if we replace $g\cdot\mathbf{d}'$ by $\mathrm{Av}^{K_n}_*(g\cdot\mathbf{d}')$.

8.3.3. Now, for a given point g of $\mathfrak{L}(G)_{x_0}$, let $m \gg 1$ be large enough so that

$$\mathrm{Ad}_g(K_m) \subset K_n.$$

In this case,

$$\mathrm{Av}^{K_n}_*(g\cdot\mathbf{d}') \simeq \mathrm{Av}^{K_n}_*(g\cdot \mathrm{Av}^{K_m}_*(\mathbf{d}')).$$

Therefore, the right-hand side in (8.11) remains unchanged if we replace $\mathbf{d}'$ by $\mathrm{Av}^{K_m}_*(\mathbf{d}')$.

8.3.4. Thus, we can assume that $\mathbf{d}'$ is also invariant with respect to some congruence subgroup.

8.4. **Proof of Main Lemma: reduction of point (a) to point (b).**

8.4.1. Given an S-point g of $\mathfrak{L}(N)_{x_0}$, we need to establish isomorphism (8.11).

By the previous subsection, we can assume that both $\mathbf{d}$ and $\mathbf{d}'$ are invariant with respect to the subgroup $K_n(N) := K_n \cap \mathfrak{L}(N)_{x_0}$ for some $n \geq 1$.

8.4.2. Let λ be a dominant coweight so that

$$\mathrm{Ad}_{t^\lambda}(\mathfrak{L}^+(N)_{x_0}) \subset K_n(N).$$

I.e.,

$$\mathfrak{L}^+(N)_{x_0} \subset \mathrm{Ad}_{t^{-\lambda}}(K_n(N)).$$

Using the $\mathfrak{L}(G)_{x_0}$-action on $\mathbf{D}^0_{X\times x_0}$, we can identify the two sides of (8.11) with ones, where we replace $\mathbf{d}$ by $t^{-\lambda}\cdot\mathbf{d}$ and $\mathbf{d}'$ by $t^{-\lambda}\cdot\mathbf{d}'$, and g by $\mathrm{Ad}_{t^{-\lambda}}(g)$.

However, now $t^{-\lambda}\cdot\mathbf{d}$ and $t^{-\lambda}\cdot\mathbf{d}'$ are $\mathrm{Ad}_{t^{-\lambda}}(K_n(N))$-equivariant, and hence $\mathfrak{L}^+(N)_{x_0}$-equivariant, by construction.

8.4.3. This reduces the assertion of point (a) of Lemma 8.2.7 to point (b) (for G replaced by N).

8.5. **Proof of Main Lemma 8.2.7(b): the mechanism.** In this subsection we will explain the main geometric idea behind the proof of Main Lemma 8.2.7(b).

We will describe a paradigm in which one obtains an isomorphism (8.11).

8.5.1. Assume that $\mathbf{d}$ is invariant with respect to a subgroup $K \subset \mathfrak{L}(G)_{x_0}$. Note that using the averaging functor as in Sect. 8.3.2, we can assume that $\mathbf{d}'$ is also K-invariant[18].

Let g be an element of $\mathfrak{L}(G)_{x_0}$, i.e., a map $\mathcal{D}_{x_0}^\times \to G$. Let g' be a map

$$g' : X \times X - \Delta \to G \tag{8.14}$$

with the following properties:

- For every $x \neq x_0$, the restriction of map
$$g'_x := g'|_{(X-x)\times x} : X - x \to G$$
along $\mathcal{D}_{x_0} \hookrightarrow X - x$ lies in K;
- The restriction of the map
$$g'_{x_0} := g'|_{(X-x_0)\times x_0} : X - x_0 \to G$$
along $\mathcal{D}_{x_0}^\times \hookrightarrow X - x_0$ equals g modulo K.

We claim that a choice of a map g' as above gives rise to an isomorphism (8.11).

8.5.2. For any map as in (8.14), its Laurent expansion in the first coordinate around the divisor

$$\Delta \cup (x_0 \times X)$$

defines a section of

$$\mathfrak{L}(G)_{X\times x_0} := \mathfrak{L}(G)_{\mathrm{Ran}_{x_0}}|_{X\times x_0}$$

over $X \simeq X \times x_0$, and hence acts by a self-equivalence on $\mathbf{D}_{X\times x_0}$.

We claim that the action of the above element g' is such that

$$g' \cdot (j_*(\mathbf{a} \boxtimes \mathbf{d})) \simeq j_*(\mathbf{a} \boxtimes \mathbf{d}) \tag{8.15}$$

and

$$g' \cdot (i_*(\mathbf{d}')) \simeq i_*(g \cdot \mathbf{d}). \tag{8.16}$$

This would give rise to an isomorphism (8.11).

8.5.3. To prove (8.15), we need to establish the corresponding isomorphism over $X - x_0$.

Note that a map (8.14) acts on an object

$$\mathbf{a} \boxtimes \mathbf{d} \in \mathbf{A}_{X-x_0} \otimes \mathbf{D}_{x_0}$$

via:

- The Laurent expansion of g' around the diagonal divisor and the action of the resulting element of $\mathfrak{L}(G)_{X-x_0}$ on $\mathbf{a}$;
- The Laurent expansion of g' around $x_0 \times X$ and the resulting action of $\mathfrak{L}(G)_{x_0}$ on $\mathbf{d}$.

Now, the condition that $\mathbf{a}$ is $\mathfrak{L}(G)_{X-x_0}$-equivariant implies that

$$g' \cdot \mathbf{a} \simeq \mathbf{a}.$$

And the condition that the $g'|_{\mathcal{D}_{x_0}^\times \times x} \in K$ combined with the assumption that $\mathbf{d}$ in K-invariant implies that $g' \cdot \mathbf{d} \simeq \mathbf{d}$.

8.5.4. To prove (8.16), we need to show that the action of

$$g'|_{\mathcal{D}_{x_0}^\times \times x_0} \in \mathfrak{L}(G)$$

on $\mathbf{d}'$ produces the same result as $g \cdot \mathbf{d}'$.

However, this follows from the assumption that

$$g'|_{\mathcal{D}_{x_0}^\times \times x_0} = g \bmod K$$

and the fact that $\mathbf{d}'$ is K-invariant.

[18]In practice, for the proof of Main Lemma 8.2.7(b), we will take K to be $\mathfrak{L}(G)_{x_0}$ itself.

8.5.5. Assume now that g' is in fact a map

$$X \times X \to G,$$

i.e., that it is regular around the diagonal.

Unwinding the construction, we obtain that the resulting isomorphism (8.11) equals

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes \mathbf{d}), i_*(\mathbf{d}')) \overset{(8.9)}{\simeq}$$
$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes (g\cdot\mathbf{d})), i_*(g\cdot\mathbf{d}')) \overset{\mathbf{d}\text{ is spherical}}{\simeq} \simeq \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes \mathbf{d}), i_*(g\cdot\mathbf{d}')).$$

8.5.6. Note also that the isomorphism (8.11) is naturally compatible with the group structure on $\mathrm{Maps}(X\times X-\Delta, G)$, i.e., for a pair of points $g_1', g_2' \in \mathrm{Maps}(X\times X-\Delta, G)$, the diagram

$$\begin{tikzcd}
\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes \mathbf{d}), i_*(\mathbf{d}')) \arrow[r, "\alpha_{g_2,\mathbf{d},\mathbf{d}'}"] \arrow[d, "\alpha_{g_1\cdot g_2,\mathbf{d},\mathbf{d}'}"'] & \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes \mathbf{d}), i_*(g_2\cdot\mathbf{d}')) \arrow[d, "\alpha_{g_1,g_2\cdot\mathbf{d},\mathbf{d}'}"] \\
\mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes \mathbf{d}), i_*((g_1\cdot g_2)\cdot\mathbf{d}')) \arrow[r, "="] & \mathcal{H}om_{\mathbf{D}_{X\times x_0}}(j_*(\mathbf{a}\boxtimes \mathbf{d}), i_*(g_1\cdot(g_2\cdot\mathbf{d}')))
\end{tikzcd}$$

commutes.

8.6. **Proof of Main Lemma 8.2.7(b): approximating the loop group.** The assertion of the Main Lemma 8.2.7(b) is étale-local, so in what follows we will assume that the pair (X, x_0) is $(\mathbb{A}^1, 0)$. We will implement the mechanism from Sect. 8.5 or $K = \mathfrak{L}^+(G)_{x_0}$.

8.6.1. We will construct a group ind-scheme (ind-affine, of ind-finite type) Γ, equipped with a map

$$\phi : \Gamma \to \mathrm{Maps}(X\times X-\Delta, G),$$

such that the composite map

$$\Gamma \overset{\phi}{\to} \mathrm{Maps}(X\times X-\Delta, G) \to \mathfrak{L}(G)_{x_0} \tag{8.17}$$

satisfies:

- For $\Gamma^+ := \Gamma \underset{\mathfrak{L}(G)_{x_0}}{\times} \mathfrak{L}^+(G)_{x_0}$ the resulting map

$$\Gamma/\Gamma^+ \to \mathfrak{L}(G)_{x_0}/\mathfrak{L}^+(G)_{x_0} = \mathrm{Gr}_{G,x_0} \tag{8.18}$$

  is an isomorphism.

Given such a map, the construction in Sect. 8.5 will imply the assertion of Main Lemma 8.2.7(b).

8.6.2. We take

$$\Gamma := G(t,t^{-1}) := \mathbf{Maps}(\mathbb{A}^1-0, G).$$

We let ϕ be defined by precomposition with the projection

$$X\times X-\Delta = \mathbb{A}^1\times\mathbb{A}^1-\Delta \overset{t_1,t_2\mapsto t_1-t_2}{\longrightarrow} \mathbb{A}^1-0.$$

8.6.3. Note that the map (8.17) is the standard map corresponding to $k[t,t^{-1}] \hookrightarrow k(\!(t)\!)$. Hence,

$$\Gamma^+ = \mathbf{Maps}(\mathbb{A}^1, G) = G[t].$$

Hence, the map (8.18) is the map

$$G(t,t^{-1})/G[t] \to G(\!(t)\!)/G[\![t]\!],$$

which is known to be an isomorphism[19].

9. Proof of Theorem 4.7.3

9.1. **What to we need to show?**

[19]This is a consequence of the Beauville-Laszlo theorem, combined with the fact that every G-bundle on $\mathbb{P}^1$ can be trivialized on $\mathbb{A}^1$, étale-locally with respect to the scheme of parameters.

9.1.1. Recall that by Corollary 8.1.8, the category

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \tag{9.1}$$

carries an action of $\mathfrak{L}(G)_{x_0}$.

Since the forgetful functor

$$\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \to \mathbf{C}$$

is conservative and compatible with the $\mathfrak{L}(G)_{x_0}$-actions, by Lemma 4.6.4, it suffices to show that the functor

$$\text{alm-inv}_{\mathfrak{L}(G)_{x_0}}\left(\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\right) \to \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \tag{9.2}$$

is an equivalence.

9.1.2. The strategy of the proof will be as follows:

(1) We will show that the category (9.1) is *spherically generated* (see Sect. 4.5.9 for what this means);

(2) We will show that the functor $\mathrm{Av}_!^{\mathfrak{L}^+(G)_{x_0}\to\mathfrak{L}(G)_{x_0}}$ on the spherical subcategory of (9.1) (see Sect. 4.6.11) is conservative. By Proposition 4.6.12, this will imply that (9.2) is an equality.

9.2. **Functoriality revisted.** In this subsection we briefly revisit the set-up of Sect. 2.2.

9.2.1. Note that we have a canonical map of factorization algebras in $\mathrm{D}(\mathrm{Gr}_G)$

$$\omega_\phi : (\mathrm{Gr}_\phi)_*(\omega_{\mathrm{Gr}_{G'}}) \to \omega_{\mathrm{Gr}_G}.$$

Hence, we obtain a functor

$$\begin{aligned}\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} &\overset{\mathrm{Res}_{\omega_\phi}}{\longrightarrow} (\mathrm{Gr}_\phi)_*(\omega_{\mathrm{Gr}_{G'}})\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \simeq \\ &\simeq \omega_{\mathrm{Gr}_{G'}}\text{-mod}^{\mathrm{fact}}(\mathbf{Res}_{\mathrm{Gr}_\phi}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}))_{x_0} \simeq \omega_{\mathrm{Gr}_{G'}}\text{-mod}^{\mathrm{fact}}((\mathbf{C}')^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_{G'})})_{x_0}.\end{aligned} \tag{9.3}$$

Note that the source (resp., target) of the above functor carries a natural action of $\mathfrak{L}(G)_{x_0}$ (resp., $\mathfrak{L}(G')_{x_0}$), see Corollary 8.1.8.

9.2.2. Unwinding the constructions, we obtain:

Lemma 9.2.3. *The functor* (9.3) *intertwines the $\mathfrak{L}(G')_{x_0}$-action on the target with the $\mathfrak{L}(G')_{x_0}$-action on the source obtained from the $\mathfrak{L}(G)_{x_0}$-action by precomposing with ϕ.*

9.3. **Proof of spherical generation.**

9.3.1. Replacing $\mathbf{C}$ by the kernel of the right adjoint to (4.25), we can assume that $\mathbf{C}^{\mathfrak{L}^+(G)_{x_0}} = 0$.

We will show that the category (9.1) is zero in this case.

9.3.2. By lemmas 2.2.4 and 9.2.3, and using the fact that Theorem 3.1.7 has already been proved for the group T, we know that the action of $\mathfrak{L}(T)_{x_0}$ (and in particular of $\mathfrak{L}^+(T)_{x_0}$) has been canonically trivialized.

In particular, the inclusion

$$\text{alm-inv}_{\mathfrak{L}^+(T)_{x_0}}\left(\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\right) \hookrightarrow \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}$$

is an equality.

9.3.3. We claim that the inclusion

$$\text{alm-inv}_{\mathfrak{L}^+(N)_{x_0}}\left(\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\right) \hookrightarrow \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}$$

is also an equality.

To prove this, it suffices to show that the functor $\mathrm{Av}_*^{\mathfrak{L}^+(N)_{x_0}}$ is conservative on (9.1). Let

$$\mathbf{c} \in \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}$$

be an object in the kernel of $\mathrm{Av}_*^{\mathfrak{L}^+(N)_{x_0}}$. Then by Corollary 8.2.11(a), we obtain that

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}\Big(j_*(\omega_{\mathrm{Gr}_G}\boxtimes \mathbf{c}_{x_0}), i_*(\mathbf{c}_{x_0})\Big) = 0, \tag{9.4}$$

where:

- $\mathbf{D} := \mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)}$;
- $\mathbf{c}_{x_0} = \mathrm{oblv}_{\omega_{\mathrm{Gr}_G}}(\mathbf{c})$.

9.3.4. We claim that (9.4) implies that $\mathbf{c}_{x_0} = 0$. Indeed, the structure of factorization ω_{Gr_G}-module on $\mathbf{c}$ gives rise to a map

$$j_*(\omega_{\mathrm{Gr}_G}\boxtimes \mathbf{c}_{x_0}) \to i_*(\mathbf{c}_{x_0})[1],$$

and we claim that if this map is zero, then $\mathbf{c}_{x_0}$ is zero.

Indeed, this follows from the fact that $\mathbf{c}$ was a *unital* factorization module, and hence the composition

$$j_*(\mathbf{1}_{\mathrm{D}(\mathrm{Gr}_G)}\boxtimes \mathbf{c}_{x_0}) \to j_*(\omega_{\mathrm{Gr}_G}\boxtimes \mathbf{c}_{x_0}) \to i_*(\mathbf{c}_{x_0})[1]$$

is the canonical morphism.

9.3.5. We now claim that the inclusion

$$\mathrm{inv}_{K_1}\left(\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\right) \overset{\sim}{\to} \text{alm-inv}_{K_1}\left(\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\right) \hookrightarrow$$
$$\hookrightarrow \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}$$

is also an equality.

Indeed, this follows from the triangular decomposition

$$K_1 = (K_1\cap \mathfrak{L}^+(N)_{x_0})\cdot (K_1\cap \mathfrak{L}^+(T)_{x_0})\cdot (K_1\cap \mathfrak{L}^+(N^-)_{x_0}),$$

and the fact that the equality holds for each of the factors.

9.3.6. Thus, the action of $\mathfrak{L}^+(G)_{x_0}$ on

$$\mathbf{C}' := \omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}$$

factors through

$$\mathfrak{L}^+(G)_{x_0} \twoheadrightarrow G,$$

while we have:

- $(\mathbf{C}')^N = \mathbf{C}'$
- The action of T on $\mathbf{C}'$ is trivialized;
- $(\mathbf{C}')^G = 0$.

In Sect. 9.4 below we will show that any $\mathbf{C}'$ with the above properties is zero.

Remark 9.3.7. Note that if we weakened the second assumption to $\text{alm-inv}_T(\mathbf{C}') = \mathbf{C}'$, the assertion that $\mathbf{C}' = 0$ would be false: a counterexample is provided by Sect. 4.2.6.

Remark 9.3.8. As we shall see, the first condition (i.e., that $(\mathbf{C}')^N = \mathbf{C}'$) is actually superfluous. So, in fact we are proving the following:

Proposition 9.3.9. *Let us be a given a categorical representation of G, such that its restriction to T can be trivialized. Then the initial categorical representation is almost trivial.*

9.4. **The toric localization argument.** To simplify the notation, for the duration of this subsection we will perform the notational change,

$$\mathbf{C}' \rightsquigarrow \mathbf{C}.$$

9.4.1. Let $\mathbf{D}$ be a category equipped with an action of T. Then the category $\mathbf{D}^T$ is acted on by Vect^T.

In particular, the (commutative algebra)

$$\mathrm{C}^{\cdot}(BT) \simeq \mathrm{Sym}(\mathfrak{t}^*[-2])$$

maps to the Bernstein center of $\mathbf{D}^T$.

9.4.2. Let $\mathbf{D}^T_0 \subset \mathbf{D}$ be the *non-cocomplete* subcategory consisting of objects, on which some non-zero graded ideal in $\mathrm{Sym}(\mathfrak{t}^*[-2])$ acts trivially.

Denote by $\widetilde{\mathbf{D}}$ the quotient

$$\mathbf{D}^T/\mathbf{D}^T_0,$$

taken in the world of non-cocomplete categories.

9.4.3. We consider $\mathbf{C}^N$ and $\mathbf{C}$ itself as acted on by T (note that by [BZGO, Remark 1.2], if $\mathbf{C} \neq 0$, then $\mathbf{C}^N \neq 0$). Let $\mathcal{F}$ be an object of $\mathrm{D}(T\backslash G/B)$. Convolution with $\mathcal{F}$ can be thought of as a functor

$$\mathbf{C}^B = (\mathbf{C}^N)^T \to \mathbf{C}^T.$$

The following assertion is an abstract version of the toric localization principle:

Lemma 9.4.4. *For $\mathbf{c} \in \mathbf{C}^B$, the image of $\mathcal{F} \star \mathbf{c} \in \mathbf{C}^T$ under*

$$\mathbf{C}^T \to \widetilde{\mathbf{C}}$$

is canonically isomorphic to the image of

$$\underset{w \in W}{\oplus} (w \cdot \mathbf{c}) \otimes \mathcal{F}_w,$$

where:

- *W denotes the Weyl group;*
- *For $w \in W$ we denote by the same symbol the corresponding T-fixed point in G/B;*
- *$w \cdot \mathbf{c} := \delta_w \star \mathbf{c}$, where δ_w is viewed as an object of $\mathrm{D}(G/B)^T = \mathrm{D}(T\backslash G/B)$.*
- *$\mathcal{F}_w$ is the !-fiber of $\mathcal{F}$ at $w \in G/B$.*

The proof is given in Sect. 9.5 below.

Corollary 9.4.5. *The image of $\mathbf{c}$ under $\mathbf{C}^T \to \widetilde{\mathbf{C}}$ is a retract of the image of*

$$\underline{k}_{G/B} \star \mathbf{c}.$$

9.4.6. Note now that the assumption that $\mathbf{C}^G = 0$ implies that the functor

$$\underline{k}_{G/B} \star (-) : \mathbf{C}^B \to \mathbf{C}^T$$

vanishes.

Hence, from Corollary 9.4.5 we obtain that this assumption forces that the inclusion

$$(\mathbf{C}^N)^T_0 \subset (\mathbf{C}^N)^T = \mathbf{C}^B$$

is an equality.

9.4.7. We now use the assumption that the T-action on $\mathbf{C}$ is trivialized. Hence so is the T-action on $\mathbf{C}^N$. This assumption implies that
$$(\mathbf{C}^N)^T \simeq \mathbf{C}^N \otimes \mathrm{Vect}^T,$$
where the action of Vect^T is via the second factor.

Now, for $0 \neq \mathbf{c} \in \mathbf{C}^N$, the object
$$\mathbf{c} \otimes k \in \mathbf{C}^N \otimes \mathrm{Vect}^T$$
does *not* belong to $(\mathbf{C}^N)^T_0$.

□

9.5. **Proof of Lemma 9.4.4.**

9.5.1. It is enough to show that the image of $\mathcal{F} \in \mathrm{D}(T\backslash G/B) \simeq \mathrm{D}(G/B)^T$ under
$$\mathrm{D}(G/B)^T \to \widetilde{\mathrm{D}(G/B)}$$
is isomorphic to the image of
$$\underset{w}{\oplus}\, (\delta_w \otimes \mathcal{F}_w).$$

We have a canonical map
$$\underset{w}{\oplus}\, (\delta_w \otimes \mathcal{F}_w) \to \mathcal{F},$$
whose cone has the property that its !-fibers at the points w are zero.

Hence, it is enough to show that if $\mathcal{F}$ has vanishing !-fibers at all the points w, then its projection to $\widetilde{\mathrm{D}(G/B)}$ vanishes.

9.5.2. Using Cousin decomposition, we can assume that $\mathcal{F}$ is the *-extension from an object on a single Schuber cell $(G/B)_w \in G/B$. Furthermore, by assumption, it is the *-extension from the open subscheme $(G/B)_w - w$. Hence, it is enough to show that the inclusion
$$\mathrm{D}((G/B)_w - w)^T_0 \subseteq \mathrm{D}((G/B)_w - w)^T$$
is an equality.

We claim that this is the case for any scheme Y on which a torus T acts without fixed points.

9.5.3. First, in order to show that the inclusion
$$\mathrm{D}(Y)^T_0 \subseteq \mathrm{D}(Y)^T$$
is an equality, we can replace the action of the original T by the action of any $\mathbb{G}_m$ that maps to T.

The fact that T has no fixed points on Y implies that we can find $\mathbb{G}_m \to T$ that acts on Y with finite stabilizers. We will show that for this copy of $\mathbb{G}_m$, the action of
$$\mathrm{C}^\cdot(B\mathbb{G}_m) \simeq k[\eta], \quad \deg(\eta) = 2$$
on $\mathrm{D}(Y)^{\mathbb{G}_m}$ factors though a non-zero ideal.

9.5.4. The action of $\mathrm{C}^\cdot(B\mathbb{G}_m)$ on
$$\mathrm{D}(Y)^{\mathbb{G}_m} \simeq \mathrm{D}(Y/\mathbb{G}_m)$$
factors via a homomorphism
$$\mathrm{C}^\cdot(B\mathbb{G}_m) \to \mathrm{C}^\cdot(Y/\mathbb{G}_m). \tag{9.5}$$

Hence, it is enough to show that this homomorphism factors though a non-zero ideal.

9.5.5. The assumption on the $\mathbb{G}_m$-action on Y implies that $Y/\mathbb{G}_m$ is a Deligne-Mumford stack. Hence, $\mathrm{C}^\cdot(Y/\mathbb{G}_m)$ is finite-dimensional.

Hence, the homomorphism (9.5) has a non-trivial kernel.

□

9.6. **Proof of the conservativity of** $\mathrm{Av}_!^{\mathfrak{L}^+(G)_{x_0} \to \mathfrak{L}(G)_{x_0}}$**.**

9.6.1. Let

$$\mathbf{c} \in \left(\omega_{\mathrm{Gr}_G}\text{-mod}^{\mathrm{fact}}(\mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}\right)^{\mathfrak{L}^+(G)_{x_0}}$$

be an object in the kernel of $\mathrm{Av}_!^{\mathfrak{L}^+(G)_{x_0}\to\mathfrak{L}(G)_{x_0}}$.

Then by Corollary 8.2.11(b), we obtain that

$$\mathcal{H}om_{\mathbf{D}_{X\times x_0}}\Big(j_*(\omega_{\mathrm{Gr}_G}\boxtimes \mathbf{c}_{x_0}), i_*(\mathbf{c}_{x_0})\Big) = 0. \tag{9.6}$$

9.6.2. This implies that $\mathbf{c}_{x_0} = 0$ by the same argument as in Sect. 9.3.4.

□[Theorem 4.7.3]

10. An application: integrable Kac-Moody representations

In this section we discuss an application of Theorem 2.1.6 (rather, of its incarnation as Theorem 3.1.7). Namely, we establish an equivalence between integrable Kac-Moody representations and representations of the integrable quotient of the Kac-Moody chiral algebra.

10.1. **Integrable Kac-Moody representations and the integrable quotient.**

10.1.1. We define integrable Kac-Moody representations, following the framework of [Ro, Sect. 7.4]. Namely, we let $\mathcal{L}_{\mathrm{Gr}_G}$ be a factorization line bundle on Gr_G.

Consider the factorization functor

$$\mathrm{D}(\mathrm{Gr}_G) \to \mathrm{Vect}, \quad \mathcal{F} \mapsto \Gamma^{\mathrm{IndCoh}}(\mathrm{Gr}_G, \mathcal{F}\otimes \mathcal{L}_{\mathrm{Gr}_G}). \tag{10.1}$$

We define the *integrable Kac-Moody factorization algebra* in Vect to be

$$\mathbb{V}^{\mathrm{Int}}_{G,\kappa} := \Gamma^{\mathrm{IndCoh}}(\mathrm{Gr}_G, \omega_{\mathrm{Gr}_G}\otimes \mathcal{L}_{\mathrm{Gr}_G}).$$

Remark 10.1.2. Recall that to the datum of $\mathcal{L}_{\mathrm{Gr}_G}$ there corresponds a discrete invariant, called a level, and denoted κ, which is a W-invariant $\mathbb{Z}$-valued quadratic form on the coweight lattice Λ of G.

It is shown in [Ro, Theorem 7.4.3] that if the restriction of κ to one of the simple factors of G is negative-definite, then $\mathbb{V}^{\mathrm{Int}}_{G,\kappa} = 0$.

If the restriction of κ to all simple factors of G is non-negative definite, then

$$\mathbb{V}^{\mathrm{Int,ch}}_{G,\kappa} := \mathbb{V}^{\mathrm{Int}}_{G,\kappa}|_X[-1] \in \mathrm{D}(X)$$

lives in single cohomological degree 0, so can be regarded as a classical chiral algebra.

In the latter case, if G is semi-simple and simply-connected, $\mathbb{V}^{\mathrm{Int,ch}}_{G,\kappa}$ is what is usually called *the integrable quotient* of the Kac-Moody chiral algebra at level κ, denoted

$$\mathbb{V}^{\mathrm{ch}}_{\mathfrak{g},\kappa} := \mathbb{V}_{\mathfrak{g},\kappa}|_X[-1]$$

(per our conventions, we reserve the symbol $\mathbb{V}_{\mathfrak{g},\kappa}$ for the corresponding factorization algebra).

In the opposite case, namely, when $G = T$ is a torus, $\mathbb{V}^{\mathrm{Int,ch}}_{G,\kappa}$ is the lattice chiral algebra.

10.1.3. Recall that the pullback of $\mathcal{L}_{\mathrm{Gr}_{G,\mathrm{Ran}}}$ along the unit section

$$\mathrm{unit}_{\mathrm{Gr}_G} : \mathrm{Ran} \to \mathrm{Gr}_G$$

is canonically trivialized (as a factorization line bundle over pt).

We quote the following result (see [Zhao, Corollary C]):

Theorem 10.1.4. *The pullback $\mathcal{L}_{\mathfrak{L}(G)}$ of $\mathcal{L}_{\mathrm{Gr}_G}$ along the projection*

$$\mathfrak{L}(G) \to \mathrm{Gr}_G$$

carries a uniquely defined multiplicative structure, compatible with the trivialization of its restriction to $\mathfrak{L}^+(G)$, given by the trivialization of $\mathrm{unit}^*_{\mathrm{Gr}_G}(\mathcal{L}_{\mathrm{Gr}_G})$.

10.1.5. Consider the resulting central extension of $\mathfrak{L}(G)_{x_0}$; denote it by

$$1 \to \mathbb{G}_m \to \widehat{\mathfrak{L}(G)}_{\kappa,x_0} \to \mathfrak{L}(G)_{x_0} \to 1.$$

The datum of such a central extension is equivalent to that of a *weak* action of $\mathfrak{L}(G)_{x_0}$ on Vect. Denote the resulting object[20] of $\mathfrak{L}(G)_{x_0}$-$\mathbf{mod}^{\text{weak}}$ by Vect_κ.

The category $\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa)$ of integrable $\mathfrak{L}(G)$-representations at level κ is by definition

$$\text{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}^{\text{weak}}}(\text{Vect}, \text{Vect}_\kappa).$$

Remark 10.1.6. The category $\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa)$ is well-defined for *any* κ. However, when κ is negative-definite, the natural forgetful functor

$$\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa) \to \text{Vect} \tag{10.2}$$

is zero[21].

Indeed, for an object of $\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa)$, the n-th cohomology of its image under (10.2) would be an *integrable representation* of $\mathfrak{L}(G)_{x_0}$ at level κ in the classical sense. But for κ negative-definite, there are non-zero such, because the sign of κ makes the dominance condition on the highest weight impossible to satisfy.

Yet, the category $\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa)$ is non-zero. In fact, for any κ, we have a natural identification

$$\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa') \simeq (\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa))^\vee, \quad \kappa' = -\kappa - \kappa_{\text{Killing}}.$$

10.1.7. In this section we will prove:

Theorem 10.1.8. *Suppose that the level κ is non-negative definite on each simple factor. Then there is a canonical equivalence*

$$\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa) \simeq \mathbb{V}^{\text{Int}}_{G,\kappa}\text{-mod}^{\text{fact}}_{x_0},$$

commuting with the tautological forgetful functors of both sides to Vect.

The theorem will be proved in Sects. 10.3-10.5.

Remark 10.1.9. The statement of Theorem 10.1.8 is false for κ negative-definite:

Indeed, according to Remark 10.1.2, in this case $\mathbb{V}^{\text{Int}}_{G,\kappa} = 0$, so the right-hand side in Theorem 10.1.8 is zero. However, according to Remark 10.1.6, the left-hand side is non-zero.

Remark 10.1.10. The statement of Theorem 10.1.8 is known at the level of abelian categories, in two cases: either when G is semi-simple and simply-connected or when G is a torus.

When G is semi-simple and simply-connected, this is the statement that for a module $\mathcal{M}$ over the affine Kac-Moody Lie algebra at level κ, the following two conditions are equivalent:

(i) The action of the affine Kac-Moody Lie algebra on $\mathcal{M}$ integrates to an action of $\widehat{\mathfrak{L}(G)}_{\kappa,x_0}$;

(ii) When we view $\mathcal{M}$ as a chiral module over $\mathbb{V}^{\text{ch}}_{\mathfrak{g},\kappa}$ (i.e., a factorization module over $\mathbb{V}_{\mathfrak{g},\kappa}$ at x_0), the action factors through the integrable quotient chiral module over

$$\mathbb{V}^{\text{ch}}_{\mathfrak{g},\kappa} \twoheadrightarrow \mathbb{V}^{\text{Int,ch}}_{G,\kappa}.$$

When $G = T$ is a torus, the statement of Theorem 10.1.8 at the abelian level is equivalent to that of [BD1, Theorem 3.10.14].

[20]We refer the reader to [GLC2, Sect. B.14.19], where weak actions of loop groups on categories are discussed.

[21]However, the forgetful functor $\text{Rep}(\mathfrak{L}(G)_{x_0}, \kappa) \to \text{KL}(G, \kappa)$ is conservative. There is no contradiction here, since the forgetful functor $\text{KL}(G, \kappa) \to \text{Vect}$ is not conservative.

10.1.11. Under the assumption on the level κ, the category $\mathrm{Rep}(\mathfrak{L}(G)_{x_0}, \kappa)$ is known to be semi-simple (e.g., this can be proved by the same method as in [Ro, Appendix D]). Hence, from Theorem 10.1.8, we obtain:

Corollary 10.1.12. *The category $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$ is semi-simple.*

Remark 10.1.13. The statement of Corollary 10.1.12 is known at the abelian level. The innovation here is that it continues to hold at the derived level, i.e., that there are no higher Exts between objects in the heart.

One may (somewhat recklessly) conjecture that the same holds for any rational VOA.

10.2. **Factorization categories with an action of the loop group.** Recall the setting of Sects. 8.1.1-8.1.3.

Thus, let $\mathbf{A}$ be a factorization algebra category, equipped with an action of $\mathfrak{L}(G)_{\mathrm{Ran}}$, compatible with the factorization structure.

In particular, $\mathbf{A}_{x_0}$ is a category acted on by $\mathfrak{L}(G)_{x_0}$, and we can form

$$(\mathbf{A}_{x_0})^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)} \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}. \tag{10.3}$$

In this subsection we will discuss an additional feature of the above construction in the unital setting.

10.2.1. Let us assume that $\mathbf{A}$ is unital and that its unit object $\mathbf{1}_{\mathbf{A}}$ is $\mathfrak{L}^+(G)$-equivariant. In this case, the action on the unit gives rise to a unital factorization functor

$$\Phi : \mathrm{D}(\mathrm{Gr}_G) \to \mathbf{A}.$$

In particular, we can form the object

$$\mathbf{Res}_{\Phi}(\mathbf{A}^{\mathrm{fact}_{x_0}}) \in \mathrm{D}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}. \tag{10.4}$$

10.2.2. We now add the following technical condition: we assume that $\mathbf{A}$ is *tight*, i.e., the functor of the insertion of the unit

$$\mathrm{ins.\,unit}_{\underline{x}_1 \subseteq \underline{x}_2} : \mathbf{A}_{\underline{x}_1} \to \mathbf{A}_{\underline{x}_2}, \quad \underline{x}_1 \subseteq \underline{x}_2$$

admits a colimit-preserving right adjoint (see [GLC2, Sect. C.16.1]).

In particular, this implies that for any $\underline{x}$, the object $(\mathbf{1}_{\mathbf{A}})_{\underline{x}} \in \mathbf{A}_{\underline{x}}$ is compact. We impose an even stronger condition, namely, that $(\mathbf{1}_{\mathbf{A}})_{\underline{x}}$ is compact as an object of $(\mathbf{A}_{\underline{x}})^{\mathfrak{L}^+(G)_{\underline{x}}}$.

Proposition-Construction 10.2.3. *Under the above circumstances, the objects* (10.3) *and* (10.4) *are canonically isomorphic.*

The rest of this subsection is devoted to the proof of this proposition.

10.2.4. We first construct a map in one direction, (10.3) $\to$ (10.4). By the definition of factorization restriction, the datum of such a map is equivalent to the datum of a functor

$$\Phi_m : (\mathbf{A}_{x_0})^{\mathrm{fact}_{x_0}, \mathrm{D}(\mathrm{Gr}_G)}_{\underline{x}} \to \mathbf{A}^{\mathrm{fact}_{x_0}}_{\underline{x}} \simeq \mathbf{A}_{\underline{x}}, \quad \underline{x} \in \mathrm{Ran}_{x_0} \tag{10.5}$$

compatible with factorization via Φ.

10.2.5. Recall the group ind-scheme $\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\mathrm{Ran}_{x_0}}$, see (8.3). It projects onto $\mathfrak{L}(G)_{x_0}$, and let us denote by $\overline{\mathfrak{L}}^+(G)_{\mathrm{Ran}_{x_0}}$ the kernel. In particular, we have a short exact sequence

$$1 \to \overline{\mathfrak{L}}^+(G)_{\underline{x}} \to \mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}} \to \mathfrak{L}(G)_{x_0} \to 1. \tag{10.6}$$

(Note that $\overline{\mathfrak{L}}^+(G)_{\underline{x}}$ identifies also with the kernel of $\mathfrak{L}^+(G)_{\underline{x}} \to \mathfrak{L}^+(G)_{x_0}$.)

We can identify

$$\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}} \simeq \mathfrak{L}(G)_{\underline{x}}/\overline{\mathfrak{L}}^+(G)_{\underline{x}},$$

where the *right* action of $\mathfrak{L}(G)_{x_0}$ on $\mathrm{Gr}^{\mathrm{level}^\infty_{x_0}}_{G,\underline{x}}$ comes from the short exact sequence (10.6).

10.2.6. Consider the functor

$$\text{ins.unit}_{\{x_0\}\subseteq \underline{x}} : \mathbf{A}_{x_0} \to \mathbf{A}_{\underline{x}}. \tag{10.7}$$

Both categories are acted on by $\mathfrak{L}(G)_{x_0}$, and the functor (10.7) is compatible with these actions. Moreover, (10.7) factors via a functor

$$\text{ins.unit}_{\{x_0\}\subseteq \underline{x}} : \mathbf{A}_{x_0} \to (\mathbf{A}_{\underline{x}})^{\overline{\mathfrak{L}}^+(G)_{\underline{x}}}. \tag{10.8}$$

The functor (10.8) is also compatible with the $\mathfrak{L}(G)_{x_0}$-actions, where the action on the right-hand side is via the short exact sequence (10.6).

10.2.7. For future reference we note that it follows from the assumptions on $\mathbf{A}$ in Sect. 10.2.2 that the functor (10.8) admits a colimit-preserving right adjoint.

10.2.8. Recall that

$$(\mathbf{A}_{x_0})_{\underline{x}}^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)} = \text{D}(\text{Gr}_{G,\underline{x}}^{\text{level}^\infty_{x_0}}) \underset{\mathfrak{L}(G)_{x_0}}{\otimes} \mathbf{A}_{x_0}.$$

Thus, from (10.8) we obtain a functor

$$(\mathbf{A}_{x_0})_{\underline{x}}^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)} \to \text{D}(\text{Gr}_{G,\underline{x}}^{\text{level}^\infty_{x_0}}) \underset{\mathfrak{L}(G)_{x_0}}{\otimes} (\mathbf{A}_{\underline{x}})^{\overline{\mathfrak{L}}^+(G)_{\underline{x}}} \tag{10.9}$$

10.2.9. We rewrite the right-hand side in (10.9) as

$$\text{D}(\mathfrak{L}(G)_{\underline{x}})^{\overline{\mathfrak{L}}^+(G)_{\underline{x}}} \underset{\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}(G)_{\underline{x}}/\overline{\mathfrak{L}}^+(G)_{\underline{x}}}{\otimes} (\mathbf{A}_{\underline{x}})^{\overline{\mathfrak{L}}^+(G)_{\underline{x}}} \simeq \text{D}(\mathfrak{L}(G)_{\underline{x}}) \underset{\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}(G)_{\underline{x}}}{\otimes} (\mathbf{A}_{\underline{x}})^{\overline{\mathfrak{L}}^+(G)_{\underline{x}}}. \tag{10.10}$$

Now, the action of $\mathfrak{L}(G)_{\underline{x}}$ on $\mathbf{A}_{\underline{x}}$ gives rise to a functor

$$\text{D}(\mathfrak{L}(G)_{\underline{x}}) \underset{\mathfrak{L}^{\text{mer}\rightsquigarrow\text{reg}}(G)_{\underline{x}}}{\otimes} (\mathbf{A}_{\underline{x}})^{\overline{\mathfrak{L}}^+(G)_{\underline{x}}} \to \mathbf{A}_{\underline{x}}. \tag{10.11}$$

Composing (10.9), (10.10) and (10.11), we obtain the sought-for functor (10.5).

The compatibility with factorization against Φ follows from the construction.

10.2.10. We now show that the resulting functor

$$(\mathbf{A}_{x_0})^{\text{fact}_{x_0},\text{D}(\text{Gr}_G)} \to \mathbf{Res}_\Phi(\mathbf{A}^{\text{fact}_{x_0}}) \tag{10.12}$$

is an equivalence.

In order to do se, we apply Proposition C.10.20. We need to show:

- The functor (10.12) induces an equivalence between the fibers at x_0;
- The functor Φ admits a colimit-preserving right adjoint;
- The functor Φ_m admits a colimit-preserving right adjoint.

The fact that the first condition is satisfied is automatic. Indeed, at the level of fibers at x_0, the functor (10.12) is the identity functor $\mathbf{A}_{x_0} \to \mathbf{A}_{x_0}$.

The fact that Φ admits a colimit-preserving right adjoint follows from the fact that $(\mathbf{1}_\mathbf{A})_{\underline{x}}$ is compact as an object of $\mathbf{A}_{\underline{x}}^{\mathfrak{L}^+(G)}$, combined with the ind-properness of $\text{Gr}_{G,\underline{x}}$.

Finally, let us show that Φ admits a colimit-preserving right adjoint. For that it is sufficient show that both functors (10.9) and (10.11) admit a colimit-preserving right adjoints.

10.2.11. The fact that (10.9) admits a colimit-preserving right adjoint follows from the corresponding fact for (10.8), see Sect. 10.2.7.

For the functor (10.11), we write it as a composition

$$\mathrm{D}(\mathfrak{L}(G)_{\underline{x}})^{\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}}} \otimes (\mathbf{A}_{\underline{x}})^{\overline{\mathfrak{L}}^{+}(G)_{\underline{x}}} \to \mathrm{D}(\mathfrak{L}(G)_{\underline{x}})^{\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}}} \otimes \mathbf{A}_{\underline{x}} \to \mathbf{A}_{\underline{x}},$$

and it suffices to show that the second arrow admits a colimit-preserving right adjoint.

Using the $\mathfrak{L}(G)_{\underline{x}}$-action on $\mathbf{A}_{\underline{x}}$, we rewrite this arrow as

$$\mathrm{D}(\mathfrak{L}(G)_{\underline{x}})^{\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}}} \otimes \mathbf{A}_{\underline{x}} \to \mathbf{A}_{\underline{x}},$$

induced by the functor

$$\mathrm{D}(\mathfrak{L}(G)_{\underline{x}})^{\mathfrak{L}^{\mathrm{mer}\rightsquigarrow\mathrm{reg}}(G)_{\underline{x}}} \simeq \mathrm{D}(\mathrm{Gr}_{G,\underline{x}}/\operatorname{Hecke}_{x_0}) \overset{\mathrm{C}^{\cdot}(\mathrm{Gr}_{G,\underline{x}}/\operatorname{Hecke}_{x_0},-)}{\longrightarrow} \mathrm{Vect}\,.$$

This implies the desired assertion since $\mathrm{Gr}_{G,\underline{x}}$ is ind-proper and $\operatorname{Hecke}_{x_0}$ is pseudo-proper.
$\square$[Proposition 10.2.3]

10.3. **Proof of Theorem 10.1.8.**

10.3.1. We consider the factoriztion category $\mathrm{KM}(\mathfrak{g},\kappa)$ of Kac-Moody representations at level κ, see [GLC2, Sect. B.14.22]. It is naturally equipped with a (strong) action of $\mathfrak{L}(G)$ *at level* κ.

Now, the datum of the central extension $\widehat{\mathfrak{L}(G)}_\kappa$ allows us to modify the weak action of $\mathfrak{L}(G)$ on $\mathrm{KM}(\mathfrak{g},\kappa)$, so that the resulting strong action occurs at level 0. The resulting object

$$\mathrm{KM}(\mathfrak{g},\kappa) \in \mathfrak{L}(G)\textbf{-mod}$$

has the universal property that

$$\operatorname{Funct}_{\mathfrak{L}(G)_{\underline{x}}}(\mathbf{C},\mathrm{KM}(\mathfrak{g},\kappa)_{\underline{x}}) \simeq \operatorname{Funct}_{\mathfrak{L}(G)_{\underline{x}}\text{-weak}}(\mathbf{C},\mathrm{Vect}_\kappa), \quad \mathbf{C}\in\mathfrak{L}(G)_{\underline{x}}\textbf{-mod}, \quad \underline{x}\in\mathrm{Ran}, \tag{10.13}$$

see [GLC2, Sect. B.14.12].

10.3.2. Taking $\mathbf{C}$ in (10.13) to be Vect, we obtain

$$\operatorname{Rep}(\mathfrak{L}(G)_{x_0},\kappa) := \operatorname{Funct}_{\mathfrak{L}(G)_{x_0}\text{-weak}}(\mathrm{Vect},\mathrm{Vect}_\kappa) \simeq \operatorname{Funct}_{\mathfrak{L}(G)_{x_0}\textbf{-mod}}(\mathrm{Vect},\mathrm{KM}(\mathfrak{g},\kappa)_{x_0}). \tag{10.14}$$

10.3.3. Consider the object

$$\mathrm{KM}(\mathfrak{g},\kappa)^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)} \in \mathrm{D}(\mathrm{Gr}_G)\textbf{-mod}^{\mathrm{fact}}_{x_0}. \tag{10.15}$$

We now perform the crucial step in the proof of Theorem 10.1.8. Namely, we combine (10.14) with Theorem 3.1.7, and obtain an equivalence

$$\operatorname{Rep}(\mathfrak{L}(G)_{x_0},\kappa) \simeq \omega_{\mathrm{Gr}_{G,\mathrm{Ran}}}\text{-mod}^{\mathrm{fact}}(\mathrm{KM}(\mathfrak{g},\kappa)^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0}. \tag{10.16}$$

The equivalence (10.16) commutes with the forgetful functors to Vect, where in the right-hand side, the corresponding functor is

$$\omega_{\mathrm{Gr}_{G,\mathrm{Ran}}}\text{-mod}^{\mathrm{fact}}(\mathrm{KM}(\mathfrak{g},\kappa)^{\mathrm{fact}_{x_0},\mathrm{D}(\mathrm{Gr}_G)})_{x_0} \to \mathrm{KM}(\mathfrak{g},\kappa)_{x_0} \overset{\mathrm{oblv}_{\mathrm{KM}}}{\longrightarrow} \mathrm{Vect},$$

where the second arrow is the tautological forgetful functor.

10.3.4. Note now that $\mathrm{KM}(\mathfrak{g},\kappa)$, viewed as a factorization category equipped with an action of $\mathfrak{L}(G)$, fits into the paradigm of Sect. 10.2.

The corresponding factorization functor

$$\Phi : \mathrm{D}(\mathrm{Gr}_G) \to \mathrm{KM}(\mathfrak{g},\kappa)$$

is the functor of $\mathcal{L}_{\mathrm{Gr}_G}$-twisted IndCoh sections, to be denoted $\Gamma^{\mathrm{enh}}_\kappa$. I.e., its composition with the forgetful functor

$$\mathrm{oblv}_{\mathrm{KM}} : \mathrm{KM}(\mathfrak{g},\kappa) \to \mathrm{Vect} \tag{10.17}$$

is the functor (10.1), to be denoted Γ_κ.

Thus, applying Proposition 10.2.3, we obtain that the object (10.15) identifies with

$$\mathbf{Res}_{\Gamma^{\mathrm{enh}}_\kappa}(\mathrm{KM}(\mathfrak{g},\kappa)^{\mathrm{fact}_{x_0}}).$$

10.3.5. Let $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}$ be the factorization algebra in $\mathrm{KM}(\mathfrak{g},\kappa)$ equal to the image of ω_{Gr_G} under the functor $\Gamma^{\mathrm{enh}}_\kappa$.

Applying (1.9), we rewrite

$$\omega_{\mathrm{Gr}_{G,\mathrm{Ran}}}\text{-mod}^{\mathrm{fact}}\left(\mathbf{Res}_{\Gamma^{\mathrm{enh}}_\kappa}(\mathrm{KM}(\mathfrak{g},\kappa)^{\mathrm{fact}_{x_0}})\right)_{x_0} \simeq \mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{KM}(\mathfrak{g},\kappa)^{\mathrm{fact}_{x_0}})_{x_0}.$$

10.3.6. Thus, combining, we obtain an equivalence

$$\mathrm{Rep}(\mathfrak{L}(G)_{x_0},\kappa) \simeq \mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}. \tag{10.18}$$

(see Sect. 1.5.5 for the notation), which commutes with the natural forgetful functors of both sides to Vect.

Remark 10.3.7. Note that the equivalence (10.18) did not use the assumption that κ is non-negative definite.

10.3.8. By construction

$$\mathbb{V}^{\mathrm{Int}}_{G,\kappa} \simeq \mathrm{oblv}_{\mathrm{KM}}(\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}).$$

Hence, in order to prove Theorem 10.1.8, it remains to show the following:

Proposition 10.3.9. *The functor*

$$\mathrm{oblv}^{\mathrm{Int}}_{\mathrm{KM}} : \mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0},$$

induced by the factorization functor $\mathrm{oblv}_{\mathrm{KM}}$*, is an equivalence.*

□[Theorem 10.1.8]

Remark 10.3.10. As we will see, Proposition 10.3.9 would be almost tautological, if not not for some homological algebra "issues" (the idea is that the category $\mathrm{KM}(\mathfrak{g},\kappa)$ is "almost" the same as $\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}$).

Yet, this issues become fatal when κ is negative-definite. (Indeed, as was remarked above, the equivalence (10.18) holds for any κ.)

10.4. Proof of Proposition 10.3.9: reduction to the bounded below category.

10.4.1. Note that the chiral algebra $\mathbb{V}^{\mathrm{Int,ch}}_{G,\kappa}$ is concentrated in non-positive cohomological degrees[22]. Hence, the category $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$ acquires a t-structure, characterized by the property that the forgetful functor

$$\mathrm{oblv}_{\mathbb{V}^{\mathrm{Int}}_{G,\kappa}} : \mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathrm{Vect}$$

(which is by definition conservative) is t-exact. Moreover, $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}$ is left-complete in its t-structure, see [GLC2, Proposition B.9.18].

[22] In fact, according to Remark 10.1.2, it is actually concentrated in degree 0, a fact that will be used later.

10.4.2. Note that the factorization category $\mathrm{KM}(\mathfrak{g},\kappa)$ carries a t-structure in the sense of [GLC2, Sect. B.11.11] (see [GLC2, Sect. B.14.22]).

Since the functor $\mathrm{oblv}_{\mathrm{KM}}$ is t-exact and conservative *on the bounded below category*, we obtain that the object

$$\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}|_X[-1] \in \mathrm{KM}(\mathfrak{g},\kappa)_X$$

is connective.

Hence, by [GLC2, Sect. B.11.16], the category $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$ also acquires a t-structure, characterized by the property that the forgetful functor

$$\mathrm{oblv}_{\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}} : \mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathrm{KM}(\mathfrak{g},\kappa)_{x_0}$$

(which is by definition conservative) is t-exact.

Remark 10.4.3. For future reference, we note that the object $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}|_X[-1]$ lies in the heart of the t-structure of $\mathrm{KM}(\mathfrak{g},\kappa)_X$.

Let us consider $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}|_X[-1]$ as an object of $\mathrm{KL}(G,\kappa)_X$, where

$$\mathrm{KL}(G,\kappa) = \mathrm{KM}(\mathfrak{g},\kappa)^{\mathfrak{L}^+(G)}.$$

Recall that the level was assumed *non-negative definite* (on every simple factor of $\mathfrak{g}$). We claim that in this case, the functor $\mathrm{oblv}_{\mathrm{KM}}$, restricted to $\mathrm{KL}(G,\kappa)_{x_0}$, is actually conservative[23].

Indeed, in this case, the compact generators of $\mathrm{KL}(G,\kappa)_{x_0}$ have a finite cohomological dimension (this can be seen, e.g., from the Kashiwara-Tanisaki localization); hence $\mathrm{KL}(G,\kappa)_{x_0}$ has no non-zero infinitely connective objects.

Hence, the fact that $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}|_X[-1] = \mathbb{V}^{\mathrm{Int,ch}}_{G,\kappa}$ lies in the heart of the t-structure implies that the same is true for $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}|_X[-1]$.

10.4.4. In particular, we obtain that the functor $\mathrm{oblv}^{\mathrm{Int}}_{\mathrm{KM}}$, appearing in Proposition 10.3.9, is t-exact (since its composition with $\mathrm{oblv}_{\mathbb{V}^{\mathrm{Int}}_{G,\kappa}}$, which is t-exact and conservative, is t-exact).

Recall that the source category in Proposition 10.3.9, being equivalent to $\mathrm{Rep}(\mathfrak{L}(G)_{x_0},\kappa)$, is semi-simple (see Sect. 10.1.11). Moreover, the same argument shows that its irreducible objects are bounded below (in fact, that they lie in the heart of the t-structure).

Since the target category is left-complete in its t-structure, we obtain that in order to prove that $\mathrm{oblv}^{\mathrm{Int}}_{\mathrm{KM}}$ is an equivalence, it suffices to show that it induces an equivalence on the corresponding bounded below categories.

10.5. **Proof of Proposition 10.3.9: the bounded below part.**

10.5.1. Let $\mathrm{Vac}_{\mathfrak{g},\kappa}$ denote the factorization unit in $\mathrm{KM}(\mathfrak{g},\kappa)$. Denote

$$\mathbb{V}_{\mathfrak{g},\kappa} := \mathrm{oblv}_{\mathrm{KM}}(\mathrm{Vac}_{\mathfrak{g},\kappa}).$$

This is the usual factorization algebra attached to $\mathfrak{g}$ at level κ.

[23] This observation is due to G. Dhillon.

10.5.2. The unit for $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}$, which is a map

$$\mathrm{Vac}_{\mathfrak{g},\kappa} \to \mathrm{Vac}^{\mathrm{Int}}_{G,\kappa},$$

induces a homomorphism

$$\mathbb{V}_{\mathfrak{g},\kappa} \to \mathbb{V}^{\mathrm{Int}}_{G,\kappa} \tag{10.19}$$

as unital factorization algebras in Vect.

Let us regard $\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}$ as a unital *lax factorization category*, see [GLC2, Sect. B.11.12]. The homomorphism (10.19) allows us to upgrade $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}$ to a unital factorization algebra in $\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}$ (see [GLC2, Setct. C.11.18]); when viewed as such we will denote it by $\mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}$.

The forgetful factorization functor

$$\mathrm{oblv}_{\mathbb{V}_{\mathfrak{g},\kappa}} : \mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}} \to \mathrm{Vect}$$

induces a functor

$$\mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathbb{V}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}, \tag{10.20}$$

to be denoted $\mathrm{oblv}^{\mathrm{Int}}_{\mathbb{V}_{\mathfrak{g},\kappa}}$.

According to [GLC2, Lemma C.11.19], the functor (10.20) is an equivalence.

10.5.3. The functor $\mathrm{oblv}_{\mathrm{KM}}$ induces a factorization functor

$$\mathrm{KM}(\mathfrak{g},\kappa) \simeq \mathrm{Vac}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}(\mathrm{KM}(\mathfrak{g},\kappa)) \to \mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}, \tag{10.21}$$

to be denoted $\mathrm{oblv}^{\mathrm{enh}}_{\mathrm{KM}}$.

We have

$$\mathrm{oblv}^{\mathrm{enh}}_{\mathrm{KM}}(\mathbb{V}^{\mathrm{int}}_{G,\kappa}) \simeq \mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}$$

as factorization algebras in $\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}$.

In particular, $\mathrm{oblv}^{\mathrm{enh}}_{\mathrm{KM}}$ induces a functor

$$\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}, \tag{10.22}$$

to be denoted $\mathrm{oblv}^{\mathrm{Int,enh}}_{\mathrm{KM}}$.

Unwinding, we obtain that the functor $\mathrm{oblv}^{\mathrm{Int}}_{\mathrm{KM}}$ identifies with the composition

$$\mathrm{oblv}^{\mathrm{Int}}_{\mathbb{V}_{\mathfrak{g},\kappa}} \circ \mathrm{oblv}^{\mathrm{Int,enh}}_{\mathrm{KM}}.$$

10.5.4. By [GLC2, Sect. B.11.15], the lax factorization category $\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}$ carries a t-structure in the sense of [GLC2, Sect. B.11.11]. It is characterized by the property that the (conservative) forgetful functor $\mathrm{oblv}_{\mathbb{V}_{\mathfrak{g},\kappa}}$ is t-exact.

Since $\mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}$ is connective, by [GLC2, Sect. B.11.16], we obtain that the category $\mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$ acquires a t-structure, characterized by the property that the (conservative) forgetful functor

$$\mathbb{V}^{\mathrm{Int,enh}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}_{x_0} \to \mathrm{Vect}$$

is t-exact.

In particular, we obtain that the equivalence $\mathrm{oblv}^{\mathrm{Int}}_{\mathbb{V}_{\mathfrak{g},\kappa}}$ of (10.20) t-exact.

10.5.5. Hence, in order to prove Proposition 10.3.9, it remains to show that the functor $\mathrm{oblv}_{\mathrm{KM}}^{\mathrm{Int,enh}}$ of (10.22) induces an equivalence on the bounded below subcategories of the two sides.

Recall (see Remark 10.4.3) that the object $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}|_X[-1]$ lies in the heart of the t-structure[24]. Hence, its values over all powers of X are bounded below. Hence, for an eventually coconnective object in $\mathrm{KM}(\mathfrak{g},\kappa)_{x_0}$, the morphisms that define on it a structure of object of $\mathrm{Vac}^{\mathrm{Int}}_{G,\kappa}\text{-mod}^{\mathrm{fact}}_{x_0}$ take place in the bounded below subcategories of values $\mathrm{KM}(\mathfrak{g},\kappa)$ on powers of X.

The same is true for $\mathbb{V}^{\mathrm{Int}}_{G,\kappa}|_X[-1]$ and $\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod}^{\mathrm{fact}}$.

Therefore, in order to prove that $\mathrm{oblv}_{\mathrm{KM}}^{\mathrm{Int,enh}}$ induces an equivalence on the bounded below subcategories, it suffices to show that the functor $\mathrm{oblv}_{\mathrm{KM}}^{\mathrm{enh}}$ of (10.21) induces an equivalence between the bounded below subcategories of the two sides (evaluated on powers of X).

However, the latter is the assertion of [GLC2, Lemma 4.2.3(a)].

$\square$[Proposition 10.3.9]

10.6. An addendum: failure of the coherent version of Theorem 2.1.6.

10.6.1. Parallel to the setting of Sect. 2.1, one can consider:

- The category $\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}^{\mathrm{weak}}$ of DG categories, equipped with a weak action of $\mathfrak{L}(G)_{x_0}$;
- The factorization category $\mathrm{IndCoh}(\mathrm{Gr}_G)$;
- The functor

$$\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}^{\mathrm{weak}} \to \mathrm{IndCoh}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}, \quad \mathbf{C} \mapsto \mathbf{C}^{\mathrm{fact}_{x_0},\mathrm{IndCoh}(\mathrm{Gr}_G)}. \tag{10.23}$$

10.6.2. However, we claim that, unlike Theorem 2.1.6, the functor (10.23) fails to be fully faithful. Namely, as we shall presently explain, the functor

$$\begin{aligned}\mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}^{\mathrm{weak}}}(\mathbf{C}_1,\mathbf{C}_2) \to \\ \to \mathrm{Funct}_{\mathrm{IndCoh}(\mathrm{Gr}_G)\text{-}\mathbf{mod}^{\mathrm{fact}}_{x_0}}\left((\mathbf{C}_1)^{\mathrm{fact}_{x_0},\mathrm{IndCoh}(\mathrm{Gr}_G)},(\mathbf{C}_2)^{\mathrm{fact}_{x_0},\mathrm{IndCoh}(\mathrm{Gr}_G)}\right)\end{aligned} \tag{10.24}$$

fails to be an equivalence.

Namely, the right-hand side admits a natural conservative functor to $\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}_1,\mathbf{C}_2)$, whose composition with the functor in (10.24) is the natural forgetful functor

$$\mathrm{Funct}_{\mathfrak{L}(G)_{x_0}\text{-}\mathbf{mod}^{\mathrm{weak}}}(\mathbf{C}_1,\mathbf{C}_2) \to \mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}_1,\mathbf{C}_2). \tag{10.25}$$

However, we claim that we can find $\mathbf{C}_1,\mathbf{C}_2$ so that the functor (10.25) fails to be conservative.

10.6.3. Namely, we take $\mathbf{C}_1 = \mathrm{Vect}$ and $\mathbf{C}_2 = \mathrm{Vect}_\kappa$, where κ is negative-definite. The left-hand side in (10.25) is $\mathrm{Rep}(\mathfrak{L}(G)_{x_0},\kappa)$, and (10.25) is the natural forgetful functor

$$\mathrm{Rep}(\mathfrak{L}(G)_{x_0},\kappa) \to \mathrm{Vect}\,.$$

However, the above functor is zero for negative-definite (see Remark 10.1.6).

Appendix A. D-modules in infinite type

In this section we (re)collect some facts pertaining to the extension of the theory of D-modules to algebro-geometric objects of infinite type. The goal is to make sense of the category of D-modules on the loop group $\mathfrak{L}(G)_{x_0}$.

A.1. **The case of affine schemes.** In this subsection we develop the theory of D-modules on affine schemes (not necessarily of finite type). We will mostly follow [GLC2, Sect. A.4-A.5].

[24]It is here that we crucially use the assumption that κ is non-negative definite.

A.1.1. Let S be a scheme (not necessarily of finite type). Set

$$\mathrm{D}^!(S) := \underset{S\to S_0}{\mathrm{colim}}\ \mathrm{D}(S_0), \tag{A.1}$$

where:

- The colimit is taken over the (opposite of the) category of affine schemes of finite type that receive a map from S;
- The transition functors are given by !-pullback.

Equivalently, for a fixed presentation

$$S \simeq \lim_{\alpha} S_{0,\alpha}, \tag{A.2}$$

where $S_{0,\alpha}$ are affine schemes of finite type, we have

$$\mathrm{D}^!(S) \simeq \underset{\alpha}{\mathrm{colim}}\ \mathrm{D}(S_{0,\alpha}). \tag{A.3}$$

For an arbitrary S, there is no reason for $\mathrm{D}^!(S)$ defined in the above way to be compactly generated or dualizable.

A.1.2. We shall say that an object $\mathrm{D}^!(S)$ is *ind-holonomic* it lies in the essential image of

$$\underset{\alpha}{\mathrm{colim}}\ \mathrm{D}^{\mathrm{hol}}(S_{0,\alpha}) \to \underset{\alpha}{\mathrm{colim}}\ \mathrm{D}(S_{0,\alpha}) = \mathrm{D}^!(S).$$

A.1.3. We set

$$\mathrm{D}_*(S) := \mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}^!(S), \mathrm{Vect}).$$

Using Verdier duality, we obtain:

$$\mathrm{D}_*(S) \simeq \lim_{S\to S_0} \mathrm{D}(S_0),$$

where the limit is taken with respect to the *-pushforward functors.

In terms of the presentation (A.2), we have

$$\mathrm{D}_*(S) \simeq \lim_{\alpha} \mathrm{D}(S_{0,\alpha}).$$

A.1.4. Since the transition functors in (A.1) are symmetric monoidal, the category $\mathrm{D}^!(S)$ carries a natural symmetric monoidal structure. Its unit object, denoted ω_S, is the image of ω_{S_0} for any $S \to S_0$.

Note that we have a canonical isomorphism

$$\mathrm{C}^{\cdot}(S) := \underset{S\to S_0}{\mathrm{colim}}\ \mathrm{C}^{\cdot}(S_0) \simeq \underset{S\to S_0}{\mathrm{colim}}\ \mathrm{End}_{\mathrm{D}(S_0)}(\omega_{S_0}) \simeq \mathrm{End}_{\mathrm{D}^!(S)}(\omega_S).$$

In addition, we have a natural action of $\mathrm{D}^!(S)$ on $\mathrm{D}_*(S)$.

The category $\mathrm{D}_*(S)$ has a distinguished object, denoted $\underline{k}_S$; it corresponds to the compatible family of functors

$$\mathrm{D}(S_0) \to \mathrm{Vect}, \quad S \to S_0$$

equal to

$$\underset{S\to S_0' \xrightarrow{f} S_0}{\mathrm{colim}}\ \mathcal{H}om(\omega_{S_0'}, f^!(\mathcal{F})).$$

Remark A.1.5. One can describe the object $\underline{k}_S$ more explicitly as follows: let

$$\mathrm{D}_*^{\mathrm{hol}}(S) \subset \mathrm{D}_*(S)$$

be the full subcategory equal to

$$\lim_{S\to S_0} \mathrm{D}^{\mathrm{hol}}(S_0) \subset \lim_{S\to S_0} \mathrm{D}(S_0),$$

where the limits are taken with respect to the *-pushforward functors.

Since on the holonomic category, the *-pushforward admit left adjoints, we can rewrite $\mathrm{D}^{\mathrm{hol}}_*(S)$ also as

$$\underset{S\to S_0}{\mathrm{colim}}\ \mathrm{D}^{\mathrm{hol}}(S_0),$$

where the colimit is taken with respect to the *-pullbacks.

In terms of the latter presentation, $\underline{k}_S$ equals the image of $\underline{k}_{S_0}$ for some/any S_0.

A.1.6. For a map $f: S_1 \to S_2$ between affine schemes, we have a tautologically defined functor

$$f^!: \mathrm{D}^!(S_2) \to \mathrm{D}^!(S_1).$$

Dually, we have a functor

$$f_*: \mathrm{D}_*(S_1) \to \mathrm{D}_*(S_2).$$

A.1.7. Assume now that f is of finite presentation. In this case we can define a functor

$$f_*: \mathrm{D}^!(S_1) \to \mathrm{D}^!(S_2),$$

which satisfies base change against !-pullbacks.

Dually, we can define

$$f^!: \mathrm{D}_*(S_2) \to \mathrm{D}_*(S_1),$$

which satisfies base change against *-pushforwards. When f is étale, this functor sends $\underline{k}_{S_2} \mapsto \underline{k}_{S_1}$.

A.1.8. Assume now that f is a closed embedding of finite presentation. In this case the functors

$$f_*: \mathrm{D}^!(S_1) \rightleftarrows \mathrm{D}^!(S_2): f^!$$

and

$$f_*: \mathrm{D}_*(S_1) \rightleftarrows \mathrm{D}_*(S_2): f^!$$

form adjoint pairs.

A.2. **Extension to prestacks–the !-version.**

A.2.1. The functoriality of $\mathrm{D}^!(-)$ on affine schemes with respect to !-pullbacks allows us to extend the functor

$$\mathrm{D}(-)^!: (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat}$$

to arbitrary prestacks by the *procedure of right Kan extension*.

Explicitly, for $\mathcal{Y} \in \mathrm{PreStk}$, we have

$$\mathrm{D}^!(\mathcal{Y}) = \lim_{S\to\mathcal{Y}} \mathrm{D}^!(S), \tag{A.4}$$

where:

- The limit is taken over the (opposite of the) category of affine schemes mapping to $\mathcal{Y}$;
- The transition functors are given by !-pullback.

A.2.2. We shall say that an object of $\mathrm{D}^!(\mathcal{Y})$ is *ind-holonomic* if its value on each $S \to \mathcal{Y}$ belongs to the ind-holonomic subcategory (see Sect. A.1.2).

A.2.3. Since the transition functors in (A.4) are symmetric monoidal, the category $\mathrm{D}^!(\mathcal{Y})$ acquires a symmetric monoidal structure.

Its unit is the object $\omega_{\mathcal{Y}} \in \mathrm{D}^!(\mathcal{Y})$, whose value on every $S \to \mathcal{Y}$ is $\omega_S \in \mathrm{D}^!(S)$.

A.2.4. By Sect. A.1.7, if $f: \mathcal{Y}_1 \to \mathcal{Y}_2$ is map between prestacks that is affine and of finite presentation, we have a well-defined functor

$$f_*: \mathrm{D}(\mathcal{Y}_1) \to \mathrm{D}(\mathcal{Y}_2)$$

that satisfies base change against !-pullbacks.

Assume now that f is a closed embedding of finite presentation, In this the functors

$$f_*: \mathrm{D}^!(\mathcal{Y}_1) \rightleftarrows \mathrm{D}^!(\mathcal{Y}_2): f^!$$

are an adjoint pair.

A.2.5. Let f again be arbitrary. The functor $f^!$ has a partially defined left adjoint, to be denoted $f_!$.

In particular, we can consider the partially defined functor

$$\mathrm{C}^{\cdot}_c(\mathcal{Y}, -) : \mathrm{D}^!(\mathcal{Y}) \to \mathrm{Vect},$$

left adjoint to

$$k \mapsto \omega_{\mathcal{Y}}.$$

Lemma A.2.6. *The functor* $\mathrm{C}^{\cdot}_c(\mathcal{Y}, -)$ *is defined on ind-holonomic objects.*

Proof. We first consider the case when $\mathcal{Y} = S$ is an affine scheme. Let $\mathcal{F} \in \mathrm{D}^!(S)$ be obtained as $g^!(\mathcal{F}_0)$ for $g : S \to S_0$, where S_0 is an affine scheme of finite type and $\mathcal{F}_0 \in \mathrm{D}^{\mathrm{hol}}(S_0)$.

Then the sought-for value of $\mathrm{C}^{\cdot}_c(S, \mathcal{F})$ is given by

$$\underset{S \to S'_0 \xrightarrow{f} S_0}{\mathrm{colim}} \mathrm{C}^{\cdot}_c(S'_0, f^!(\mathcal{F}_0)).$$

Let now $\mathcal{Y}$ be a general prestack and $\mathcal{F} \in \mathrm{D}^!(\mathcal{Y})$ be ind-holonomic. Then it is easy to see that

$$\underset{f: S \to \mathcal{Y}}{\mathrm{colim}} \mathrm{C}^{\cdot}_c(S, f^!(\mathcal{F}))$$

provides the value of the sought-for left adjoint.

□

A.2.7. We will denote

$$\mathrm{C}_{\cdot}(\mathcal{Y}) := \mathrm{C}^{\cdot}_c(\mathcal{Y}, \omega_{\mathcal{Y}}).$$

Note that the dual of $\mathrm{C}_{\cdot}(\mathcal{Y})$ identifies with

$$\mathrm{C}^{\cdot}(\mathcal{Y}) \simeq \mathrm{End}_{\mathrm{D}^!(\mathcal{Y})}(\omega_{\mathcal{Y}}) \simeq \lim_{S \to \mathcal{Y}} \mathrm{End}_{\mathrm{D}^!(S)}(\omega_S) \simeq \lim_{S \to \mathcal{Y}} \mathrm{C}^{\cdot}(S).$$

A.3. **The case of (non-affine) schemes.** In this subsection we study/define the categories $\mathrm{D}^!(-)$ and $\mathrm{D}_*(-)$ on *schemes*.

A.3.1. Let now Y be a (not necessarily affine) scheme, assumed quasi-compact and separated. Let $S \to Y$ be a Zariski cover, where S is affine, and let $S^\bullet$ be its Cech nerve. Consider $\mathrm{D}^!(S^\bullet)$ as a cosimplicial category under !-pullbacks.

A standard argument shows that the restriction functor

$$\mathrm{D}^!(Y) \to \mathrm{Tot}(\mathrm{D}^!(S^\bullet))$$

is an equivalence.

In addition, since the morphism in $S^\bullet$ are of finite presentation, we can view $\mathrm{D}^!(S^\bullet)$ as a *simplicial* category under *-pushforwards. The functor of *-pushforward gives rise to a functor

$$|\,\mathrm{D}^!(S^\bullet)| \to \mathrm{D}^!(Y), \tag{A.5}$$

and another standard argument shows that (A.5) is also an equivalence.

A.3.2. We continue to assume that Y is a scheme. Note that we can consider $\mathrm{D}_*(S^\bullet)$ as simplicial (resp., cosimplicial) category via *-pushforwards (resp., !-pullbacks).

Set

$$\mathrm{D}_*(Y) := |(\mathrm{D}_*(S^\bullet)|.$$

A standard argument shows that the restriction functor

$$\mathrm{D}_*(Y) \to \mathrm{Tot}(\mathrm{D}_*(S^\bullet)), \tag{A.6}$$

given by !-pullback, is an equivalence.

The category $\mathrm{D}_*(Y)$ contains a canonically defined object, denoted $\underline{k}_Y$, whose value on the terms of $\mathrm{D}_*(S^\bullet)$ is $\underline{k}_{S^\bullet}$ (see the last sentence in Sect. A.1.7).

From the equivalences (A.5) and (A.6), we obtain that we have a canonical equivalence

$$\mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}^!(Y), \mathrm{Vect}) \simeq \mathrm{D}_*(Y),$$

One shows that the above constructions is canonically independent of the choice of the cover $S \to Y$.

A.3.3. For a map $f : Y_1 \to Y_2$ we have a naturally defined functor

$$f_* : \mathrm{D}_*(Y_1) \to \mathrm{D}_*(Y_2).$$

This functor can be also thought of as obtained from

$$f^! : \mathrm{D}_*(Y_2) \to \mathrm{D}_*(Y_1)$$

by duality.

A.3.4. We have a natural action of $\mathrm{D}^!(Y)$ on $\mathrm{D}_*(Y)$.

We shall call an object

$$\omega_Y^{\mathrm{fake},*} \in \mathrm{D}_*(Y)$$

a *fake* dualizing sheaf, if the functor

$$\mathrm{D}^!(Y) \to \mathrm{D}_*(Y), \quad \mathcal{F} \mapsto \mathcal{F} \overset{!}{\otimes} \omega_Y^{\mathrm{fake},*}$$

is an equivalence.

A.3.5. Let Y be written

$$Y \simeq \lim_\alpha Y_\alpha, \tag{A.7}$$

where the transition maps

$$Y_\alpha \overset{f_{\alpha,\beta}}{\to} Y_\beta$$

are affine.

In this case, it is easy to see that the functor

$$\underset{\alpha}{\mathrm{colim}}\, \mathrm{D}^!(Y_\alpha) \to \mathrm{D}^!(Y), \tag{A.8}$$

defined by !-pullback, is an equivalence.

Similarly, the functor

$$\mathrm{D}_*(Y) \to \lim_\alpha \mathrm{D}_*(Y_\alpha), \tag{A.9}$$

defined by *-pushforward, is an equivalence.

A.3.6. Let $f : Y_1 \to Y_2$ be a map of finite presentation between schemes.

Using the equivalence (A.8), we construct a functor

$$f_* : \mathrm{D}^!(Y_1) \to \mathrm{D}^!(Y_2), \tag{A.10}$$

which satisfies base change against !-pullbacks.

This construction allows us to extend the definition of *-pushforward on $\mathrm{D}^!(-)$ for schematic maps between prestacks.

Similarly, we construct a functor

$$f^! : \mathrm{D}^*(Y_2) \to \mathrm{D}^*(Y_1),$$

which can also be thought of as obtained from (A.10) by duality.

When f is a closed embedding, the functors

$$f_* : \mathrm{D}_*(Y_1) \rightleftarrows \mathrm{D}_*(Y_2) : f^!$$

form an adjoint pair.

A.3.7. We shall say that Y is *placid* if there exists a presentation (A.7), where:

- The schemes Y_α are of finite type;
- The maps $f_{\alpha,\beta}$ are smooth,

In this case, the transition functors in (A.8) preserve compactness, so $\mathrm{D}^!(Y)$ is compactly generated, and in particular, dualizable. Hence, in this case $\mathrm{D}_*(Y)$ is also compactly generated and

$$\mathrm{D}_*(Y) \simeq (\mathrm{D}^!(Y))^\vee.$$

For a presentation (A.7) as above, we can rewrite $\mathrm{D}_*(Y)$ as

$$\mathrm{D}_*(Y) \simeq \underset{\alpha}{\mathrm{colim}}\, \mathrm{D}(Y_\alpha), \tag{A.11}$$

where the limit is taken with respect to the *-pullback functors (they are defined thanks to the smoothness assumption).

In terms of this presentation, the object $\underline{k}_Y$ equals to the image of $\underline{k}_{Y_\alpha}$ for any α.

A.3.8. Suppose that Y is placid, and assume that in the presentation (A.7), the transition maps $f_{\alpha,\beta}$ are equidimensional.

A dimension theory for such a presentation is an assignment

$$\alpha \mapsto d_\alpha \in \mathbb{Z}$$

such that for a pair of indices and an arrow $\alpha \to \beta$ in the category of indices, we have

$$d_\alpha - d_\beta = \mathrm{dim.rel.}(f_{\alpha,\beta}).$$

Note that a choice of a dimension theory gives rise to a fake dualizing object $\omega_Y^{\mathrm{fake},*} \in \mathrm{D}_*(Y)$. Namely, in terms of (A.11), $\omega_Y^{\mathrm{fake},*}$ is the image of $\omega_{Y_\alpha}[-2d_\alpha]$ for some/any α.

A.3.9. Note that if the schemes Y_α are smooth and connected, we have a distinguished dimension theory, given by $d_\alpha = \dim(Y_\alpha)$. In this case $\omega_Y^{\mathrm{fake},*} = \underline{k}_Y$.

A.4. **The case of ind-schemes.**

A.4.1. Let $\mathcal{Y}$ be an ind-scheme. Write

$$\mathcal{Y} = \text{“}\underset{i}{\operatorname{colim}}\text{”}\, Y_i, \tag{A.12}$$

where Y_i are schemes, and the transition maps $Y_i \overset{f_{i,j}}{\to} Y_j$ are closed embeddings, and the index category is filtered.

Recall that the category $\mathrm{D}^!(\mathcal{Y})$ is a priori defined, see Sect. A.2.1. Note, however, that the functor

$$\mathrm{D}^!(\mathcal{Y}) \to \lim_{i} \mathrm{D}^!(Y_i), \tag{A.13}$$

given by !-pullback is an equivalence, where the limit is formed also using the !-pullback functors.

A.4.2. We define

$$\mathrm{D}_*(\mathcal{Y}) := \underset{i}{\operatorname{colim}}\, \mathrm{D}_*(Y_i), \tag{A.14}$$

where the colimit is taken with respect to *-pushforwards.

It is clear that the definition does not depend on the presentation of $\mathcal{Y}$ as in Sect. A.12.

A.4.3. For a map between ind-schemes

$$f : \mathcal{Y}_1 \to \mathcal{Y}_2,$$

we obtain a well-defined functor

$$f_* : \mathrm{D}_*(\mathcal{Y}_1) \to \mathrm{D}_*(\mathcal{Y}_2).$$

In particular, we have a well-defined functor

$$\mathrm{C}^\cdot(\mathcal{Y}, -) : \mathrm{D}(\mathcal{Y}) \to \mathrm{D}(\mathrm{pt}) = \mathrm{Vect}\,.$$

Suppose now that f is schematic of finite presentation. In this case we also obtain a functor

$$f^! : \mathrm{D}_*(\mathcal{Y}_2) \to \mathrm{D}_*(\mathcal{Y}_1).$$

When f is a closed embedding, the functors

$$f_* : \mathrm{D}_*(\mathcal{Y}_1) \rightleftarrows \mathrm{D}(\mathcal{Y}_2) : f^!$$

form an adjoint pair.

A.4.4. We have a naturally defined action of $\mathrm{D}^!(\mathcal{Y})$ on $\mathrm{D}_*(\mathcal{Y})$.

We shall call an object

$$\omega_{\mathcal{Y}}^{\mathrm{fake},*} \in \mathrm{D}_*(\mathcal{Y})$$

a *fake* dualizing sheaf, if the functor

$$\mathrm{D}^!(\mathcal{Y}) \to \mathrm{D}_*(\mathcal{Y}), \quad \mathcal{F} \mapsto \mathcal{F} \overset{!}{\otimes} \omega_{\mathcal{Y}}^{\mathrm{fake},*}$$

is an equivalence.

A.4.5. Following [BD2, Sect. 7], we shall say that $\mathcal{Y}$ is *reasonable* if it admits a presentation (A.12), where the transition maps $f_{i,j}$ are of finite presentation.

If $\mathcal{Y}$ is reasonable, and Z is a closed subscheme of $\mathcal{Y}$, we shall say that Z is *reasonable* if for some/any i such that $Z \subset Y_i$, this closed embedding is of finite presentation.

Reasonable subschemes of $\mathcal{Y}$ form a filtered category. A presentation of $\mathcal{Y}$ as in (A.12) with Y_i reasonable will be called a *reasonable presentation*.

A.4.6. Let (A.12) be a reasonable presentation. In this case the functors

$$f_{i,j}^! : \mathrm{D}^!(Y_j) \to \mathrm{D}^!(Y_i)$$

admit left adjoints.

Hence, using the equivalence (A.13), we can rewrite

$$\mathrm{D}^!(\mathcal{Y}) \simeq \underset{i}{\mathrm{colim}}\, \mathrm{D}^!(Y_i), \tag{A.15}$$

where the colimit is taken with respect to *-pushforwards.

Similarly, in the formation of the colimit in (A.14), the transition functors $(f_{i,j})_*$ admit right adjoints. Hence, we can rewrite

$$\mathrm{D}^*(\mathcal{Y}) \simeq \lim_i \mathrm{D}^*(Y_i), \tag{A.16}$$

where the limit is formed using the !-pullback functors.

Using (A.15) and (A.16), we obtain that for a reasonable ind-scheme $\mathcal{Y}$, we still have an identification

$$\mathrm{Funct}_{\mathrm{DGCat}}(\mathrm{D}^!(\mathcal{Y}), \mathrm{Vect}) \simeq \mathrm{D}_*(\mathcal{Y}).$$

A.4.7. We shall say that an ind-scheme $\mathcal{Y}$ is *ind-placid* if it admits a reasonable presentations who terms are placid schemes.

Note that using (A.15) we obtain that in this case $\mathrm{D}^!(\mathcal{Y})$ is compactly generated (and hence dualizable) and $\mathrm{D}_*(\mathcal{Y})$ is also compactly generated, and we have

$$\mathrm{D}_*(\mathcal{Y}) \simeq (\mathrm{D}^!(\mathcal{Y}))^\vee.$$

A.5. **D-modules on the loop and arc groups.** In this section we let G be a connected affine algebraic group (i.e., we are not assuming G to be reductive).

For the duration of this section we will unburden the notation and replace

$$\mathfrak{L}^+(G)_{x_0} \rightsquigarrow \mathfrak{L}^+(G),\ \mathfrak{L}(G)_{x_0} \rightsquigarrow \mathfrak{L}(G),\ \mathrm{Gr}_{G,x_0} \rightsquigarrow \mathrm{Gr}_G.$$

A.5.1. Write

$$\mathfrak{L}^+(G) \simeq \lim_n \mathfrak{L}^+(G)/K_n.$$

This presentation exhibits $\mathfrak{L}^+(G)$ as a placid scheme. Thus we have:

$$\mathrm{D}^!(\mathfrak{L}^+(G)) \simeq \underset{n,(-)^!}{\mathrm{colim}}\, \mathrm{D}(\mathfrak{L}^+(G)/K_n)$$

and

$$\mathrm{D}_*(\mathfrak{L}^+(G)) \simeq \lim_{n,(-)_*} \mathrm{D}(\mathfrak{L}^+(G)/K_n) \simeq \underset{n,(-)^*}{\mathrm{colim}}\, \mathrm{D}(\mathfrak{L}^+(G)/K_n).$$

We have a canonical identification

$$\mathrm{D}_*(\mathfrak{L}^+(G)) \simeq (\mathrm{D}^!(\mathfrak{L}^+(G)))^\vee. \tag{A.17}$$

A.5.2. That said, by Sect. A.3.9, we have a canonical identification

$$\mathrm{D}^!(\mathfrak{L}^+(G)) \overset{\sim}{\to} \mathrm{D}_*(\mathfrak{L}^+(G)), \quad \omega_{\mathfrak{L}^+(G)} \mapsto \underline{k}_{\mathfrak{L}^+(G)}.$$

Using this identification, we will simply write

$$\mathrm{D}^!(\mathfrak{L}^+(G) =: \mathrm{D}(\mathfrak{L}^+(G)) := \mathrm{D}_*(\mathfrak{L}^+(G)).$$

A.5.3. The group structure on $\mathfrak{L}^+(G)$ makes $\mathrm{D}^!(\mathfrak{L}^+(G))$ into a commutative Hopf algebra in DGCat (using !-pullbacks), and it makes $\mathrm{D}_*(\mathfrak{L}^+(G))$ into a cocommutative Hopf algebra in DGCat (under *-pushforwards).

The counit and unit in $\mathrm{D}^!(\mathfrak{L}^+(G))$ are given by

$$\mathrm{D}^!(\mathfrak{L}^+(G)) \overset{\text{!-fiber at } 1}{\longrightarrow} \mathrm{D}(\mathrm{pt}) = \mathrm{Vect} \text{ and } \mathrm{Vect} \overset{k \mapsto \omega_{\mathfrak{L}^+(G)}}{\longrightarrow} \mathrm{D}^!(\mathfrak{L}^+(G)),$$

respectively.

The unit and counit in $\mathrm{D}_*(\mathfrak{L}^+(G))$ are given by

$$\mathrm{Vect} \overset{k\mapsto \delta_1}{\longrightarrow} \mathrm{D}_*(\mathfrak{L}^+(G)) \text{ and } \mathrm{D}_*(\mathfrak{L}^+(G)) \overset{\mathrm{C}^\cdot(\mathfrak{L}^+(G),-)}{\longrightarrow} \mathrm{Vect},$$

respectively, where δ_1 is the *-direct image of $k \in \mathrm{D}(\mathrm{pt})$ under the unit map $\mathrm{pt} \to \mathfrak{L}^+(G)$.

These two structures are obtained from one another by duality, using (A.17).

Remark A.5.4. When using the notation $\mathrm{D}(\mathfrak{L}^+(G))$, we will view it either as $\mathrm{D}^!(\mathfrak{L}^+(G))$ (in the comonoidal incarnation) or $\mathrm{D}_*(\mathfrak{L}^+(G))$ (in the monoidal incarnation), depending on the context.

A.5.5. We now consider the case of $\mathfrak{L}(G)$. First off, we claim that it is ind-placid as an ind-scheme. Indeed, write

$$\mathrm{Gr}_G = \text{“}\underset{i}{\mathrm{colim}}\text{”}\, Y_i,$$

where $Y_i \subset \mathrm{Gr}_G$ are closed $\mathfrak{L}^+(G)$-invariant subschemes.

Denote by $\widetilde{Y}_i$ the preimage of Y_i in $\mathfrak{L}(G)$. The closed embeddings

$$\widetilde{Y}_i \to \widetilde{Y}_j,$$

being obtained by base change from $Y_i \to Y_j$, are automatically of finite presentation.

We claim that each $\widetilde{Y}_i$ is placid. Indeed, we can write it as

$$\lim_n \widetilde{Y}_i/K_n, \tag{A.18}$$

and/or

$$\lim_n K_n\backslash\widetilde{Y}_i. \tag{A.19}$$

Note that these two inverse families are automatically equivalent: for every n there exists n' such that the projection

$$\widetilde{Y}_i \to \widetilde{Y}_i/K_n$$

factors as

$$\widetilde{Y}_i \to K_{n'}\backslash\widetilde{Y}_i \to \widetilde{Y}_i/K_n \tag{A.20}$$

and vice versa.

A.5.6. Hence, we obtain that the categories

$$\mathrm{D}^!(\mathfrak{L}(G)) \text{ and } \mathrm{D}_*(\mathfrak{L}(G))$$

are compactly generated and dual to each other.

Explicitly,

$$\mathrm{D}^!(\mathfrak{L}(G)) \simeq \lim_{i,(-)^!} \mathrm{D}^!(\widetilde{Y}_i) \simeq \underset{i,(-)_*}{\mathrm{colim}}\, \mathrm{D}^!(\widetilde{Y}_i),$$

while for every i,

$$\mathrm{D}^!(\widetilde{Y}_i) \simeq \underset{n,(-)^!}{\mathrm{colim}}\, \mathrm{D}(\widetilde{Y}_i/K_n),$$

and

$$\mathrm{D}_*(\mathfrak{L}(G)) \simeq \lim_{i,(-)^!} \mathrm{D}_*(\widetilde{Y}_i) \simeq \underset{i,(-)_*!}{\mathrm{colim}}\, \mathrm{D}_*(\widetilde{Y}_i),$$

while for every i,

$$\mathrm{D}_*(\widetilde{Y}_i) \simeq \lim_{n,(-)_*} \mathrm{D}(\widetilde{Y}_i/K_n) \simeq \operatorname{colim}_{n,(-)^*} \mathrm{D}(\widetilde{Y}_i/K_n).$$

In the above presentations of $\mathrm{D}^!(\widetilde{Y}_i)$ and $\mathrm{D}_*(\widetilde{Y}_i)$, one can replace the family $\widetilde{Y}_i/K_n$ by $K_n\backslash\widetilde{Y}_i$.

A.5.7. Note also that we have

$$\mathrm{D}^!(\mathfrak{L}(G)) \simeq \operatorname{colim}_{n,(-)^!} \mathrm{D}(\mathfrak{L}(G)/K_n) \tag{A.21}$$

and

$$\mathrm{D}_*(\mathfrak{L}(G)) \simeq \lim_{n,(-)_*} \mathrm{D}(\mathfrak{L}(G)/K_n) \simeq \operatorname{colim}_{n,(-)^*} \mathrm{D}(\mathfrak{L}(G)/K_n), \tag{A.22}$$

and we can also replace the family

$$n \mapsto \mathfrak{L}(G)/K_n$$

by

$$n \mapsto K_n\backslash\mathfrak{L}(G).$$

A.5.8. Let

$$\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)} \in \mathrm{D}_*(\mathfrak{L}(G))$$

be the object equal to the image of

$$\omega_{\mathfrak{L}(G)/K_n}[-2\dim(\mathfrak{L}^+(G)/K_n)] \in \mathrm{D}(\mathfrak{L}(G)/K_n)$$

for some/any n under the presentation

$$\mathrm{D}_*(\mathfrak{L}(G)) \simeq \operatorname{colim}_{n,(-)^*} \mathrm{D}(\mathfrak{L}(G)/K_n).$$

It is clear that $\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)}$ is indeed a fake dualizing sheaf, i.e., it gives rise to an equivalence

$$\mathrm{D}^!(\mathfrak{L}(G)) \to \mathrm{D}_*(\mathfrak{L}(G)), \quad \mathcal{F} \mapsto \mathcal{F} \overset{!}{\otimes} \omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)}.$$

We define

$$\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)} \in \mathrm{D}_*(\mathfrak{L}(G))$$

similarly, using the presentation

$$\mathrm{D}_*(\mathfrak{L}(G)) \simeq \operatorname{colim}_{n,(-)^*} \mathrm{D}(K_n\backslash\mathfrak{L}(G)).$$

A.5.9. Let $\mu : G \to \mathbb{G}_m$ be the modular character (i.e., the determinant of the adjoint action). Let $\deg(\mu)$ be the corresponding function

$$\pi_0(\mathfrak{L}(G)) \overset{\mu}{\to} \pi_0(\mathfrak{L}(\mathbb{G}_m)) \simeq \mathbb{Z}.$$

Proposition A.5.10. *The objects* $\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)}$ *and* $\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)}[2\deg(\mu)]$ *are canonically isomorphic.*

Proof. For every i, let

$$\omega^{\mathrm{fake},*,L}_{\widetilde{Y}_i} \text{ and } \omega^{\mathrm{fake},*,R}_{\widetilde{Y}_i} \tag{A.23}$$

be the objects of $\mathrm{D}_*(\widetilde{Y}_i)$ defined by a similar procedure, using the presentation of $\widetilde{Y}_i$ as (A.18) and (A.19), respectively.

It suffices to exhibit a compatible family of isomorphisms

$$\omega^{\mathrm{fake},*,L}_{\widetilde{Y}_i} \simeq \omega^{\mathrm{fake},*,R}_{\widetilde{Y}_i}[2\deg(\mu)]. \tag{A.24}$$

Note that the objects (A.23) are associated to the two dimension theories on $\widetilde{Y}_i$: one attaches to

$$\widetilde{Y}_i \to \widetilde{Y}_i/K_n$$

the integer $d_n := \dim(\mathfrak{L}^+(G)/K_n)$, and another attaches the same integer to

$$\widetilde{Y}_i \to K_n\backslash\widetilde{Y}_i.$$

It suffices to show that these two dimension theories are in fact equivalent up to the shift by $\deg(\mu)$.

To do this, fix an integer n and let n' be as in (A.20). The required equality follows from the fact that the resulting map

$$K_{n'}\backslash\widetilde{Y}_i \to \widetilde{Y}_i/K_n$$

is smooth of relative dimension $\dim(K_n/K_{n'}) + \deg(\mu)$.

$\square$

A.5.11. Assume that G is unimodular. In this case, using the identification of Proposition A.5.10, we will use the notation

$$\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)} =: \omega^{\mathrm{fake},*}_{\mathfrak{L}(G)} := \omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)}.$$

Thus, $\omega^{\mathrm{fake},*}_{\mathfrak{L}(G)}$ gives rise to a canonical identification

$$\mathrm{D}^!(\mathfrak{L}(G)) \simeq \mathrm{D}^*(\mathfrak{L}(G)). \tag{A.25}$$

We will use the notation:

$$\mathrm{D}^!(\mathfrak{L}(G)) =: \mathrm{D}(\mathfrak{L}(G)) := \mathrm{D}^*(\mathfrak{L}(G)).$$

A.5.12. The structure of group-object in ind-schemes defines on $\mathrm{D}^!(\mathfrak{L}(G))$ a structure of commutative Hopf algebra, and on $\mathrm{D}_*(\mathfrak{L}(G))$ a structure of cocommutative Hopf algebra.

These two structures are obtained from one another by duality.

Remark A.5.13. When using the notation $\mathrm{D}(\mathfrak{L}(G))$, we will view it either as $\mathrm{D}^!(\mathfrak{L}(G))$ (in the comonoidal incarnation) or $\mathrm{D}_*(\mathfrak{L}(G))$ (in the monoidal incarnation), depending on the context.

Appendix B. Categorical representations of (loop) groups

In this section we (re)collect some facts pertaining to the notion of action of a group on a category. We start with groups of finite type, and then develop the theory for the arc and loop groups.

B.1. **The case of groups of finite type.** In this subsection, we let H be an affine algebraic group (of finite type).

B.1.1. We consider $\mathrm{D}(H)$ as a monoidal category in DGCat under convolution. We set

$$H\text{-}\mathbf{mod} := \mathrm{D}(H)\text{-}\mathbf{mod}.$$

Note that Verdier duality identifies $\mathrm{D}(H)$ with its own dual. This allows us to view $\mathrm{D}(H)$ as comonoidal category via

$$\mathrm{D}(H) \xrightarrow{\mathrm{mult}^!} \mathrm{D}(H \times H) \simeq \mathrm{D}(H) \otimes \mathrm{D}(H).$$

We can tautologically interpret H-**mod** as $\mathrm{D}(H)$-**comod** for this structure.

B.1.2. Push-forward (resp., pullback) with respect to the diagonal map extends the above monoidal (resp., comonoidal) structure on DGCat to a structure of cocommutative (resp., commutative) Hopf algebra object in DGCat. This structure endows H-**mod** with a symmetric monoidal structure, compatible with the forgetful functor

$$H\text{-}\mathbf{mod} \to \mathrm{DGCat}\,. \tag{B.1}$$

The unit of this symmetric monoidal structure is a copy of Vect, equipped with the trivial action of H, i.e. the coaction map

$$\mathrm{Vect} \to \mathrm{D}(H) \otimes \mathrm{Vect} \simeq \mathrm{D}(H)$$

is the functor $k \mapsto \omega_H$.

The corresponding action map

$$\mathrm{D}(H) \simeq \mathrm{D}(H) \otimes \mathrm{Vect} \to \mathrm{Vect}$$

is the functor of de Rham cochains, denoted $\mathrm{C}^\cdot(H)$.

B.1.3. As is the case for modules over Hopf algebras, an object of H-**mod** is dualizable if and only if its image under (B.1) is dualizable, and the functor (B.1), being symmetric monoidal, commutes with duality.

B.1.4. Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be 1-morphism in H-**mod**, and suppose that F admits a left (resp., right) adjoint, when viewed as a 1-morphism in DGCat. We claim that this adjoint then exists in H-**mod**.

Indeed, a priori F^L (resp., F^R) will be op-lax (resp., lax) compatible with the action of $\mathrm{D}(H)$. However, it is easy to see that the fact that H is is a group (as opposed to a monoid) forces any op-lax (resp., lax) compatible functor between module categories to be strictly compatible.

Indeed, for a point S-point h of H and a lax-compatible functor $F : \mathbf{C}_1 \to \mathbf{C}_2$, the morphism

$$F \circ h \overset{\alpha_h}{\to} h \circ F$$

(viewed as a natural transformation beween functors $\mathbf{C}_1 \otimes \mathrm{D}(S) \rightrightarrows \mathbf{C}_2 \otimes \mathrm{D}(S)$) admits an inverse given by

$$h \circ F \simeq h \circ F \circ h^{-1} \circ h \overset{\alpha_{h^{-1}}}{\to} h \circ h^{-1} \circ F \circ h \simeq F \circ h,$$

and similarly for op-lax functors.

B.1.5. Note that the Verdier duality identification

$$\mathrm{D}(H)^\vee \simeq \mathrm{D}(H) \tag{B.2}$$

is compatible with the structure of object of $(H \times H)$-**mod** on the two sides.

In particular, the left and right adjoints of the forgetful functor (B.1) are canonically isomorphic.

Remark B.1.6. Note that we can also interpret (B.2) as follows: the functor

$$\mathrm{D}(H) \otimes \mathrm{D}(H) \overset{\text{convolution}}{\longrightarrow} \mathrm{D}(H) \overset{\text{counit}}{\longrightarrow} \mathrm{Vect}$$

is a duality pairing, which differs from the Verdier one by the inversion operation on H.

This pairing makes $\mathrm{D}(H)$ into a Frobenius algebra in DGCat.

B.1.7. For $\mathbf{C} \in H$-**mod**, set

$$\mathbf{C}^H := \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C}).$$

We have a pair of adjoint functors

$$\mathrm{oblv}_H : \mathbf{C}^H \rightleftarrows \mathbf{C} : \mathrm{Av}^H_* .$$

B.1.8. Consider the category Vect. The forgetful functor

$$\mathrm{oblv}_H : \mathrm{Vect}^H \to \mathrm{Vect}$$

is comonadic with the comonad in question is given by tensor product with $\mathrm{C}^\cdot(H)$, on which the coalgebra structure is induced by the group structure on H.

Equivalently,

$$\mathrm{Vect}^H \simeq \mathrm{C}_\cdot(H)\text{-mod},$$

where $\mathrm{C}_\cdot(H)$ is an algebra via the group structure on H.

The category Vect^H carries a natural (symmetric) monoidal structure. It corresponds to the structure on $\mathrm{C}^\cdot(H)$ (resp., $\mathrm{C}_\cdot(H)$) of commutative (resp., cocommutative) Hopf algebra.

B.1.9. For any $\mathbf{C}$, the category $\mathbf{C}^H$ is naturally tensored over Vect^H. The comonad

$$\mathrm{oblv}_H \circ \mathrm{Av}^H_*$$

on $\mathbf{C}^H$ is given by tensoring with the coalgebra object

$$\mathrm{Av}^H_*(k) \in \mathrm{Vect}^H .$$

Note that the underlying object of Vect, i.e., $\mathrm{oblv}_H \circ \mathrm{Av}^H_*(k)$, identifies with $\mathrm{C}^\cdot(H)$.

In particular, this implies that the functor $\mathrm{oblv}_H : \mathbf{C}^H \to \mathbf{C}$ is fully faithful if H is unipotent.

B.1.10. For $\mathbf{C} \in H\text{-}\mathbf{mod}$, let $\mathbf{C}_H \in \mathrm{DGCat}$ be the object such that

$$\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}_H, \mathbf{C}_0) = \mathrm{Funct}_{H\text{-}\mathbf{mod}}(\mathbf{C}, \mathbf{C}_0), \quad \mathbf{C}_0 \in \mathrm{DGCat},$$

where H acts trivially on $\mathbf{C}_0$.

Note that the functor

$$\mathrm{Av}_*^H : \mathbf{C} \to \mathbf{C}^H$$

is H-invariant, and hence gives rise to a functor

$$\mathbf{C}_H \to \mathbf{C}^H. \tag{B.3}$$

B.1.11. We have the following basic assertion, see, e.g., [Ga4, Theorem B.1.2]:

Proposition B.1.12. *The functor* (B.3) *is an equivalence.*

Corollary B.1.13. *Let $\mathbf{C} \in H\text{-}\mathbf{mod}$ be dualizable as a DG category. Then $\mathbf{C}^H$ is also dualizable and we have a canonical equivalence*

$$(\mathbf{C}^H)^\vee \simeq (\mathbf{C}^\vee)^H.$$

Corollary B.1.14. *For $\mathbf{C}_1, \mathbf{C}_2 \in H\text{-}\mathbf{mod}$, we have a canonical equivalence*

$$\mathbf{C}_1 \underset{\mathrm{D}(H)}{\otimes} \mathbf{C}_2 \simeq (\mathbf{C}_1 \otimes \mathbf{C}_2)^H.$$

Corollary B.1.15. *The functor*

$$\mathbf{C} \mapsto \mathbf{C}^H, \quad H\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

commutes with colimits and tensor products by objects of DGCat.

B.2. **The case of arc groups.** In this and the next subsections we keep the notational change from Sect. A.5.

B.2.1. We set

$$\mathfrak{L}^+(G)\text{-}\mathbf{mod} := \mathrm{D}^!(\mathfrak{L}^+(G))\text{-}\mathbf{comod} \simeq \mathrm{D}_*(\mathfrak{L}^+(G))\text{-}\mathbf{mod}.$$

The entire discussion in Sect. B.1 is applicable in this case. Moreover, as we shall see below, the study of $\mathfrak{L}^+(G)$-**mod** reduces to that of H-**mod** for finite-dimensional quotients $\mathfrak{L}^+(G) \rightrightarrows H$.

B.2.2. For $\mathbf{C} \in \mathfrak{L}^+(G)\text{-}\mathbf{mod}$ and an integer n, let $\mathbf{e}_n$ be the endofunctor of $\mathbf{C}$ equal to $\mathrm{oblv}_{K_n} \circ \mathrm{Av}_*^{K_n}$.

Note that $\mathbf{e}_n$ can be thought of as the action of

$$\underline{k}_{K_n} \in \mathrm{D}_*(\mathfrak{L}^+(G)).$$

The essential image of $\mathbf{e}_n$ is the *full subcategory*

$$\mathbf{C}^{K_n} \subset \mathbf{C}.$$

Since K_n is normal in $\mathfrak{L}^+(G)$, the category $\mathbf{C}^{K_n}$ is stable under the $\mathfrak{L}^+(G)$-action, i.e., forms a subobject in $\mathfrak{L}^+(G)$-**mod**. Moreover, the action of $\mathfrak{L}^+(G)$ on $\mathbf{C}^{K_n}$ factors through $\mathfrak{L}^+(G)/K_n$.

B.2.3. Consider the colimit

$$\underset{n}{\mathrm{colim}}\, \mathbf{C}^{K_n}, \tag{B.4}$$

formed in $\mathfrak{L}^+(G)$-**mod**, where the transition functors are the natural inclusions.

Since the index family (i.e., $\mathbf{Z}^{\geq 0}$) is filtered, the underlying DG category of (B.4) is a similar colimit taken in DGCat.

Note also that since for $n_1 \leq n_2$, the embedding

$$\mathbf{C}^{K_{n_1}} \hookrightarrow \mathbf{C}^{K_{n_2}}$$

admits a right adjoint, we can rewrite (B.4) also as

$$\lim_n \mathbf{C}^{K_n}, \tag{B.5}$$

where the limit is taken with respect to the above averaging functors.

B.2.4. We have a tautologically defined map in $\mathfrak{L}^+(G)$-**mod**

$$\underset{n}{\operatorname{colim}}\, \mathbf{C}^{K_n} \to \mathbf{C}. \tag{B.6}$$

We claim:

Proposition B.2.5. *The map* (B.6) *is an isomorphism.*

Proof. Since the forgetful functor

$$\mathfrak{L}^+(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

is conservative, it suffices to show that the functor (B.6) is an equivalence of DG categories.

Since the category of indices is filtered, and each $\mathbf{C}^{K_n} \to \mathbf{C}$ is fully faithful admitting a colimit-preserving right adjoint, it follows automatically that the functor (B.6) is fully faithful.

Hence, it remains to show that it is essentially surjective. I.e., it suffices to exhibit any object of $\mathbf{C}$ as a colimit of objects belonging to $\mathbf{C}^{K_n}$ for some n. In fact, we claim that for any $\mathbf{c} \in \mathbf{C}$, the naturally defined map

$$\underset{n}{\operatorname{colim}}\, \mathbf{e}_n(\mathbf{c}) \to \mathbf{C} \tag{B.7}$$

is an isomorphism.

Indeed, (B.7) follows from the isomorphism

$$\underset{n}{\operatorname{colim}}\, \underline{k}_{K_n} \simeq \delta_1$$

in $\mathrm{D}_*(\mathfrak{L}^+(G))$.

$\square$

B.3. **The case of loop groups.** We finally consider the case of $\mathfrak{L}(G)$, and define what we mean by categories acted on by it.

B.3.1. We define

$$\mathfrak{L}(G)\text{-}\mathbf{mod} := \mathrm{D}^!(\mathfrak{L}(G))\text{-}\mathbf{comod} = \mathrm{D}_*(\mathfrak{L}(G))\text{-}\mathbf{mod}.$$

Remarks as in Sect. B.1.2-B.1.4 apply equally well to $\mathfrak{L}(G)$-**mod**.

B.3.2. For $\mathbf{C} \in \mathfrak{L}(G)$-**mod**, set

$$\mathbf{C}^{\mathfrak{L}(G)} := \mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathrm{Vect}, \mathbf{C}).$$

In particular, we can consider the (symmetric monoidal) category $\mathrm{Vect}^{\mathfrak{L}(G)}$.

B.3.3. We claim:

Lemma-Construction B.3.4. *There is a canonical equivalence of symmetric monoidal categories*

$$\mathrm{Vect}^{\mathfrak{L}(G)} \simeq \mathrm{C}_\cdot(\mathfrak{L}(G))\text{-mod},$$

compatible with the forgeftul functors to Vect.

Proof. For $V \in \mathrm{Vect}$, its lift to an object of $\mathrm{Vect}^{\mathfrak{L}(G)}$ is a datum of structure of an object of

$$\mathrm{Tot}(\mathrm{D}^!(\mathfrak{L}(G)^\bullet))$$

with terms $\omega_{\mathfrak{L}(G)^\bullet} \otimes V$, where $\mathfrak{L}(G)^\bullet$ is the simplicial ind-scheme constructed out of $\mathfrak{L}(G)$, viewed as a group-object in ind-schemes.

Such a datum amounts to an isomorphism

$$\omega_{\mathfrak{L}(G)} \otimes V \simeq \omega_{\mathfrak{L}(G)} \otimes V, \tag{B.8}$$

in $\mathrm{D}^!(\mathfrak{L}(G))$, satisfying the natural associativity conditions.

The map $\to$ in (B.8) gives rise by adjunction to a map

$$\mathrm{C}_\cdot(\mathfrak{L}(G)) \otimes V = \mathrm{C}_c^\cdot(\mathfrak{L}(G), \omega_{\mathfrak{L}(G)}) \otimes V \to V, \tag{B.9}$$

and the associativity datum on (B.8) is equivalent to the associativity datum for (B.9).

Vice versa, starting from (B.9) we construct a map → (B.8), and the associativity for (B.9) implies that (B.8) is automtically an isomorphism.

□

B.3.5. Consider $\mathrm{D}_*(\mathfrak{L}(G))$ as an object of $(\mathfrak{L}(G)\times\mathfrak{L}(G))$-**mod**. This can be either done by viewing $\mathrm{D}_*(\mathfrak{L}(G))$ as a bimodule over itself. Or, equivalently, we can view $\mathrm{D}_*(\mathfrak{L}(G))$ as the dual (inside $(\mathfrak{L}(G)\times\mathfrak{L}(G))$-**mod**) of $\mathrm{D}^!(\mathfrak{L}(G))$, viewed as a bi-comodule over itself.

Let us now assume that G is unimodular. Recall the object

$$\omega^{\mathrm{fake},*}_{\mathfrak{L}(G)}\in \mathrm{D}_*(\mathfrak{L}(G)),$$

see Sect. A.5.11.

Its interpretation as $\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)}$ implies that it is naturally an object of

$$\mathrm{D}_*(\mathfrak{L}(G))^{\mathfrak{L}(G)-L},$$

where the superscript "L" refers to the left action of $\mathfrak{L}(G)$ on $\mathrm{D}_*(\mathfrak{L}(G))$.

Similarly, its interpretation as $\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)}$ implies that it is naturally an object of

$$\mathrm{D}_*(\mathfrak{L}(G))^{\mathfrak{L}(G)-R}.$$

However, we claim:

Lemma B.3.6. *The object $\omega^{\mathrm{fake},*}_{\mathfrak{L}(G)}$ naturally lifts to an object of*

$$\mathrm{D}_*(\mathfrak{L}(G))^{\mathfrak{L}(G)\times\mathfrak{L}(G)}.$$

Proof. We have to show that for points $g_1,g_2\in\mathfrak{L}(G)$, the diagram

$$\begin{tikzcd}
(g_1\cdot\omega^{\mathrm{fake},*}_{\mathfrak{L}(G)})\cdot g_2 \arrow[r, "\sim"] \arrow[d, "\sim"'] & g_1\cdot(\omega^{\mathrm{fake},*}_{\mathfrak{L}(G)})\cdot g_2) \arrow[d, "\sim"] \\
(g_1\cdot\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)})\cdot g_2 \arrow[d, "\sim"'] & g_1\cdot(\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)})\cdot g_2) \arrow[d, "\sim"] \\
\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)}\cdot g_2 \arrow[d, "\sim"'] & g_1\cdot\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)} \arrow[d, "\sim"] \\
\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)}\cdot g_2 \arrow[d, "\sim"'] & g_1\cdot\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)} \arrow[d, "\sim"] \\
\omega^{\mathrm{fake},*,R}_{\mathfrak{L}(G)} \arrow[r] & \omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)}
\end{tikzcd} \tag{B.10}$$

commutes, along with the higher compatibilities.

First, we claim that the higher compatibilities hold automatically, since

$$\mathcal{H}om(\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)},\omega^{\mathrm{fake},*,L}_{\mathfrak{L}(G)})\simeq \mathrm{C}^\cdot(\mathfrak{L}(G))$$

is coconnective.

Hence, we only need to check the commutation of (B.10) up to homotopy. However, the latter follows immediately from the construction of the isomorphism of Proposition A.5.10.

□

As a corollary, we obtain:

Corollary B.3.7. *The isomorphism* (A.25) *lifts to an identification of objects in* $(\mathfrak{L}(G)\times\mathfrak{L}(G))$-**mod**.

Corollary B.3.8. *The left and right adjoints to the forgetful functor*

$$\mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

are canonically isomorphic.

Remark B.3.9. Remark B.1.6 applies here as well for the pairing

$$\mathrm{D}_*(\mathfrak{L}(G)) \otimes \mathrm{D}_*(\mathfrak{L}(G)) \overset{\text{convolution}}{\longrightarrow} \mathrm{D}_*(\mathfrak{L}(G)) \simeq \mathrm{D}^!(\mathfrak{L}(G)) \overset{\text{counit}}{\longrightarrow} \mathrm{Vect}\,.$$

This pairing makes $\mathrm{D}_*(\mathfrak{L}(G))$ into a Frobenius algebra in DGCat.

B.4. **The case of a reductive G.** Up to now, the discussion of $\mathfrak{L}(G)$ was valid for *any* affine algebraic group G. We will now assume that G is reductive, and exploit some special features of this case. The key to what we are about to say is that for G reductive, the affine Grassmannian

$$\mathrm{Gr}_G := \mathfrak{L}(G)/\mathfrak{L}^+(G)$$

is ind-proper.

B.4.1. We claim:

Proposition B.4.2. *The forgetful functor*

$$\mathrm{oblv}_{\mathfrak{L}(G)\to\mathfrak{L}^+(G)} : \mathbf{C}^{\mathfrak{L}(G)} \to \mathbf{C}^{\mathfrak{L}^+(G)}$$

admits a left adjoint, to be denoted $\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}$.

Proof. Note that we can interpret $\mathbf{C}^{\mathfrak{L}^+(G)}$ as

$$(\mathrm{D}(\mathrm{Gr}_G) \otimes \mathbf{C})^{\mathfrak{L}(G)}.$$

Hence, it suffices to show that the functor

$$\mathrm{Vect} \to \mathrm{D}(\mathrm{Gr}_G), \quad k \mapsto \omega_{\mathrm{Gr}_G}$$

admits a left adjoint in $\mathfrak{L}(G)$-**mod**, or equivalently, in Vect.

However, the left adjoint in question is provided by $\mathrm{C}^\cdot(\mathrm{Gr}_G, -)$.

□

B.4.3. Set

$$\mathrm{Sph}_G := \mathrm{D}_*(\mathfrak{L}(G))^{\mathfrak{L}^+(G)\times\mathfrak{L}^+(G)}.$$

The usual Hecke algebra construction shows that the monoidal structure on $\mathrm{D}_*(\mathfrak{L}(G))$ induces one on Sph_G. Moreover, for every $\mathbf{C} \in \mathfrak{L}(G)$-**mod**, we have a canonical action of Sph_G on $\mathbf{C}^{\mathfrak{L}^+G}$.

B.4.4. Note that we can consider $\omega^{\text{fake},*}_{\mathfrak{L}(G)}$ as an associative algebra object of Sph_G; when viewed as such, let us denote it by $\omega^{\text{fake},*}_{\text{Hecke}}$.

Unwinding, we obtain that the monad

$$\mathrm{oblv}_{\mathfrak{L}(G)\to\mathfrak{L}^+(G)} \circ \mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)} \tag{B.11}$$

acting on $\mathbf{C}^{\mathfrak{L}^+(G)}$ is given by the action of $\omega^{\text{fake},*}_{\text{Hecke}}$.

Therefore, by the Barr-Beck-Lurie theorem, we can identify

$$\mathbf{C}^{\mathfrak{L}(G)} \simeq \omega^{\text{fake},*}_{\text{Hecke}}\text{-mod}(\mathbf{C}^{\mathfrak{L}^+(G)}).$$

B.4.5. Let $\mathbf{C}$ be an object $\mathfrak{L}(G)$-**mod**. Let $\mathbf{C}_{\mathfrak{L}(G)}$ be the object of DGCat such that

$$\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}_{\mathfrak{L}(G)}, \mathbf{C}_0) \simeq \mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathbf{C}, \mathbf{C}_0), \quad \mathbf{C}_0 \in \mathrm{DGCat},$$

where in the right-hand side, $\mathbf{C}_0$ is considered as equipped with the trivial $\mathfrak{L}(G)$-action.

B.4.6. For $\mathbf{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$ and $\mathbf{C}_0 \in \mathrm{DGCat}$, let us view $\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}, \mathbf{C}_0)$ as an object of $\mathfrak{L}(G)\text{-}\mathbf{mod}$ via the action on the source.

By Proposition B.3 (applied to $\mathfrak{L}^+(G)$), we have

$$\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}, \mathbf{C}_0)^{\mathfrak{L}^+(G)} \simeq \mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}^{\mathfrak{L}^+(G)}, \mathbf{C}_0). \tag{B.12}$$

Unwinding, we obtain that the action of $\omega^{\mathrm{fake},*}_{\mathrm{Hecke}}$ on the left-hand side corresponds to the action of $\omega^{\mathrm{fake},*}_{\mathrm{Hecke}}$ on the source in the right-hand side.

B.4.7. Take $\mathbf{C}_0 = \mathbf{C}^{\mathfrak{L}(G)}$, and consider the object of the category $\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}^{\mathfrak{L}^+(G)}, \mathbf{C}^{\mathfrak{L}(G)})$ given by $\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}$. Since the functor $\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}$ is acted on by the monad (B.11), we obtain that the above object lifts to an object of

$$\omega^{\mathrm{fake},*}_{\mathrm{Hecke}}\text{-mod}(\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}^{\mathfrak{L}^+(G)}, \mathbf{C}_0)) \simeq$$
$$\simeq \omega^{\mathrm{fake},*}_{\mathrm{Hecke}}\text{-mod}(\mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}, \mathbf{C}_0)^{\mathfrak{L}^+(G)}) \simeq \mathrm{Funct}_{\mathrm{DGCat}}(\mathbf{C}, \mathbf{C}_0)^{\mathfrak{L}(G)}.$$

Thus, the functor $\mathrm{Av}_!^{\mathfrak{L}^+(G)\to\mathfrak{L}(G)}$ may be viewed as a point in

$$\mathrm{Funct}_{\mathfrak{L}(G)\text{-}\mathbf{mod}}(\mathbf{C}, \mathbf{C}^{\mathfrak{L}(G)}) \simeq \mathrm{Funct}(\mathbf{C}_{\mathfrak{L}(G)}, \mathbf{C}^{\mathfrak{L}(G)}).$$

B.4.8. We have (see [Ga4, Theorem D.1.4(b)]):

Proposition B.4.9. *The above functor*

$$\mathbf{C}_{\mathfrak{L}(G)} \to \mathbf{C}^{\mathfrak{L}(G)}.$$

is an equivalence.

B.4.10. From Proposition B.4.9 we obtain:

Corollary B.4.11. *Let $\mathbf{C} \in \mathfrak{L}(G)\text{-}\mathbf{mod}$ be dualizable as a category. Then $\mathbf{C}^{\mathfrak{L}(G)}$ is also dualizable and we have a canonical equivalence*

$$(\mathbf{C}^{\mathfrak{L}(G)})^\vee \simeq (\mathbf{C}^\vee)^{\mathfrak{L}(G)}.$$

Corollary B.4.12. *For $\mathbf{C}_1, \mathbf{C}_2 \in \mathfrak{L}(G)\text{-}\mathbf{mod}$, we have a canonical isomorphism*

$$\mathbf{C}_1 \underset{\mathrm{D}_*(\mathfrak{L}(G))}{\otimes} \mathbf{C}_2 \simeq (\mathbf{C}_1 \otimes \mathbf{C}_2)^{\mathfrak{L}(G)}.$$

Corollary B.4.13. *The functor*

$$\mathbf{C} \mapsto \mathbf{C}^{\mathfrak{L}(G)}, \quad \mathfrak{L}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}$$

commutes with colimits and tensor products by objects of DGCat.

Remark B.4.14. One can rewrite the natural transformation in Proposition B.4.9 also as follows.

Let G be an arbitrary unimodular group. As is the case for an arbitrary augmented algebra A, which is isomorphic to its own dual as a bimodule, there is a canonical natural transformation

$$M_A \to M^A, \quad M \in A\text{-mod}. \tag{B.13}$$

Unwinding, one can see that (B.14) is the map (B.14) for $A := \mathrm{D}_*(\mathfrak{L}(G)) \in \mathrm{DGCat}$.
Note, however, that the resulting natural transformation

$$\mathbf{C}_{\mathfrak{L}(G)} \to \mathbf{C}^{\mathfrak{L}(G)} \tag{B.14}$$

will *not* in general be an isomorphism, unless G is reductive.

For example, if $G = N$ is unipotent and $\mathbf{C} = \mathrm{Vect}$, the natural transformation (B.14) will be zero.

Appendix C. Factorization categories and modules

Most of the material in this appendix, except Sect. C.11 - Sect. C.13, follows [GLC2, Sect. C]. The reader is referred to *loc.cit.* for more details.

C.1. **Unital factorization spaces.**

C.1.1. A *categorical prestack* is a functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Cat}\,.$$

Let

$$\mathrm{CatPreStk} := \mathrm{Fun}((\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathrm{Cat})$$

be the *2-category* of categorical prestacks.

C.1.2. Given a categorical prestack $\mathcal{Y}$, its value at an affine scheme S is denoted by $\mathcal{Y}(S)$. By Yoneda's lemma, objects in $\mathcal{Y}(S)$ are identified with morphisms $S \to \mathcal{Y}$, while morphisms in $\mathcal{Y}(S)$ can be identified with 2-morphisms

$$\begin{tikzcd} S \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D}] & \mathcal{Y} \arrow[Rightarrow, from=U, to=D, "\varphi"] \end{tikzcd}$$

in CatPreStk.

C.1.3. The *unital Ran space* is the categorical prestack $\mathrm{Ran}^{\mathrm{untl}}$ that attaches to an affine test scheme S the *category* of finite subsets of $\mathrm{Hom}(S, X)$, where morphisms are given by inclusions of subsets.

In particular, a k-point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$ is just a finite subset of closed points of X.

C.1.4. *Remark.* We warn the readers that we do not require $\underline{x}$ to be nonempty. In particular, there is a canonical point $\emptyset \in \mathrm{Ran}^{\mathrm{untl}}$ corresponding to the empty subset. See Sect. C.13 for the reason we make this choice.

Note that $\mathrm{Ran}^{\mathrm{untl}}$ is denoted $\mathrm{Ran}^{\mathrm{untl},*}$ in [GLC2, Sect. C.5].

C.1.5. Let $\mathcal{Y}$ be a categorical prestack. A *coCartesian space over* $\mathcal{Y}$ is a categorical prestack $\mathcal{Z} \to \mathcal{Y}$ such that for any affine scheme S, the functor $\mathcal{Z}(S) \to \mathcal{Y}(S)$ is a left fibration, i.e., a coCartesian fibration in groupoids.

Dually, we define the notion of *Cartesian spaces over* $\mathcal{Y}$.

C.1.6. *Remark.* Roughly speaking, a coCartesian space $\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ is an assignment as follows.

- For any point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$, a (usual) prestack $\mathcal{T}_{\underline{x}}$;
- For any $\underline{x} \subseteq \underline{x}'$, a morphism $\mathcal{T}_{\underline{x}} \to \mathcal{T}_{\underline{x}'}$ that is compatible with compositions.

The above data should depend "algebraically" on $\underline{x}$ and $\underline{x}'$.

C.1.7. *Example.* We have a coCartesian space $\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ such that $\mathrm{Gr}_{G,\underline{x}}$ is as in Sect. 1.2.4, while the morphism

$$\mathrm{Gr}_{G,\underline{x}} \to \mathrm{Gr}_{G,\underline{x}'}$$

(for $\underline{x} \subseteq \underline{x}'$) sends $(\mathcal{P}_G^{\mathrm{glob}}, \beta)$ to $(\mathcal{P}_G^{\mathrm{glob}}, \beta|_{X \setminus \underline{x}'})$.

C.1.8. *Example.* We have a Cartesian space $\mathfrak{L}^+(G)_{\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ such that $\mathfrak{L}^+(G)_{\underline{x}} := G(\mathcal{D}_{\underline{x}})$, while the morphism

$$\mathfrak{L}^+(G)_{\underline{x}'} \to \mathfrak{L}^+(G)_{\underline{x}}$$

(for $\underline{x} \subseteq \underline{x}'$) is induced by restriction along $\mathcal{D}_{\underline{x}} \subseteq \mathcal{D}_{\underline{x}'}$.

C.1.9. Let $\mathcal{Y}$ be a categorical prestack. There is a common generalization of coCartesian spaces and Cartesian spaces over $\mathcal{Y}$. A *corr-space over* $\mathcal{Y}$ is a categorical prestack $\mathcal{Z} \to \mathcal{Y}$ such that for any affine scheme S, the functor $\mathcal{Z}(S) \to \mathcal{Y}(S)$ is a *fibration in correspondences* (see [GLC2, C.10]).

C.1.10. *Remark.* Roughly speaking, a corr-space $\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ is an assignment as follows.

- For any point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$, a (usual) prestack $\mathcal{T}_{\underline{x}}$;
- For any $\underline{x} \subseteq \underline{x}'$, a correspondence $\mathcal{T}_{\underline{x}} \leftarrow \mathcal{T}_{\underline{x}\subseteq\underline{x}'} \to \mathcal{T}_{\underline{x}'}$ or prestacks that is compatible with compositions.

In particular, we have

$$\mathcal{T}_{\underline{x}} \simeq \mathcal{T}_{\underline{x}\subseteq\underline{x}}$$

and

$$\mathcal{T}_{\underline{x}\subseteq\underline{x}'} \times_{\mathcal{T}_{\underline{x}'}} \mathcal{T}_{\underline{x}'\subseteq\underline{x}''} \simeq \mathcal{T}_{\underline{x}\subseteq\underline{x}''}.$$

The corr-space $\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}}$ is coCartesian (resp. Cartesian) over $\mathrm{Ran}^{\mathrm{untl}}$ when the morphisms $\mathcal{T}_{\underline{x}} \leftarrow \mathcal{T}_{\underline{x}\subseteq\underline{x}'}$ (resp. $\mathcal{T}_{\underline{x}\subseteq\underline{x}'} \to \mathcal{T}_{\underline{x}'}$) are invertible.

C.1.11. *Example.* We have a corr-space $\mathfrak{L}(G)_{\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ such that $\mathfrak{L}(G)_{\underline{x}} := G(\overset{\circ}{\mathcal{D}}_{\underline{x}})$, $\mathfrak{L}(G)_{\underline{x}\subseteq\underline{x}'} := G(\mathcal{D}_{\underline{x}'} \setminus \underline{x})$, while the correspondence

$$\mathfrak{L}(G)_{\underline{x}} \leftarrow \mathfrak{L}(G)_{\underline{x}\subseteq\underline{x}'} \to \mathfrak{L}(G)_{\underline{x}'}$$

(for $\underline{x} \subseteq \underline{x}'$) is induced by restriction along

$$\overset{\circ}{\mathcal{D}}_{\underline{x}} \to \mathcal{D}_{\underline{x}'} \setminus \underline{x} \leftarrow \overset{\circ}{\mathcal{D}}_{\underline{x}'}.$$

C.1.12. Note that $\mathrm{Ran}^{\mathrm{untl}}$ is an abelian monoid object in CatPreStk, with the addition map given by

$$\mathrm{union} : \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}},\ (\underline{x}, \underline{y}) \mapsto \underline{x} \cup \underline{y}.$$

For any finite set I, we obtain a map

$$\mathrm{union}_I : \prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}} \to \mathrm{Ran}^{\mathrm{untl}}.$$

Note that when $I = \emptyset$, this is the map $\emptyset : \mathrm{pt} \to \mathrm{Ran}^{\mathrm{untl}}$.

C.1.13. As in the non-unital case, we consider the disjoint locus

$$(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}})_{\mathrm{disj}} \subseteq \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}},$$

which sends an affine scheme S to the category

$$\{(\underline{x}, \underline{y}) \in (\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}})(S) \mid \mathrm{Graph}_{\underline{x}} \cap \mathrm{Graph}_{\underline{y}} = \emptyset\}.$$

Similarly, for any finite set I, we can define

$$\big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}} \subseteq \prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}.$$

C.1.14. Now a *unital* (resp. *counital*, *corr-unital*) factorization space $\mathcal{T}$ is a *coCartesian space* (resp. *Cartesian space*, *corr-space*) $\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ equipped with a *multiplicative structure over the disjoint loci.* In other words, for any finite set I, we have an isomorphism

$$\mathrm{union_I}^{-1}(\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}})|_{\left(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}\right)_{\mathrm{disj}}} \simeq \big(\prod_{i\in I} \mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}}\big)|_{\left(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}\right)_{\mathrm{disj}}}$$

and a homotopy-coherent datum of associativity and commutativity. Here union_I^{-1} is the change-of-base along the map union_I.

C.1.15. *Remark.* Roughly speaking, a corr-unital factorization space $\mathcal{T}$ consists of the following data

- Prestacks $\mathcal{T}_{\underline{x}}$ and correspondences $\mathcal{T}_{\underline{x}} \leftarrow \mathcal{T}_{\underline{x}\subseteq\underline{x}'} \to \mathcal{T}_{\underline{x}'}$ as in Sect. C.1.10;
- For any finite collection $\underline{x}_i \subseteq \underline{x}'_i$ $(i \in I)$ such that $\underline{x}'_i \cap \underline{x}'_j = \emptyset$ $(i \neq j)$, ca ommutative diagram

$$\begin{tikzcd}
\prod_{i\in I} \mathcal{T}_{\underline{x}_i} \arrow[d, "\mathrm{mult}_{(\underline{x}_i)}"'] & \prod_{i\in I} \mathcal{T}_{\underline{x}_i\subseteq\underline{x}'_i} \arrow[l] \arrow[r] \arrow[d, "\mathrm{mult}_{(\underline{x}_i\subseteq\underline{x}'_i)}"] & \prod_{i\in I} \mathcal{T}_{\underline{x}'_i} \arrow[d, "\mathrm{mult}_{(\underline{x}'_i)}"] \\
\mathcal{T}_{\sqcup\underline{x}_i} & \mathcal{T}_{\sqcup\underline{x}_i\subseteq\sqcup\underline{x}'_i} \arrow[l] \arrow[r] & \mathcal{T}_{\sqcup\underline{x}'_i}
\end{tikzcd}$$

that is compatible with compositions.

Such a $\mathcal{T}$ is unital (resp. counital) means when the leftward (resp. rightward) morphisms are invertible.

C.1.16. *Example.* The corr-space $\mathfrak{L}(G)_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{Ran}^{\mathrm{untl}}$ has a natural corr-unital factorization structure. We denote the resulting corr-unital factorization space by $\mathfrak{L}(G)$.

Similarly, we have a unital factorization space Gr_G and a counital factorization space $\mathfrak{L}^+(G)$.

C.2. **Crystal of categories over the unital Ran space.**

C.2.1. Given a categorical prestack $\mathcal{Y}$, a *crystal of categories* $\underline{\mathbf{C}}$ over $\mathcal{Y}$ is an assignment as follows.

- For an affine scheme S and a morphism $y : S \to \mathcal{Y}$, assign a $\mathrm{D}(S)$-module category $\mathbf{C}_y$;
- For affine schemes S_1, S_2 and a 2-morphism

$$\begin{tikzcd}
 & S_1 \arrow[dr, "y_1"] \arrow[d, Rightarrow, "\alpha"] & \\
S_2 \arrow[ur, "f"] \arrow[rr, "y_2"'] & {} & \mathcal{Y},
\end{tikzcd} \tag{C.1}$$

(i.e. a morphism $\alpha : y_1 \circ f \to y_2$ in $\mathcal{Y}(S_2)$), assign a functor

$$\mathbf{C}_\alpha : \mathbf{C}_{y_1} \to \mathbf{C}_{y_2}$$

intertwining the action of $f^! : \mathrm{D}(S_1) \to \mathrm{D}(S_2)$, such that the induced functor

$$\mathbf{C}_{y_1} \otimes_{\mathrm{D}(S_1)} \mathrm{D}(S_2) \to \mathbf{C}_{y_2}$$

is an equivalence when α is invertible.
- A homotopy-coherent system of compatibilities for compositions.

C.2.2. *Remark.* Let

$$\widetilde{\mathcal{Y}} \to (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$$

be the coCartesian fibration corresponding to the functor $\mathcal{Y}$. Let

$$\widetilde{\mathbf{CrysCat}} \to (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$$

be the coCartesian fibration of 2-categories corresponding to the functor $S \mapsto \mathrm{D}(S)$-mod. Then a crystal of categories $\underline{\mathbf{C}}$ over $\mathcal{Y}$ is *defined* to be a $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$-functor $\widetilde{\mathcal{Y}} \to \widetilde{\mathbf{CrysCat}}$ preserving coCartesian arrows.

C.2.3. *Example.* We have the *constant crystal of categories* $\underline{\mathrm{D}}(\mathcal{Y})$ over $\mathcal{Y}$ which assignes to $y \in \mathcal{Y}(S)$ the cateogry $\mathrm{D}(S)$ and assignes to a 2-morphism α the functor $f^! : \mathrm{D}(S_1) \to \mathrm{D}(S_2)$.

C.2.4. *Remark.* Roughly speaking, a crystal of categories $\mathbf{A}$ over $\mathrm{Ran}^{\mathrm{untl}}$ is an assignment as follows.

- For any point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$, assign a DG category $\mathbf{A}_{\underline{x}}$;
- For any inclusion $\underline{x} \subseteq \underline{x}'$, assign a functor

$$\mathrm{ins}_{\underline{x}\subseteq\underline{x}'} : \mathbf{A}_{\underline{x}} \to \mathbf{A}_{\underline{x}'}$$

  that is compatible with compositions.

The above data should depend "algebraically" on $\underline{x}$ and $\underline{x}'$. This means we also allow $\underline{x}$ to be affine points $\mathrm{Ran}^{\mathrm{untl}}(S)$, and the above data should be contravariantly functorial in S.

C.2.5. Given a corr-space $\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}}$ over $\mathrm{Ran}^{\mathrm{untl}}$, under some mild finiteness assumptions, we can construct a crystal of categories $\underline{\mathrm{D}}(\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}})$ over $\mathrm{Ran}^{\mathrm{untl}}$ such that

- For any $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$,

$$\underline{\mathrm{D}}(\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}})_{\underline{x}} \simeq \mathrm{D}(\mathcal{T}_{\underline{x}})$$

- For any inclusion $\underline{x} \subseteq \underline{x}'$, the functor

$$\mathrm{ins}_{\underline{x}\subseteq\underline{x}'} : \underline{\mathrm{D}}(\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}})_{\underline{x}} \to \underline{\mathrm{D}}(\mathcal{T}_{\mathrm{Ran}^{\mathrm{untl}}})_{\underline{x}'}$$

  is given by !-pull-$*$-push along the correspondence

$$\mathcal{T}_{\underline{x}} \leftarrow \mathcal{T}_{\underline{x}\subseteq\underline{x}'} \to \mathcal{T}_{\underline{x}'}.$$

Here the finiteness assumptions are required such that:

- The category $\mathrm{D}(\mathcal{T}_{\underline{x}})$ is well-defined[25];
- The !-pullback functor $\mathrm{D}(\mathcal{T}_{\underline{x}}) \to \mathrm{D}(\mathcal{T}_{\underline{x}\subseteq\underline{x}'})$ is well-defined, and the $*$-pushforward functor $\mathrm{D}(\mathcal{T}_{\underline{x}\subseteq\underline{x}'}) \to \mathrm{D}(\mathcal{T}_{\underline{x}'})$ is well-defined.
- The above !-pullback functors and $*$-pushforward functors have base-change isomorphisms.

C.2.6. *Example.* The coCartesian space $\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{Ran}^{\mathrm{untl}}$ produces a crystal of categories $\underline{\mathrm{D}}(\mathrm{Gr}_G)$ over $\mathrm{Ran}^{\mathrm{untl}}$, due to the fact that each $\mathrm{Gr}_{G,\underline{x}}$ is ind-finite type.

The Cartesian space $\mathfrak{L}^+(G)_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{Ran}^{\mathrm{untl}}$ produces crystals of categories $\underline{\mathrm{D}}^!(\mathfrak{L}^+(G))$ and $\underline{\mathrm{D}}_*(\mathfrak{L}^+(G))$, due to the fact that each $\mathfrak{L}^+(G)_{\underline{x}}$ is placid (see Sect. A.3.7). Note however that we can write

$$\underline{\mathrm{D}}^!(\mathfrak{L}^+(G)) =: \underline{\mathrm{D}}(\mathfrak{L}^+(G)) := \underline{\mathrm{D}}_*(\mathfrak{L}^+(G))$$

because of Sect. A.5.2.

Similarly, the corr-space $\mathfrak{L}(G)_{\mathrm{Ran}^{\mathrm{untl}}} \to \mathrm{Ran}^{\mathrm{untl}}$ produces a crystal of categories $\underline{\mathrm{D}}(\mathfrak{L}(G))$.

C.3. **Morphisms between crystals of categories.** There are two notions of morphisms between crystals of categories over a categorical prestack $\mathcal{Y}$: *lax functors* and *strict functors*.

C.3.1. A *lax functor* $F : \mathbf{C} \to \mathbf{C}'$ is an assignment as follows.

- For any $y : S \to \mathcal{Y}$, assign a $\mathrm{D}(S)$-linear functor $F_y : \mathbf{C}_y \to \mathbf{C}'_y$;
- For any 2-morphism (C.1), assign a $\mathrm{D}(S)$-linear natural transformation

$$\begin{tikzcd} \mathbf{C}_y \arrow[r, "\mathbf{C}_\alpha"] \arrow[d, "F_y"'] & \mathbf{C}_{y'} \arrow[d, "F_{y'}"] \\ \mathbf{C}'_y \arrow[r, "\mathbf{C}'_\alpha"'] \arrow[ur, Rightarrow, "F_\alpha"] & \mathbf{C}'_{y'}, \end{tikzcd}$$

  such that it is invertible if α is so.
- A homotopy-coherent system of compatibilities for compositions.

[25] In fact, we need to consider all finite type affine points $\underline{x} : S \to \mathrm{Ran}^{\mathrm{untl}}$.

A lax functor $F : \mathbf{C} \to \mathbf{C}'$ is *strict* if the above natural transformations are all invertible.

The totality of crystals of categories over $\mathcal{Y}$ and lax functors gives a 2-category $\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})$. There is a 1-full subcategory

$$\mathbf{CrysCat}^{\mathrm{strict}}(\mathcal{Y}) \subseteq \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})$$

with morphisms being strict functors.

We equip $\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})$ with the natural symmetric monoidal structure given by the formula

$$(\mathbf{C} \otimes \mathbf{D})_y := \mathbf{C}_y \otimes_{\mathrm{D}(S)} \mathbf{D}_y,\ (y : S \to Y).$$

C.3.2. *Remark.* Following Sect. C.2.2, a lax functor $F : \mathbf{C} \to \mathbf{C}'$ is *defined* to be a (right-)lax natural transformation (over $(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$) between the corresponding functors

$$\mathbf{C}, \mathbf{C}' : \widetilde{\mathcal{Y}} \to \widetilde{\mathbf{CrysCat}}$$

such that its value at any coCartesian arrow in $\widetilde{\mathcal{Y}}$ is strict, while a strict functor $F : \mathbf{C} \to \mathbf{C}'$ is just a strict natural transformation.

A 2-morphism in $\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})$ is *defined* to be a (strict) *modification*[26] between such lax natural transformations. In other words, they are 2-morphisms in $\mathrm{Fun}_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}}(\widetilde{\mathcal{Y}}, \widetilde{\mathbf{CrysCat}})$.

C.3.3. *Example.* A lax functor $\underline{\mathrm{D}}(\mathcal{Y}) \to \mathbf{C}$ is called a *lax global section* of $\mathbf{C}$. It is an assignment as follows.

- For any $y : S \to \mathcal{Y}$, assign an object $\mathcal{F}_y \in \mathbf{C}_y$;
- For any 2-morphism (C.1), assign a morphism $\mathcal{F}_\alpha : \mathbf{C}_\alpha(\mathcal{F}_y) \to \mathcal{F}_{y'}$ in $\mathbf{C}_{y'}$ such that it is invertible if α is so.
- A homotopy-coherent system of compatibilities for compositions.

The collection of lax global sections of $\mathbf{C}$ form a DG category $\mathbf{C}_{\mathcal{Y}}^{\mathrm{lax}}$, which is denoted $\Gamma^{\mathrm{lax}}(\mathcal{Y}, \mathbf{C})$ in [GLC2, Sect. C.2].

Similarly, a strict functor $\underline{\mathrm{D}}(\mathcal{Y}) \to \mathbf{C}$ is called a *strict global section* of $\mathbf{C}$. It is an assignment as above such that $\mathcal{F}_\alpha$ is *always* invertible. The collection of strict global sections of $\mathbf{C}$ form a full subcategory $\mathbf{C}_{\mathcal{Y}}^{\mathrm{strict}} \subseteq \mathbf{C}_{\mathcal{Y}}^{\mathrm{lax}}$, which is denoted by $\Gamma^{\mathrm{strict}}(\mathcal{Y}, \mathbf{C})$ in *loc.cit.*.

C.3.4. *Example.* Objects in

$$\mathrm{D}^{\mathrm{lax}}(\mathcal{Y}) := \underline{\mathrm{D}}(\mathcal{Y})_{\mathcal{Y}}^{\mathrm{lax}} \simeq \Gamma^{\mathrm{lax}}(\mathcal{Y}, \underline{\mathrm{D}}(\mathcal{Y}))$$

are called *lax D-modules* on $\mathcal{Y}$, while those in

$$\mathrm{D}^{\mathrm{strict}}(\mathcal{Y}) := \underline{\mathrm{D}}(\mathcal{Y})_{\mathcal{Y}}^{\mathrm{strict}} \simeq \Gamma^{\mathrm{strict}}(\mathcal{Y}, \underline{\mathrm{D}}(\mathcal{Y}))$$

are *strict D-modules* on $\mathcal{Y}$.

C.3.5. *Remark.* Following Sect. C.2.4, roughly speaking, a lax (resp. strict) functor $F : \mathbf{A} \to \mathbf{A}'$ over $\mathrm{Ran}^{\mathrm{untl}}$ consists of the following data:

- For any point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$, assign a functor $F_{\underline{x}} : \mathbf{A}_{\underline{x}} \to \mathbf{A}'_{\underline{x}}$;
- For any inclusion $\underline{x} \subseteq \underline{x}'$, assign a natural transformation (resp. isomorphism)

$$\begin{tikzcd} \mathbf{A}_{\underline{x}} \arrow[r, "\mathrm{ins}_{\underline{x}\subseteq\underline{x}'}"] \arrow[d, "F_{\underline{x}}"'] & \mathbf{A}_{\underline{x}'} \arrow[d, "F_{\underline{x}'}"] \\ \mathbf{A}'_{\underline{x}} \arrow[r, "\mathrm{ins}_{\underline{x}\subseteq\underline{x}'}"'] \arrow[ur, Rightarrow, "F_{\underline{x}\subseteq\underline{x}'}"] & \mathbf{A}'_{\underline{x}'}, \end{tikzcd}$$

that is compatible with compositions.

In particular, a lax (resp. strict) global section $\mathcal{A}$ of $\mathbf{A}$ over $\mathrm{Ran}^{\mathrm{untl}}$ consists of the following data:

- For any point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}$, assign an object $\mathcal{A}_{\underline{x}} \in \mathbf{A}_{\underline{x}}$;

[26] There is no room for laxness for modifications because 3-morphisms in $\widetilde{\mathbf{CrysCat}}$ are invertible.

- For any inclusion $\underline{x} \subseteq \underline{x}'$, assign a morphism (resp. isomorphism)
$$\mathcal{A}_{\underline{x}\subseteq\underline{x}'} : \mathrm{ins}_{\underline{x}\subseteq\underline{x}'}(\mathcal{A}_{\underline{x}}) \to \mathcal{A}_{\underline{x}'}$$
that is compatible with compositions.

C.3.6. Given a morphism $f : \mathcal{Y} \to \mathcal{Z}$ between categorical prestacks, there is a symmetric monoidal functor
$$f^* : \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Z}) \to \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})$$
given by the formula
$$(f^*\mathbf{C})_y \simeq \mathbf{C}_{f(y)},$$
where $y : S \to \mathcal{Y}$ is an affine point of $\mathcal{Y}$ and $f(y) = f \circ y$ is an affine point of $\mathcal{Z}$. Note that f^* preserves strict functors, i.e., we have a functor
$$f^* : \mathbf{CrysCat}^{\mathrm{strict}}(\mathcal{Z}) \to \mathbf{CrysCat}^{\mathrm{strict}}(\mathcal{Y})$$

C.3.7. In particular, for any $\mathbf{C} \in \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Z})$, we obtain a functor
$$f^! : \mathbf{C}^{\mathrm{lax}}_{\mathcal{Z}} \to (f^*\mathbf{C})^{\mathrm{lax}}_{\mathcal{Y}}$$
given by
$$\mathrm{Fun}_{\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Z})}(\underline{\mathrm{D}}(\mathcal{Z}), \mathbf{C}) \to \mathrm{Fun}_{\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})}(f^*\underline{\mathrm{D}}(\mathcal{Z}), f^*\mathbf{C}) \simeq \mathrm{Fun}_{\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})}(\underline{\mathrm{D}}(\mathcal{Y}), f^*\mathbf{C}).$$
Note that $f^!$ sends $\mathbf{C}^{\mathrm{strict}}_{\mathcal{Z}}$ into $(f^*\mathbf{C})^{\mathrm{strict}}_{\mathcal{Y}}$.

When $\mathbf{C} = \underline{\mathrm{D}}(\mathcal{Z})$, the functor $f^!$ sends (lax) D-modules of $\mathcal{Z}$ to (lax) D-modules of $\mathcal{Y}$. This construction generalizes the !-pullback functors for usual (i.e. non-categorical) prestacks.

C.3.8. *Remark.* Following Sect. C.2.2, the functor f^* is given by precomposing with $\widetilde{\mathcal{Y}} \to \widetilde{\mathcal{Z}}$.

C.3.9. *Example.* There is an obvious map $\mathrm{Ran} \to \mathrm{Ran}^{\mathrm{untl}}$, which induces a restriction functor
$$\mathbf{CrysCat}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathbf{CrysCat}(\mathrm{Ran}).$$
Here we do not need to distinguish $\mathbf{CrysCat}^{\mathrm{lax}}(\mathrm{Ran})$ and $\mathbf{CrysCat}^{\mathrm{strict}}(\mathrm{Ran})$ because Ran is a usual prestack.

C.3.10. Let $\mathcal{Y}$ and $\mathcal{Z}$ be categorical prestacks. The *external tensor product* functor
$$- \boxtimes - : \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y}) \times \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Z}) \to \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y} \times \mathcal{Z})$$
is defined to be
$$\mathcal{C} \boxtimes \mathcal{D} := \mathrm{pr}_1^*(\mathcal{C}) \otimes \mathrm{pr}_2^*(\mathcal{D}).$$
Note that
$$(\mathcal{C} \boxtimes \mathcal{D})_{(y,z)} \simeq \mathcal{C}_y \otimes \mathcal{D}_z$$
for $y : S \to \mathcal{Y}$ and $z : T \to \mathcal{Z}$.

C.3.11. *Warning.* For a 2-morphism $f_1 \to f_2 : \mathcal{Y} \to \mathcal{Z}$ in **CatPreStk**, we only have a *left-lax* natural transformation
$$f_1^* \to f_2^* : \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Z}) \to \mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y}).$$
In other words, given a morphism $F : \mathbf{C} \to \mathbf{C}'$ in $\mathbf{CrysCat}^{\mathrm{lax}}(\mathcal{Y})$, we have a canonical 2-morphism
$$\begin{tikzcd} f_1^*(\mathbf{C}) \arrow[r] \arrow[d] & f_2^*(\mathbf{C}) \arrow[d] \\ f_1^*(\mathbf{C}') \arrow[r] \arrow[ur, Rightarrow] & f_2^*(\mathbf{C}'). \end{tikzcd}$$

C.4. **Unital factorization categories.**

C.4.1. Recall that $\mathrm{Ran}^{\mathrm{untl}}$ is an abelian monoid object in CatPreStk (see Sect. C.1.12).

A *unital factorization category* $\mathbf{A}$ is a crystal of categories $\underline{\mathbf{A}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ equipped with a *multiplicative structure over the disjoint loci.* In other words, for any finite set I, we have an equivalence

$$\mathrm{mult}_I : (\underset{i\in I}{\boxtimes}\underline{\mathbf{A}})|_{\mathrm{disj}} \xrightarrow{\simeq} \mathrm{union}_I^*(\underline{\mathbf{A}})|_{\mathrm{disj}} \tag{C.2}$$

and a homotopy-coherent datum of associativity and commutativity. Here $(-)|_{\mathrm{disj}}$ means restriction along

$$\big(\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}} \to \prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}},$$

see Sect. C.1.13.

C.4.2. *Remark.* Roughly speaking, a unital factorization category consists of the following data:

- DG categories $\mathbf{A}_{\underline{x}}$ and functors $\mathrm{ins}_{\underline{x}\subseteq\underline{x}'}$ as in Sect. C.2.4;
- For any finite collection of *disjoint* points in $\mathrm{Ran}^{\mathrm{untl}}$, i.e.,

$$(\underline{x}_i)_{i\in I} \in \big(\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}},\ |I|<\infty$$

  assign an equivalence

$$\mathrm{mult}_{(\underline{x}_i)} : \otimes\mathbf{A}_{\underline{x}_i} \simeq \mathbf{A}_{\bigsqcup \underline{x}_i} \tag{C.3}$$

  (and a datum of associativity and commutatitvity)

such that for two collections $(\underline{x}_i)_{i\in I}$, $(\underline{x}'_i)_{i\in I}$ of disjoint points satisfying $\underline{x}_i \subseteq \underline{x}'_i$, the following diagram commutes[27]

$$\begin{tikzcd} \otimes\mathbf{A}_{\underline{x}_i} \arrow[r, "\mathrm{mult}_{(\underline{x}_i)}", "\simeq"'] \arrow[d, "\otimes\,\mathrm{ins}_{\underline{x}_i\subseteq\underline{x}'_i}"'] & \mathbf{A}_{\bigsqcup\underline{x}_i} \arrow[d, "\mathrm{ins}_{\sqcup\underline{x}_i\subset\sqcup\underline{x}'_i}"] \\ \otimes\mathbf{A}_{\underline{x}'_i} \arrow[r, "\simeq", "\mathrm{mult}_{(\underline{x}'_i)}"'] & \mathbf{A}_{\bigsqcup\underline{x}'_i}. \end{tikzcd} \tag{C.4}$$

Note that for $I=\emptyset$, we have a canonical equivalence $\mathrm{mult}_\emptyset : \mathrm{Vect} \simeq \mathbf{A}_\emptyset$[28].

C.4.3. *Example.* The constant crystal $\underline{\mathrm{D}}(\mathrm{Ran}^{\mathrm{untl}})$ is a unital factorization cateogry by identifying both sides of (C.2) with

$$\underline{\mathrm{D}}\Big(\big(\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}}\Big).$$

We often denote this factorization category by Vect (because its fiber at any $\underline{x}\in\mathrm{Ran}^{\mathrm{untl}}$ is Vect).

C.4.4. *Example.* The factorization structures on $\mathrm{Gr}_{G,\mathrm{Ran}^{\mathrm{untl}}}$, $\mathfrak{L}^+(G)_{\mathrm{Ran}^{\mathrm{untl}}}$ and $\mathfrak{L}(G)_{\mathrm{Ran}^{\mathrm{untl}}}$ upgrade the crystals of categories $\underline{\mathrm{D}}(\mathrm{Gr}_G)$, $\underline{\mathrm{D}}(\mathfrak{L}^+(G))$, $\underline{\mathrm{D}}(\mathfrak{L}(G))$ to unital factorization categories.

C.4.5. *Variant.* We define a unital *lax-factorization* cateogry to be a crystal of categories $\underline{\mathbf{A}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ equipped with a *(right-)lax* multiplicative structure over the disjoint loci. This means we replace the equivalence (C.2) with a morphism in $\mathbf{CrysCat}^{\mathrm{strict}}\big(\big(\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}}\big)$:

$$\mathrm{mult}_I : (\underset{i\in I}{\boxtimes}\underline{\mathbf{A}})|_{\mathrm{disj}} \to \mathrm{union}_I^*(\underline{\mathbf{A}})|_{\mathrm{disj}}. \tag{C.5}$$

One may also consider an even weaker notion, where (C.5) is only required to be a morphism in $\mathbf{CrysCat}^{\mathrm{lax}}\big(\big(\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}}\big)$. However, we do not see any application of such a structure.

[27]This is also a structure rather than a property, and there is a datum of compatibility with associativity and commutatitvity.

[28]Note that for $I=\{1\}$ and $\underline{x}_1=\emptyset$, we also have an equivalence $\mathrm{mult}_{(\emptyset)} : \mathbf{A}_\emptyset \simeq \mathbf{A}_\emptyset$, which is just the identity functor.

C.4.6. *Remark.* The restriction functor $\mathbf{CrysCat}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}) \to \mathbf{CrysCat}(\mathrm{Ran})$ sends unital factorization categories to non-unital ones.

C.5. **Morphisms between unital factorization categories.** There are (at least) two notions of morphisms between unital factorization categories: *lax-unital functors* and *(strictly) unital functors.*

C.5.1. Let $\mathbf{A}$ and $\mathbf{A}'$ be unital factorization categories. A *lax-unital factorization functor* $F : \mathbf{A} \to \mathbf{A}'$ is a morphism $\underline{F} : \underline{\mathbf{A}} \to \underline{\mathbf{A}}'$ in $\mathbf{CrysCat}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}})$ equipped with commutative diagrams (for any finite set I)

$$\begin{tikzcd} (\boxtimes_{i\in I}\underline{\mathbf{A}})|_{\mathrm{disj}} \arrow[r, "\simeq"] \arrow[d, "(\boxtimes\underline{F})|_{\mathrm{disj}}"'] & \mathrm{union}_I^*(\underline{\mathbf{A}})|_{\mathrm{disj}} \arrow[d, "\mathrm{union}_I^*(\underline{F})|_{\mathrm{disj}}"] \\ (\boxtimes_{i\in I}\underline{\mathbf{A}}')|_{\mathrm{disj}} \arrow[r, "\simeq"] & \mathrm{union}_I^*(\underline{\mathbf{A}}')|_{\mathrm{disj}} \end{tikzcd} \tag{C.6}$$

and a homotopy-coherent datum of associativity and commutativity. Here the horizontal arrows are the structural equivalences for $\mathbf{A}$ and $\mathbf{A}'$ (see (C.2)).

We say F is (strictly) unital if $\underline{F}$ is contained in $\mathbf{CrysCat}^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}})$.

C.5.2. Unital factorization categories and lax-unital factorization functors between them form a 2-category, which is denoted by

$$\mathbf{UntlFactCat}^{\text{lax-untl}}.$$

The 1-full subcategory of strictly unital factorization functors is denoted by

$$\mathbf{UntlFactCat} \subseteq \mathbf{UntlFactCat}^{\text{lax-untl}}.$$

C.5.3. Let $\mathbf{A}$ be a unital factorization category. A *(unital)*[29] *factorization algebra in* $\mathbf{A}$ is *defined* to be a lax-unital factorization functor $\mathcal{A} : \mathrm{Vect} \to \mathbf{A}$, where Vect is the unital factorization category defined in Sect. C.4.3.

Factorization algebras in $\mathbf{A}$ form a category, which is denoted by $\mathrm{FactAlg}(\mathbf{A})$.

C.5.4. Below is a concrete description of factorization algebras $\mathcal{A}$ in a unital factorization category $\mathbf{A}$.

For each finite set I, we have a functor

$$\prod_{i\in I}\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{A}}) \to \Gamma^{\mathrm{lax}}(\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}}, \underset{i\in I}{\boxtimes}\underline{\mathbf{A}}) \to \Gamma^{\mathrm{lax}}\big((\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}})_{\mathrm{disj}}, (\underset{i\in I}{\boxtimes}\underline{\mathbf{A}})|_{\mathrm{disj}}\big)$$

that sends

$$(\mathcal{M}_i)_{i\in I} \to (\boxtimes\mathcal{M}_i)|_{\mathrm{disj}},$$

and a functor

$$\Gamma^{\mathrm{lax}}\big(\mathrm{Ran}^{\mathrm{untl}}, \mathbf{A}\big) \to \Gamma^{\mathrm{lax}}\big((\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}})_{\mathrm{disj}}, \mathrm{union}_I^*(\mathbf{A})|_{\mathrm{disj}}\big)$$

that in turn sends

$$\mathcal{N} \mapsto \mathrm{union}_I^!(\mathcal{N})|_{\mathrm{disj}},$$

see Sect. C.3.7. Note that the structural equivalence (C.2) induces an equivalence

$$\mathrm{mult}_I : \Gamma^{\mathrm{lax}}\big((\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}})_{\mathrm{disj}}, (\underset{i\in I}{\boxtimes}\underline{\mathbf{A}})|_{\mathrm{disj}}\big) \xrightarrow{\simeq} \Gamma^{\mathrm{lax}}\big((\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}})_{\mathrm{disj}}, \mathrm{union}_I^*(\mathbf{A})|_{\mathrm{disj}}\big).$$

Unwinding the definitions, a factorization algebra $\mathcal{A}$ in $\mathbf{A}$ is an object

$$\underline{\mathcal{A}} \in \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}, \underline{\mathbf{A}}) =: \underline{\mathbf{A}}^{\mathrm{lax}}_{\mathrm{Ran}^{\mathrm{untl}}}$$

equipped with isomorphisms (for any finite set I)

$$\mathrm{mult}_I\big((\underset{i\in I}{\boxtimes}\underline{\mathcal{A}})|_{\mathrm{disj}}\big) \xrightarrow{\simeq} \mathrm{union}_I^!(\underline{\mathcal{A}})|_{\mathrm{disj}} \tag{C.7}$$

[29]When talking about factorization algebras in a *unital* factorization category, we always assume the algebra is unital.

and a homotopy-coherent datum of associativity and commutativity.

C.5.5. *Remark.* Roughly speaking, a factorization algebra $\mathcal{A}$ in a given unital factorization category $\mathbf{A}$ consists of the following data:

(i) Objects $\mathcal{A}_{\underline{x}} \in \mathbf{A}_{\underline{x}}$ and morphisms
$$\mathcal{A}_{\underline{x} \subseteq \underline{x}'} : \mathrm{ins}_{\underline{x} \subseteq \underline{x}'}(\mathcal{A}_{\underline{x}}) \to \mathcal{A}_{\underline{x}'}$$
as in Sect. C.3.5;

(ii) For any collection $(\underline{x}_i)_{i \in I}$ of *disjoint* points in $\mathrm{Ran}^{\mathrm{untl}}$, an isomorphism
$$m_{(\underline{x}_i)} : \mathrm{mult}_{(\underline{x}_i)}(\boxtimes \mathcal{A}_{\underline{x}_i}) \xrightarrow{\simeq} \mathcal{A}_{\bigsqcup \underline{x}_i} \tag{C.8}$$
that is associative and commutative, where the (invertible) functor $\mathrm{mult}_{(\underline{x}_i)}$ is as in Sect. C.4.2;

such that for two collections $(\underline{x}_i)_{i \in I}$, $(\underline{x}'_i)_{i \in I}$ of disjoints points satisfying $\underline{x}_i \subseteq \underline{x}'_i$, the following diagram commutes
$$\begin{tikzcd}
\mathrm{ins}_{\sqcup \underline{x}_i \subset \sqcup \underline{x}'_i} \circ \mathrm{mult}_{(\underline{x}_i)}(\boxtimes \mathcal{A}_{\underline{x}_i}) \arrow[rr, "\simeq"] \arrow[d, "\simeq"] & & \mathrm{ins}_{\sqcup \underline{x}_i \subset \sqcup \underline{x}'_i}(\mathcal{A}_{\bigsqcup \underline{x}_i}) \arrow[d] \\
\mathrm{mult}_{(\underline{x}'_i)}(\boxtimes \mathrm{ins}_{\underline{x}_i \subseteq \underline{x}'_i}(\mathcal{A}_{\underline{x}_i})) \arrow[r] & \mathrm{mult}_{(\underline{x}'_i)}(\boxtimes \mathcal{A}_{\underline{x}'_i}) \arrow[r, "\simeq"'] & \mathcal{A}_{\bigsqcup \underline{x}'_i},
\end{tikzcd}$$
where

- The left vertical isomorphism is due to (C.4);
- The two horizontal isomorphisms are induced by those in (ii);
- The remaining two arrows are induced by those in (i).

Note that for $I = \emptyset$, we have a canonical isomorphism
$$m_\emptyset : \mathrm{mult}_\emptyset(k) \xrightarrow{\simeq} \mathcal{A}_\emptyset,$$
where $\mathrm{mult}_\emptyset$ is the canonical identification $\mathrm{Vect} \simeq \mathbf{A}_\emptyset$.

C.5.6. *Example.* For any unital factorization category $\mathbf{A}$, there is a unique *unital* factorization functor
$$\mathcal{A} : \mathrm{Vect} \to \mathbf{A}.$$
Namely, unitality implies each $\mathcal{A}_{\underline{x}} \in \mathbf{A}_{\underline{x}}$ is canonically identified with $\mathrm{ins}_{\emptyset \subseteq \underline{x}}(\mathcal{A}_\emptyset)$, while $\mathcal{A}_\emptyset$ is canonically identified with k via the equivalence $\mathbf{A}_\emptyset \simeq \mathrm{Vect}$.

We denote this unital factorization functor by
$$\mathrm{unit}_{\mathbf{A}} : \mathrm{Vect} \to \mathbf{A}$$
and view it as an object
$$\mathrm{unit}_{\mathbf{A}} \in \mathrm{FactAlg}(\mathbf{A}).$$

The following result is obvious modulo homotopy coherence. A rigorious proof will be provided in [CFZ].

Lemma C.5.7. *Let $\mathbf{A}$ be a unital factorization category. Then $\mathrm{unit}_{\mathbf{A}}$ is an initial object in $\mathrm{FactAlg}(\mathbf{A})$.*

C.5.8. By definition, for any lax unital factorization functor $F : \mathbf{A} \to \mathbf{A}'$, we have a functor
$$\mathrm{FactAlg}(\mathbf{A}) \to \mathrm{FactAlg}(\mathbf{A}'),\ \mathcal{A} \mapsto F \circ \mathcal{A}. \tag{C.9}$$
We also write $F(\mathcal{A}) := F \circ \mathcal{A}$.

In particular, we obtain an object $F(\mathrm{unit}_{\mathbf{A}}) \in \mathrm{FactAlg}(\mathbf{A}')$. By Lemma C.5.7, there is a unique morphism
$$\mathrm{unit}_{\mathbf{A}'} \to F(\mathrm{unit}_{\mathbf{A}}).$$

The following result is obvious modulo homotopy coherence. A rigorious proof will be provided in [CFZ].

Lemma C.5.9. *Let $F : \mathbf{A} \to \mathbf{A}'$ be a lax-unital factorization functor between unital factorization categories. Then F is strictly unital iff the canonical morphism* $\mathrm{unit}_{\mathbf{A}'} \to F(\mathrm{unit}_{\mathbf{A}})$ *is invertible.*

C.5.10. *Variant.* The notion of (lax-)unital factorization functors makes sense also for unital lax-factorization categories (see Sect. C.4.5). In particular, we can define the object $\mathrm{unit}_{\mathbf{A}}$ for any unital lax-factorization cateogry $\mathbf{A}$[30].

One can consider an even weaker notion of factorization functors, where the commutative diagram (C.6) is replaced with a 2-morphism

$$\begin{tikzcd}
(\boxtimes_{i\in I}\underline{\mathbf{A}})|_{\mathrm{disj}} \arrow[r] \arrow[d, "(\boxtimes \underline{F})|_{\mathrm{disj}}"'] & \mathrm{union}_I^*(\underline{\mathbf{A}})|_{\mathrm{disj}} \arrow[d, "\mathrm{union}_I^*(\underline{F})|_{\mathrm{disj}}"] \\
(\boxtimes_{i\in I}\underline{\mathbf{A}}')|_{\mathrm{disj}} \arrow[r] \arrow[ur, Rightarrow] & \mathrm{union}_I^*(\underline{\mathbf{A}}')|_{\mathrm{disj}}.
\end{tikzcd}$$

A morphism $\underline{F} : \underline{\mathbf{A}} \to \underline{\mathbf{A}}'$ equipped with such a structure is called a *lax-unital lax-factorization functor.* Taking $\mathbf{A} :=$ Vect, we obtain the notion of *lax-factorization algebras* in a unital lax-factorization category $\mathbf{A}'$. One can describe these objects as in Sect. C.5.5 by replacing the isomorphism $m_{(\underline{x}_i)_{i\in I}}$ with a morphism

$$m_{(\underline{x}_i)} : \mathrm{mult}_{(\underline{x}_i)}(\boxtimes \mathcal{A}_{\underline{x}_i}) \to \mathcal{A}_{\bigsqcup \underline{x}_i}.$$

C.6. Unital factorization module categories.

C.6.1. Let $\underline{x}_0 \in \mathrm{Ran}^{\mathrm{untl}}(S_0)$ be an affine point. The *$\underline{x}_0$-marked unital Ran space* is the category prestack $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ over S_0 that attaches to an affine test S_0-scheme S the category of finite subsets $\underline{y} \subseteq \mathrm{Hom}(S, X)$ that contain the image of $\underline{x}_0$ under the restriction map

$$\mathrm{Hom}(S_0, X) \to \mathrm{Hom}(S, X).$$

Note that there is a canonical morphism $S_0 \to \mathrm{Ran}_{\underline{x}_0}$ corresponding to $\underline{x}_0 \subseteq \mathrm{Hom}(S_0, X)$.

Also note that $\mathrm{Ran}^{\mathrm{untl}}_{\emptyset} = \mathrm{Ran}^{\mathrm{untl}}$.

C.6.2. *Remark.* Let $\widetilde{\mathrm{Ran}^{\mathrm{untl}}} \to (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$ be the coCartesian fibration corresponding to $\mathrm{Ran}^{\mathrm{untl}}$ (see Sect. C.2.2). We can view $\underline{x}_0$ as an object $\widetilde{\mathrm{Ran}^{\mathrm{untl}}}$ lying over S_0. By definition, $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ is the functor corresponding to the coCartesian fibration

$$\big(\widetilde{\mathrm{Ran}^{\mathrm{untl}}}\big)_{\underline{x}_0/} \to (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}},$$

where the source is the slice (a.k.a. comma) category of arrows out of $\underline{x}_0$.

C.6.3. Recall $\mathrm{Ran}^{\mathrm{untl}}$ is an abelian monoid object in CatPreStk, with addition morphism given by $(\underline{x}, \underline{y}) \mapsto \underline{x} \cup \underline{y}$. It is clear that $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ is a $\mathrm{Ran}^{\mathrm{untl}}$-module object in CatPreStk, with the action morphism given by the same formula

$$\mathrm{union} : \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0},\ (\underline{x}, \underline{y}) \mapsto \underline{x} \cup \underline{y}.$$

This morphism is well-defined because if $\underline{y}$ contains (the image of) $\underline{x}_0$, so does $\underline{x} \cup \underline{y}$.

As in the non-marked case, for any *marked* finite set $I = I^\circ \sqcup \{*\}$, we define the subfunctor

$$\Big(\big(\prod_{i\in I^\circ} \mathrm{Ran}^{\mathrm{untl}}\big) \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}\Big)_{\mathrm{disj}} \subseteq \big(\prod_{i\in I^\circ} \mathrm{Ran}^{\mathrm{untl}}\big) \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0} \tag{C.10}$$

that contains those $(\underline{x}_i, \underline{y})$ such that any pair of graphs of $\underline{x}_i$ and $\underline{y}$ is disjoint.

[30]Warning: Lemma C.5.9 needs $\mathbf{A}$ to be a unital (strict-)factorization category, although $\mathbf{A}'$ can be lax.

C.6.4. Let $\mathbf{A}$ be a unital factorization category. Recall its underlying crystal of categories $\underline{\mathbf{A}}$ over $\mathrm{Ran}^{\mathrm{untl}}$ has a multiplicative structure over the disjoint loci.

A *unital factorization* $\mathbf{A}$*-module category* $\mathbf{C}$ *at* $\underline{x}_0$ is a crystal of categories $\underline{\mathbf{C}}$ over $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ equipped with a *multiplicative* $\underline{\mathbf{A}}$*-module structure over the disjoint loci, with respect to the* $\mathrm{Ran}^{\mathrm{untl}}$*-module structure on* $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$.

In other words, for any marked finite set $I = I^\circ \sqcup \{0\}$, we have an isomorphism:

$$\mathrm{act}_I : \big((\underset{i \in I^\circ}{\boxtimes} \underline{\mathbf{A}}) \boxtimes \underline{\mathbf{C}} \big)|_{\mathrm{disj}} \xrightarrow{\simeq} \mathrm{union}_I^*(\underline{\mathbf{C}})|_{\mathrm{disj}}, \tag{C.11}$$

and a homotopy-coherent datum of compatibility with (C.2). Here

$$\mathrm{union}_I : \big(\prod_{i \in I^\circ} \mathrm{Ran}^{\mathrm{untl}} \big) \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$$

is the map $(\underline{x}_i, \underline{y}) \mapsto (\cup \underline{x}_i) \cup \underline{y}$, and $(-)|_{\mathrm{disj}}$ means restriction along (C.10).

C.6.5. *Remark.* Roughly speaking, a unital factorization $\mathbf{A}$-module category consists of the following data[31]:

- For any $\underline{y} \in \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$, assign a DG category $\mathbf{C}_{\underline{y}}$;
- For any $\underline{y} \subseteq \underline{y}'$ in $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$, assign a functor $\mathrm{ins}_{\underline{y} \subseteq \underline{y}'} : \mathbf{C}_{\underline{y}} \to \mathbf{C}_{\underline{y}'}$;
- For any finite collection of *disjoint* points $(\underline{x}_i)_{i \in I^\circ}$ in $\mathrm{Ran}^{\mathrm{untl}}$ and $\underline{y}$ in $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$, assign an equivalence
$$\mathrm{act}_{(\underline{x}_i, \underline{y})} : \big(\otimes \mathbf{A}_{\underline{x}_i} \big) \otimes \mathbf{C}_{\underline{y}} \xrightarrow{\simeq} \mathbf{C}_{(\sqcup \underline{x}_i) \sqcup \underline{y}}$$
compatible with the equivalences (C.3).
- Commutative squares similar to (C.4).
- Datum of higher compatibilities.

C.6.6. *Example.* Note that the above definitions make sense even for $\underline{x}_0 = \emptyset$. In this case, it is easy to see the following data are equivalent:

(i) A unital factorization $\mathbf{A}$-module category $\mathbf{M}$;
(ii) A (plain) DG category $\mathbf{M}_0$.

Indeed, given $\mathbf{M}$, we can consider its fiber at $\emptyset \in \mathrm{Ran}^{\mathrm{untl}}_{\emptyset}$, which is a DG category; conversely given $\mathbf{M}_0$, the tensor product

$$\underline{\mathbf{A}} \otimes \mathbf{M}_0 \in \mathbf{CrysCat}(\mathrm{Ran}^{\mathrm{untl}}) \simeq \mathbf{CrysCat}(\mathrm{Ran}^{\mathrm{untl}}_{\emptyset})$$

has an obvious unital factorization $\mathbf{A}$-module structure. One can check these two constructions are inverse to each other.

C.6.7. *Example.* Let $\mathbf{A}$ be a unital factorization category. The pullback of $\underline{\mathbf{A}}$ along the morphism $\mathrm{Ran}_{\underline{x}_0} \to \mathrm{Ran}$ is naturally a factorization $\mathbf{A}$-module at $\underline{x}_0$. We denote the resulting object by $\mathbf{A}^{\mathrm{fact}_{\underline{x}_0}}$.

C.6.8. *Variant.* As in Sect. C.4.5, for a unital lax-factorization category $\mathbf{A}$, we can define the notion of unital lax-factorization $\mathbf{A}$-module categories by allowing act_I (see (C.11)) to be non-invertible.

C.7. **Morphisms between unital factorization module categories.**

[31] To simplify the notations, in below we pretend $\underline{x}_0$ is a k-point. Otherwise certain base-changes are necessary.

C.7.1. Let $F : \mathbf{A} \to \mathbf{A}'$ be a lax-unital factorization functor between unital factorization categories. For unital factorization $\mathbf{A}$-module category $\mathbf{C}$ and $\mathbf{A}'$-module cateogry $\mathbf{C}'$ at $\underline{x}_0$, a *lax-unital F-linear factorization functor* $G : \mathbf{C} \to \mathbf{C}'$ is a morphism $\underline{G} : \underline{\mathbf{C}} \to \underline{\mathbf{C}}'$ in $\mathbf{CrysCat}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})$ equipped with commutative diagrams (for any marked finite set $I = I^\circ \sqcup \{0\}$)

$$\begin{tikzcd}
\big((\boxtimes_{i\in I^\circ}\underline{\mathbf{A}}) \boxtimes \underline{\mathbf{C}}\big)|_{\mathrm{disj}} \arrow[r, "\simeq"] \arrow[d, "((\boxtimes \underline{F})\boxtimes \underline{G})|_{\mathrm{disj}}"'] & \mathrm{union}_I^*(\underline{\mathbf{C}})|_{\mathrm{disj}} \arrow[d, "\mathrm{union}_I^*(\underline{G})|_{\mathrm{disj}}"] \\
\big((\boxtimes_{i\in I^\circ}\underline{\mathbf{A}}') \boxtimes \underline{\mathbf{C}}'\big)|_{\mathrm{disj}} \arrow[r, "\simeq"] & \mathrm{union}_I^*(\underline{\mathbf{C}}')|_{\mathrm{disj}}
\end{tikzcd}$$

and a homotopy-coherent datum of compatibility with (C.6).

When F is (strictly) unital, we say G is (strictly) unital if $\underline{G}$ is contained in $\mathbf{CrysCat}^{\mathrm{strict}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})$. In fact, G is *automatically* unital.

Lemma C.7.2. *Let $F : \mathbf{A} \to \mathbf{A}'$ be a unital factorization functor, and $G : \mathbf{C} \to \mathbf{C}'$ be a lax-unital F-linear factorization functor between unital factorization module categories at $\underline{x}_0$. Then G is strictly unital.*

Sketch. For any affine points $\underline{y}, \underline{y}' : S \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ such that $\underline{y} \subseteq \underline{y}'$, we need to show the following natural transformation is invertible:

$$\begin{tikzcd}
\mathbf{C}_{\underline{y}} \arrow[r, "\mathrm{ins}_{\underline{y}\subseteq\underline{y}'}"] \arrow[d, "G_{\underline{y}}"'] & \mathbf{C}_{\underline{y}'} \arrow[d, "G_{\underline{y}'}"] \\
\mathbf{C}'_{\underline{y}} \arrow[r, "\mathrm{ins}_{\underline{y}\subseteq\underline{y}'}"'] \arrow[ur, Rightarrow, "G_{\underline{y}\subseteq\underline{y}'}"] & \mathbf{C}'_{\underline{y}'}.
\end{tikzcd}$$

Note that if $S = \cup S_\alpha$ is a finite covering by locally closed subschemes such that $G_{\underline{y}|_{S_\alpha}\subseteq\underline{y}'|_{S_\alpha}}$ is invertible for each α, then $G_{\underline{y}\subseteq\underline{y}'}$ is also invertible. Hence without lose of generality, we can assume $\underline{y}' = \underline{y} \sqcup \underline{z}$ such that $\underline{y} \cap \underline{z} = \emptyset$. Using the factorization structure, the natural transformation $G_{\underline{y}\subseteq\underline{y}'}$ can be identified with

$$\begin{tikzcd}
\mathbf{A}_\emptyset \otimes \mathbf{C}_{\underline{y}} \arrow[r, "\mathrm{ins}_{\emptyset\subseteq\underline{z}}\otimes\mathrm{Id}"] \arrow[d, "F_\emptyset\otimes G_{\underline{y}}"'] & \mathbf{A}_{\underline{z}} \otimes \mathbf{C}_{\underline{y}} \arrow[d, "F_{\underline{z}}\otimes G_{\underline{y}}"] \\
\mathbf{A}'_\emptyset \otimes \mathbf{C}'_{\underline{y}} \arrow[r, "\mathrm{ins}_{\emptyset\subseteq\underline{z}}\otimes\mathrm{Id}"'] \arrow[ur, Rightarrow, "F_{\emptyset\subseteq\underline{z}}\otimes\mathrm{Id}"] & \mathbf{A}'_{\underline{z}} \otimes \mathbf{C}'_{\underline{y}}.
\end{tikzcd}$$

This implies $G_{\underline{y}\subseteq\underline{y}'}$ is invertible because $F_{\emptyset\subseteq\underline{z}}$ is invertible by assumption.

□

C.7.3. Let

$$\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0}.$$

be the 2-category such that:

- An object is a pair $(\mathbf{A}, \mathbf{C})$, where $\mathbf{A}$ is a unital factorization category and $\mathbf{C}$ is a unital factorization $\mathbf{A}$-module category at $\underline{x}_0$;
- A morphism $(\mathbf{A}, \mathbf{C}) \to (\mathbf{A}', \mathbf{C}')$ is a pair (F, G), where F is a lax-unital factorization functor, and G is a lax-unital F-linear factorization functor.

Let

$$\mathbf{UntlFactModCat}_{\underline{x}_0} \subseteq \mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0}$$

be the 1-full subcategory such that morphisms are strictly unital factorization functors.

We have a forgetful functor

$$\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0} \to \mathbf{UntlFactCat}^{\text{lax-untl}},\ (\mathbf{A}, \mathbf{C}) \mapsto \mathbf{A}. \tag{C.12}$$

The fiber of this functor at $\mathbf{A}$ is

$$\mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0},$$

the 2-category of unital factorization $\mathbf{A}$-module categories at $\underline{x}_0$. Note that this category is also the fiber of the forgetful functor

$$\mathbf{UntlFactModCat}_{\underline{x}_0} \to \mathbf{UntlFactCat},\ (\mathbf{A},\mathbf{C}) \mapsto \mathbf{A}.$$

See Lemma C.7.2.

C.7.4. Note that $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$, together with its $\mathrm{Ran}^{\mathrm{untl}}$-module structure, is defined over S_0. It follows that the symmetric monoidal 2-category

$$\mathbf{CrysCat}(S_0) \simeq \mathrm{D}(S_0)\text{−}\mathbf{mod}$$

acts on the fibers of (C.12). In particular, it acts on $\mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0}$.

C.7.5. Let $\mathbf{A}$ be a unital factorization category and $\mathbf{C}$ be a unital factorization $\mathbf{A}$-module category at $\underline{x}_0$. Recall a factorization algebra $\mathcal{A}$ in $\mathbf{A}$ is a lax-unital factorization functor $\mathcal{A} : \mathrm{Vect} \to \mathbf{A}$ (see Sect. C.5.3).

A *factorization* $\mathcal{A}$*-module* $\mathcal{C}$ *in* $\mathbf{C}$ is *defined* to be a lax-unital factorization $\mathcal{A}$-linear functor

$$\mathcal{C} : \mathrm{Vect}^{\mathrm{fact}_{\underline{x}_0}} \to \mathbf{C}.$$

Here $\mathrm{Vect}^{\mathrm{fact}_{\underline{x}_0}}$ is the unital factorization Vect-module category in Sect. C.6.7.

Pairs $(\mathcal{A},\mathcal{C})$ of factorization algebras and modules in $(\mathbf{A},\mathbf{C})$ form a category, which is denoted by $\mathrm{FactMod}(\mathbf{A},\mathbf{C})_{\underline{x}_0}$. There is a forgetful functor

$$\mathrm{FactMod}(\mathbf{A},\mathbf{C})_{\underline{x}_0} \to \mathrm{FactAlg}(\mathbf{A}),\ (\mathcal{A},\mathcal{C}) \mapsto \mathcal{A} \tag{C.13}$$

whose fiber at $\mathcal{A}$ is the category

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0}$$

of factorization $\mathcal{A}$-modules in $\mathbf{C}$.

C.7.6. The action in Sect. C.7.4 induces an action of $\mathrm{D}(S_0)$ on the fibers of the functor (C.13). In particular $\mathrm{D}(S_0)$ acts on $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0}$.

C.7.7. For $\mathbf{C} = \mathbf{A}^{\mathrm{fact}_{\underline{x}_0}}$, we write

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}_0} := \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{A}^{\mathrm{fact}_{\underline{x}_0}})_{\underline{x}_0}.$$

See Sect. C.6.7.

C.7.8. Below is a concrete description of factorization $\mathcal{A}$-modules in a unital factorization $\mathbf{A}$-module category $\mathbf{C}$.

As in Sect. C.5.4, for each marked finite set $I = I^\circ \sqcup \{0\}$, we have a functor

$$\Big(\prod_{i\in I^\circ}\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}},\underline{\mathbf{A}})\Big)\times\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0},\underline{\mathbf{C}}) \to \Gamma^{\mathrm{lax}}\Big(\big((\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}})\times\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}\big)_{\mathrm{disj}},\big((\underset{i\in I}{\boxtimes}\underline{\mathbf{A}})\boxtimes\underline{\mathbf{C}}\big)|_{\mathrm{disj}}\Big)$$

that sends

$$(\mathcal{M}_i,\mathcal{F}) \to \big((\boxtimes\mathcal{M}_i)\boxtimes\mathcal{F}\big)|_{\mathrm{disj}},$$

and a functor

$$\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0},\underline{\mathbf{C}}) \to \Gamma^{\mathrm{lax}}\Big(\big((\prod_{i\in I}\mathrm{Ran}^{\mathrm{untl}})\times\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}\big)_{\mathrm{disj}},\mathrm{union}^*_I(\mathbf{C})|_{\mathrm{disj}}\Big)$$

that sends

$$\mathcal{N} \mapsto \mathrm{union}^!_I(\mathcal{N})|_{\mathrm{disj}}.$$

The factorization structure on $\mathbf{C}$ provides an equivalence

$$\mathrm{act}_I : \Gamma^{\mathrm{lax}}\big(((\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}) \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}}, ((\underset{i\in I}{\boxtimes} \underline{\mathbf{A}}) \boxtimes \underline{\mathbf{C}})|_{\mathrm{disj}}\big) \to$$
$$\to \Gamma^{\mathrm{lax}}\big(((\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}) \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}}, \mathrm{union}^*_I(\mathbf{C})|_{\mathrm{disj}}\big).$$

Then a factorization $\mathcal{A}$-module $\mathcal{C}$ in $\mathbf{C}$ is an object

$$\underline{\mathcal{C}} \in \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}, \underline{\mathbf{C}}) =: \underline{\mathbf{C}}^{\mathrm{lax}}_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}}$$

equipped with isomorphisms (for any marked finite set $I = I^\circ \sqcup \{0\}$)

$$\mathrm{act}_I\,\big(((\underset{i\in I^\circ}{\boxtimes} \underline{\mathcal{A}}) \boxtimes \underline{\mathcal{C}})|_{\mathrm{disj}}\big) \xrightarrow{\simeq} \mathrm{union}^!_I(\underline{\mathcal{C}})|_{\mathrm{disj}}$$

and a homotopy-coherent datum of compatibility with (C.7).

C.7.9. Unwinding the definitions, we have the following result. A rigorious proof will be provided in [CFZ].

Lemma C.7.10. *Let* $\mathbf{A}$ *be a unital factorization category and* $\mathbf{C}$ *be a unital factorization* $\mathbf{A}$*-module category at* $\underline{x}_0$*. For a factorization algebra* $\mathcal{A}$ *in* $\mathbf{A}$*, the category* $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0}$ *is naturally a DG* $\mathrm{D}(S_0)$*-module category*[32] *and the forgetful functors*

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0} \to \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}, \underline{\mathbf{C}}) \to \mathbf{C}_{\underline{x}_0},\ \mathcal{C} \mapsto \underline{\mathcal{C}} \mapsto \mathcal{C}_{\underline{x}_0}$$

are $\mathrm{D}(S_0)$*-linear and colimit-preserving.*

C.7.11. *Example.* Let $(\mathbf{A}, \mathbf{A}^{\mathrm{fact}_{\underline{x}_0}})$ be as in Sect. C.6.7. For a factorization algebra $\mathcal{A}$ in $\mathbf{A}$, its restriction along $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0} \to \mathrm{Ran}^{\mathrm{untl}}$ defines a factorization $\mathcal{A}$-module in $\mathbf{A}^{\mathrm{fact}_{\underline{x}_0}}$. We denote this object by

$$\mathcal{A}^{\mathrm{fact}_{\underline{x}_0}} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0}.$$

C.7.12. The following result appears in [GLC2, Lemma C.14.10]. We will provide a more homotopy-coherent proof in [CFZ].

Lemma C.7.13. *Let* $\mathbf{A}$ *be a unital factorization category and* $\mathbf{C}$ *be a unital factorization* $\mathbf{A}$*-module category at* $\underline{x}_0$*. Then the forgetful functor*

$$\mathrm{unit}_{\mathbf{A}}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0} \to \mathbf{C}_{\underline{x}_0},\ \mathcal{C} \mapsto \mathcal{C}_{\underline{x}_0}$$

is an equivalence.

C.7.14. By definition, for any lax unital factorization functors $(F, G) : (\mathbf{A}, \mathbf{C}) \to (\mathbf{A}', \mathbf{C}')$ between unital factorization categories and their modules, we have a functor

$$\mathrm{FactMod}(\mathbf{A}, \mathbf{C})_{\underline{x}_0} \to \mathrm{FactMod}(\mathbf{A}', \mathbf{C}')_{\underline{x}_0},\ (\mathcal{A}, \mathcal{C}) \mapsto (F \circ \mathcal{A}, G \circ \mathcal{C})$$

compatible with (C.9). In particular, we obtain a functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0} \to F(\mathcal{A})\text{-mod}^{\mathrm{fact}}(\mathbf{C}')_{\underline{x}_0},\ \mathcal{C} \mapsto G \circ \mathcal{C}.$$

We also write $G(\mathcal{C}) := G \circ \mathcal{C}$.

C.7.15. *Variant.* As in Sect. C.5.10, we can define *lax-factorization functors* between unital factorization module categories, and use such a notion to define *lax-factorization module objects*.

C.8. **Change of base.** In Sect. C.6 and Sect. C.7, we introduced factorization *module* structures at a fixed affine point $\underline{x}_0 \in \mathrm{Ran}^{\mathrm{untl}}(S_0)$. In this subsection, we explian how such structures depend on $\underline{x}_0$.

[32] In other words, $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0}$ is presentable and stable, and the acting functors are compatible with colimits.

C.8.1. Throughout this subsection, we fix the following notations.

Let $\underline{x}_0 \in \mathrm{Ran}^{\mathrm{untl}}(S_0)$ and $\underline{x}'_0 \in \mathrm{Ran}^{\mathrm{untl}}(S'_0)$ be two affine points and

(C.14)
$$\begin{tikzcd} & S_0 \arrow[dr, "\underline{x}_0"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ S'_0 \arrow[ur, "f_\alpha"] \arrow[rr, "\underline{x}'_0"'] & & \mathrm{Ran}^{\mathrm{untl}} \end{tikzcd}$$

be a 2-morphism in CatPreStk. In other words, α is a morphism in $\widetilde{\mathrm{Ran}}^{\mathrm{untl}}$ (see Sect. C.2.2).

C.8.2. By construction, there is a canonical morphism

$$\mathrm{Ran}_\alpha : \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$$

defined over the morphism $f_\alpha : S'_0 \to S_0$ and compatible with the forgetful morphisms to $\mathrm{Ran}^{\mathrm{untl}}$. Moreover, this morphism is compatible with the $\mathrm{Ran}^{\mathrm{untl}}$-module structures on $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ and $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0}$ (see Sect. C.6.3).

It follows that pullback along $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ defines a functor

$$\alpha_\dagger : \mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0} \to \mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}'_0} \tag{C.15}$$

compatible with the forgetful functors to $\mathbf{UntlFactCat}^{\text{lax-untl}}$[33].

In particular, for $\mathbf{A} \in \mathbf{UntlFactCat}^{\text{lax-untl}}$, we obtain a functor

$$\alpha_\dagger : \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0} \to \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}'_0}.$$

We write

$$\mathbf{C}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0}} := \alpha_\dagger(\mathbf{C}).$$

In the case when $f_\alpha = \mathrm{id}_{S_0}$ is the identity morphism, we write

$$\mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}'_0} := \alpha_\dagger : \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0} \to \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}'_0}$$

and call $\mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}'_0}(\mathbf{C})$ the *propogation of* $\mathbf{C}$ *along* $\underline{x}_0 \subseteq \underline{x}'_0$.

C.8.3. *Remark.* The construction $\alpha \mapsto \alpha_\dagger$ is compatible with compositions. In fact, one can construct a coCartesian fibration of 2-categories

$$\mathbf{UntlFactModCat}^{\text{lax-untl}} \to \widetilde{\mathrm{Ran}}^{\mathrm{untl}}$$

such that its fiber at $\underline{x}_0$ is $\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0}$, and the covariant transport functor along α is $\alpha_\dagger$. A rigorous construction of this coCartesian fibration will be provided in [CFZ].

C.8.4. *Example.* Let $\mathbf{A}$ be a unital factorization category. The functor

$$\alpha_\dagger : \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0} \to \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}'_0}$$

sends $\mathbf{A}^{\mathrm{fact}_{\underline{x}_0}}$ to $\mathbf{A}^{\mathrm{fact}_{\underline{x}'_0}}$.

[33] Note that the definition of factorization (algebra) categories is independent of the points $\underline{x}_0$.

C.8.5. Let $\mathbf{A}$ be a unital factorization category and $\mathbf{C} \in \mathbf{A}\text{--}\mathbf{mod}^{\text{fact}}_{\underline{x}_0}$. Write

$$\mathbf{C}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}} := \alpha_\dagger(\mathbf{C}) \in \mathbf{A}\text{--}\mathbf{mod}^{\text{fact}}_{\underline{x}'_0}.$$

The functor (C.15) induces a functor

$$\alpha_\dagger : \text{Fun}_{\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0}}((\text{Vect}, \text{Vect}^{\text{fact}_{\underline{x}_0}}), (\mathbf{A}, \mathbf{C})) \to$$
$$\to \text{Fun}_{\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}'_0}}((\text{Vect}, \text{Vect}^{\text{fact}_{\underline{x}'_0}}), (\mathbf{A}, \mathbf{C}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}})).$$

By definition, this is a functor

$$\alpha_\dagger : \text{FactMod}(\mathbf{A}, \mathbf{C})_{\underline{x}_0} \to \text{FactMod}(\mathbf{A}, \mathbf{C}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}})_{\underline{x}'_0} \tag{C.16}$$

compatible with forgetful functors to $\text{FactAlg}(\mathbf{A})$.

In particular, for $\mathcal{A} \in \text{FactAlg}(\mathbf{A})$, we obtain a functor

$$\alpha_\dagger : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\underline{x}_0} \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}})_{\underline{x}'_0}. \tag{C.17}$$

When $\mathbf{C} = \mathbf{A}^{\text{fact}_{\underline{x}_0}}$, this gives

$$\alpha_\dagger : \mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}_0} \to \mathcal{A}\text{-mod}^{\text{fact}}_{\underline{x}'_0} \tag{C.18}$$

In the case when $f_\alpha = \text{id}_{S_0}$ is the identity morphism, recall

$$\mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}'_0}(\mathbf{C}) := \mathbf{C}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}}.$$

Hence we also denote the functor (C.17) by

$$\text{prop}_{\underline{x}_0 \subseteq \underline{x}'_0} := \alpha_\dagger : \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\underline{x}_0} \to \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}'_0}(\mathbf{C}))_{\underline{x}'_0}$$

and call $\text{prop}_{\underline{x}_0 \subseteq \underline{x}'_0}(\mathcal{C})$ the *propogation of* $\mathcal{C}$ *along* $\underline{x}_0 \subseteq \underline{x}'_0$.

C.8.6. *Example.* The functor (C.18) sends $\mathcal{A}^{\text{fact}_{\underline{x}_0}}$ to $\mathcal{A}^{\text{fact}_{\underline{x}'_0}}$. See Sect. C.7.11.

C.8.7. By construction, the functor (C.17) fits into a canonical commutative diagram

$$\begin{tikzcd}
\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C})_{\underline{x}_0} \arrow[r] \arrow[d] & \mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{C}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}})_{\underline{x}'_0} \arrow[d] \\
\Gamma^{\text{lax}}(\text{Ran}^{\text{untl}}_{\underline{x}_0}, \underline{\mathbf{C}}) \arrow[r, "!\text{-pull}"] & \Gamma^{\text{lax}}(\text{Ran}^{\text{untl}}_{\underline{x}'_0}, \underline{\mathbf{C}}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}}),
\end{tikzcd}$$

where the vertical arrows are the forgetful functors.

On the other hand, the 2-morphim (C.14) induces a 2-morphism

$$\begin{tikzcd}
S'_0 \arrow[r, "f_\alpha"] \arrow[d, "\underline{x}'_0"'] & S_0 \arrow[d, "\underline{x}_0"] \arrow[dl, Rightarrow] \\
\text{Ran}^{\text{untl}}_{\underline{x}'_0} \arrow[r, "\text{Ran}_\alpha"'] & \text{Ran}^{\text{untl}}_{\underline{x}_0},
\end{tikzcd}$$

which induces a natural transformation

$$\begin{tikzcd}
\Gamma^{\text{lax}}(\text{Ran}^{\text{untl}}_{\underline{x}_0}, \underline{\mathbf{C}}) \arrow[r, "!\text{-pull}"] \arrow[d, "!\text{-pull}"'] & \Gamma^{\text{lax}}(\text{Ran}^{\text{untl}}_{\underline{x}'_0}, \underline{\mathbf{C}}|_{\text{Ran}^{\text{untl}}_{\underline{x}'_0}}) \arrow[d, "!\text{-pull}"] \\
\mathbf{C}_{\underline{x}_0} \arrow[r, "\mathbf{C}_{\underline{x}_0 \subseteq \underline{x}'_0}"'] \arrow[ur, Rightarrow] & \mathbf{C}_{\underline{x}'_0}
\end{tikzcd}$$

(by the definition of lax sections). Combining these squares, we obtain a canonical natural transformation

$$
\begin{tikzcd}
\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0} \arrow[r] \arrow[d] & \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0}})_{\underline{x}'_0} \arrow[d] \\
\mathbf{C}_{\underline{x}_0} \arrow[r, "\mathbf{C}_{\underline{x}_0\subseteq \underline{x}'_0}"'] \arrow[ur, Rightarrow] & \mathbf{C}_{\underline{x}'_0}
\end{tikzcd}
\tag{C.19}
$$

such that the vertical arrows are the forgetul functors.

C.8.8. *Remark.* The natural transformation (C.19) can be concretely described as follows. For simplicity, we assume $S_0 = S'_0 = \mathrm{pt}$.

Let $\mathcal{C} \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0}$ be a factorization $\mathcal{A}$-module in $\mathbf{C}$.

The top horizontal arrow sends $\mathcal{C}$ to its !-pullback $\mathcal{C}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0}}$, which is lax global section $\mathbf{C}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0}}$ equipped with a canonical factorization $\mathcal{A}$-module structure. Hence the clockwise arch sends $\mathcal{C}$ to its fiber

$$\underline{\mathcal{C}}|'_{\underline{x}_0} \in \mathbf{C}_{\underline{x}'_0}$$

at the point $\underline{x}'_0 \in \mathrm{Ran}_{\underline{x}_0}$

On the other hand, the counterclockwise arch sends $\mathcal{C}$ to the object

$$\mathrm{ins}_{\underline{x}_0\subseteq \underline{x}'_0}(\underline{\mathcal{C}}_{\underline{x}_0}) \in \mathbf{C}_{\underline{x}'_0},$$

where $\mathrm{ins}_{\underline{x}_0\subseteq \underline{x}'_0} : \mathbf{C}_{\underline{x}_0} \to \mathbf{C}'_{\underline{x}_0}$ is part of the structure of $\underline{\mathbf{C}}$ as a crystal of categories (see Sect. C.2.4).

Now the value of (C.19) at $\mathcal{C}$ is given by the morphism

$$\mathrm{ins}_{\underline{x}_0\subseteq \underline{x}'_0}(\underline{\mathcal{C}}_{\underline{x}_0}) \to \underline{\mathcal{C}}|'_{\underline{x}_0},$$

which is part of the structure of $\underline{\mathcal{C}}$ as a lax global section (see Sect. C.3.5).

C.8.9. Since the morphism

$$\mathrm{Ran}_\alpha : \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$$

is defined over the morphism $f_\alpha : S'_0 \to S_0$. The functor (C.15) intertwines the action of the symmetric monoidal functor

$$f^*_\alpha : \mathbf{CrysCat}(S_0) \to \mathbf{CrysCat}(S'_0),$$

see Sect. C.7.4. It follows that the functor (C.17) intertwines the action of the symmetric monoidal functor

$$f^!_\alpha : \mathrm{D}(S_0) \to \mathrm{D}(S'_0).$$

In particular, we have a canonical functor

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})_{\underline{x}_0} \otimes_{\mathrm{D}(S_0)} \mathrm{D}(S'_0) \to \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}'_0}})_{\underline{x}'_0}. \tag{C.20}$$

The following result is stated without proof in [GLC2, Sect. C.11.9] (but its non-unital analog is proved, see [GLC2, Lemma B.9.11]). We will provide a proof in [CFZ].

Lemma C.8.10. *In the above setting, the functor* (C.20) *is invertible if* α *is so. In particular, the functor*

$$\widetilde{\mathrm{Ran}}^{\mathrm{untl}}_{\underline{x}_0} \to \widetilde{\mathbf{CrysCat}},\ \underline{y} \mapsto \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{y}}})_{\underline{y}}$$

defines a crystal of category

$$\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{C})$$

over $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$ *(see Sect. C.2.2).*

C.8.11. By construction, (C.19) provides a morphism

$$\underline{\mathrm{oblv}}_{\mathcal{A}}: \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{C}) \to \underline{\mathbf{C}} \tag{C.21}$$

in $\mathbf{CrysCat}^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})$.

C.8.12. In particular, for $\underline{x}_0 = \emptyset \in \mathrm{Ran}^{\mathrm{untl}}$, we obtain a crystal of category

$$\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}} := \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}(\mathbf{A}^{\mathrm{fact}_\emptyset})$$

over $\mathrm{Ran}^{\mathrm{untl}}$, equipped with a forgetful morphism

$$\underline{\mathrm{oblv}}_{\mathcal{A}}: \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}} \to \underline{\mathbf{A}}.$$

C.9. **External fusion.** The main goal of this subsection is to explain $\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}$ (see Lemma C.8.10) is naturally a unital *lax-factorization* category via *external fusion*.

The *construction* of external fusion for factorization modules was sketched in [Ra, Sect. 6.22] and [GLC2, Sect. B.11.14]. However, to work with external fusion, especially in a homotopy-coherent way, it is better to characterize them via universal properties. Recall the tensor product of usual modules can be *defined* as the object that corepresents multilinear morphisms. Following this idea, we will *define* the fusion product of factorization modules as the object that corepresents *factorization multi-functors*. In fact, this approach to external fusion was alluded to in [Ra, Sect. 6.26].

C.9.1. Let S_0 be an affine scheme and $\underline{x}_i \in \mathrm{Ran}^{\mathrm{untl}}(S_0)$ $(i \in I)$ be a finite collection of *disjoint* S_0-points. Consider the S_0-morphism

$$\mathrm{union}_{(\underline{x}_i)}: \big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0} \to \mathrm{Ran}^{\mathrm{untl}}_{\sqcup \underline{x}_i},\ (\underline{y}_i) \mapsto \cup \underline{y}_i, \tag{C.22}$$

where the source is the fiber product of $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}$ relative to S_0. By definition, (C.22) is a $\mathrm{Ran}^{\mathrm{untl}}$-multilinear morphism, i.e., it is $\mathrm{Ran}^{\mathrm{untl}}$-linear in each factor of the source.

Let

$$\big(\big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0}\big)_{\mathrm{disj}} \subseteq \big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0} \tag{C.23}$$

be the subfunctor containing *disjoint* points $(\underline{y}_i)_{i\in I}$.

C.9.2. Let $\mathbf{A}$ and $\mathbf{A}'$ be unital factorization categories and

$$\mathbf{C}_i \in \mathbf{A}\text{−}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_i}\ \ (i \in I),\ \ \mathbf{C}' \in \mathbf{A}'\text{−}\mathbf{mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i},$$

where I is a finite set.

Consider the external product of $\underline{\mathbf{C}}_i$ relative to S_0:

$$(\boxtimes \underline{\mathbf{C}}_i)_{/S_0} := \otimes\, \mathrm{pr}_i^*(\underline{\mathbf{C}}_i) \in \mathbf{CrysCat}^{\mathrm{lax}}\big(\big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0}\big)$$

and its restriction to the disjoint locus (C.23):

$$\big((\boxtimes \underline{\mathbf{C}}_i)_{/S_0}\big)|_{\mathrm{disj}}.$$

Note that for each $i \in I$, the factorization $\mathbf{A}$-module structure on $\underline{\mathbf{C}}_i$ induces a factorization $\mathbf{A}$-module structure on $(\boxtimes \underline{\mathbf{C}}_i)_{/S_0}$ with respect to the $\mathrm{Ran}^{\mathrm{untl}}$-module structure on

$$\big(\big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0}\big)_{\mathrm{disj}}$$

that comes from the i-th factor[34].

[34] Using the language in Sect. C.6.4, this means $(\boxtimes \underline{\mathbf{C}}_i)_{/S_0}$ has a multiplicative $\underline{\mathbf{A}}$-module structure over the disjoint loci with respect to the i-th $\mathrm{Ran}^{\mathrm{untl}}$-module structure on $\big(\big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0}\big)_{\mathrm{disj}}$.

Similarly, for each $i \in I$, the factorization $\mathbf{A}'$-module structure on $\underline{\mathbf{C}}'$ induces a same-typed structure on

$$\mathrm{union}^*_{(\underline{x}_i)}(\underline{\mathbf{C}}')|_{\mathrm{disj}}$$

because the map (C.22) is $\mathrm{Ran}^{\mathrm{untl}}$-multilinear.

C.9.3. A *lax-unital factorization multifunctor*

$$(F, G) : (\mathbf{A}, (\mathbf{C}_i)_{i\in I}) \to (\mathbf{A}', \mathbf{C}')$$

consists of the following data:

- A lax-unital factorization functor $F : \mathbf{A} \to \mathbf{A}'$;
- A morphism
$$\underline{G} : \big((\boxtimes \underline{\mathbf{C}}_i)_{/S_0}\big)|_{\mathrm{disj}} \to \mathrm{union}^*_{(\underline{x}_i)}(\underline{\mathbf{C}}')|_{\mathrm{disj}}$$
in
$$\mathbf{CrysCat}^{\mathrm{lax}}\big(\big(\big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}\big)_{/S_0}\big)_{\mathrm{disj}}\big),$$
such that for each $i \in I$, $\underline{G}$ is a factorization F-linear functor[35] with respect to the i-th factorization module structures on the source and the target.

For a fixed F, we call

$$G : (\mathbf{C}_i)_{i\in I} \to \mathbf{C}'$$

as above a *lax-unital factorization F-linear multi-functor*.

We say (F, G) is (strictly) unital if $\underline{F}$ and $\underline{G}$ are strict morphisms.

C.9.4. One can mimic the definition of compositions of usual multilinear maps to define compositions of *factorization* multilinear functors. Namely, for a given map $\phi : I \to I'$ between finite sets and (lax-)unital factorization multifunctors

$$(F, G_{i'}) : (\mathbf{A}, (\mathbf{C}_i)_{i\in\phi^{-1}(i')}) \to (\mathbf{A}', \mathbf{C}'_{i'}) \;\; i' \in I'$$

and

$$(F', G') : (\mathbf{A}', (\mathbf{C}'_{i'})_{i'\in I'}) \to (\mathbf{A}'', \mathbf{C}''),$$

there is a canonical (lax-)unital factorization multifunctor

$$\big(F' \circ F, G' \circ_\phi (G_{i'})\big) : (\mathbf{A}, (\mathbf{C}_i)_{i\in I}) \to (\mathbf{A}'', \mathbf{C}'').$$

A homotopy-coherent construction of these compositions will be provided in [CFZ].

C.9.5. *Remark.* The notion of $\mathbf{A}$-multilinear functors and their compositions is closely related to the framework of *pseudo-tensor categories* in [BD1]. See Sect. C.13 for more details.

C.9.6. *Remark.* Roughly speaking, a lax-unital factorization F-multilinear functor $G : (\mathbf{C}_i)_{i\in I} \to \mathbf{C}'$ consists of the following data:

- For any disjoint collection of points $\underline{y}_i \in \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}$ $(i \in I)$, assign a functor
$$G_{(\underline{y}_i)} : \otimes(\mathbf{C}_i)_{\underline{y}_i} \to \mathbf{C}'_{\sqcup \underline{y}_i};$$
- For two collections $(\underline{y}_i)$ and $(\underline{y}'_i)$ as above such that $\underline{y}_i \subseteq \underline{y}'_i$, assign a natural transformation

$$\begin{tikzcd} \otimes(\mathbf{C}_i)_{\underline{y}_i} \arrow[r, "\otimes\,\mathrm{ins}_{\underline{y}_i\subseteq\underline{y}'_i}"] \arrow[d, "G_{(\underline{y}_i)}"'] & \otimes(\mathbf{C}_i)_{\underline{y}'_i} \arrow[d, "G_{(\underline{y}'_i)}"] \\ \mathbf{C}'_{\sqcup\underline{y}_i} \arrow[r, "\mathrm{ins}_{\sqcup\underline{y}_i\subseteq\sqcup\underline{y}'_i}"'] \arrow[ur, Rightarrow, "G_{(\underline{y}_i\subseteq\underline{y}'_i)}"] & \mathbf{C}'_{\sqcup\underline{y}'_i}; \end{tikzcd}$$

[35]This is a structure rather than property.

- For any disjoint collection of points $\underline{y}_i \in \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}$ $(i \in I)$, $\underline{z}_j \in \mathrm{Ran}^{\mathrm{untl}}$ $(j \in J)$ and a map $\phi : J \to I$, assign a commutative diagram

$$\begin{tikzcd}
(\otimes_j \mathbf{A}_{\underline{z}_j}) \bigotimes (\otimes_i (\mathbf{C}_i)_{\underline{y}_i}) \arrow[r, "\simeq"] \arrow[d, "(\otimes F_{\underline{z}_j}) \otimes G_{(\underline{y}_i)}"'] & \bigotimes_i \left((\otimes_{j \in \phi^{-1}(i)} \mathbf{A}_{\underline{z}_j}) \otimes (\mathbf{C}_i)_{\underline{y}_i} \right) \arrow[r, "\simeq"] & \bigotimes_i (\mathbf{C}_i)_{(\sqcup_{j \in \phi^{-1}(i)} \underline{z}_j) \sqcup \underline{y}_i} \arrow[d, "G_{(\sqcup_{j \in \phi^{-1}(i)} \underline{z}_j) \sqcup \underline{y}_i}"] \\
(\otimes_j \mathbf{A}'_{\underline{z}_j}) \otimes \mathbf{C}'_{\sqcup \underline{y}_i} \arrow[rr, "\simeq"'] & & \mathbf{C}'_{(\sqcup \underline{z}_j) \bigsqcup (\sqcup \underline{y}_i)},
\end{tikzcd}$$

where the horizontal equivalences come from the factorization module structures on $\mathbf{C}_i$ and $\mathbf{D}$.
- Higher compatibilities between the above structures.

C.9.7. *Example.* Let $\mathbf{A}$ be a unital factorization category and $\mathbf{C} \in \mathbf{A}\text{--}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0}$. For $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}(S_0)$ such that $\underline{x} \cap \underline{x}_0 = \emptyset$, consider

$$\mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}_0 \sqcup \underline{x}}(\mathbf{C}) \in \mathbf{A}\text{--}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0 \sqcup \underline{x}}$$

(see Sect. C.8.2). There is a canonical unital factorization $\mathbf{A}$-linear bifunctor

$$(\mathbf{A}^{\mathrm{fact}_{\underline{x}}}, \mathbf{C}) \to \mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}_0 \sqcup \underline{x}}(\mathbf{C})$$

constructed as follows.

By definition, we need to define a strict morphism

$$(\underline{\mathbf{A}}|_{\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}}} \underset{S_0}{\boxtimes} \underline{\mathbf{C}})|_{\mathrm{disj}} \to \underline{\mathbf{C}}|_{(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}} \times_{S_0} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}})} \tag{C.24}$$

between crystal of categories over $(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}} \times_{S_0} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}})$, such that it is compatible with the factorization $\mathbf{A}$-module structures coming from both factors. Here the RHS means pullback of $\underline{\mathbf{C}}$ along the map

$$(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}} \times_{S_0} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0},\ (\underline{y}, \underline{y}_0) \to \underline{y} \sqcup \underline{y}_0.$$

Note that this map factors as

$$(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}} \times_{S_0} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}} \to (\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}} \xrightarrow{\text{union}} \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}.$$

Hence to construct (C.24), we only need a strict morphism

$$(\underline{\mathbf{A}} \boxtimes \underline{\mathbf{C}})|_{\mathrm{disj}} \to \underline{\mathbf{C}}|_{(\mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0})_{\mathrm{disj}}} \tag{C.25}$$

compatible with the factorization $\mathbf{A}$-module structures coming from both factors. However, the factorization $\mathbf{A}$-module structure of $\mathbf{C}$ implies there is a canonical *isomorphism* (C.25).

C.9.8. Let $\mathbf{A}$ be a unital factorization category and

$$\mathbf{C}_i \in \mathbf{A}\text{--}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_i}\ \ (i \in I),\ \ \mathbf{D} \in \mathbf{A}\text{--}\mathbf{mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i}$$

with I being finite. We say a *unital* factorization $\mathbf{A}$-linear multi-functor

$$G : (\mathbf{C}_i)_{i \in I} \to \mathbf{D}$$

exhibits $\mathbf{C}'$ *as the fusion product of* $\mathbf{C}_i$ if pre-composing with $(\mathrm{id}_{\mathbf{A}}, G)$ induces an equivalence between

- the category of lax-unital factorization functors

$$(\mathbf{A}, \mathbf{D}) \to (\mathbf{A}', \mathbf{D}')$$

- the category of lax-unital factorization multi-functors

$$(\mathbf{A}, (\mathbf{C}_i)_{i \in I}) \to (\mathbf{A}', \mathbf{D}')$$

for any unital factorization category $\mathbf{A}'$ and $\mathbf{D}' \in \mathbf{A}'\text{--}\mathbf{mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i}$.

Note that $\mathbf{D}$, equipped with the multi-functor G, is essentially unique if exists. We write

$$\overset{\mathrm{fact}}{\underset{\mathbf{A}}{\boxtimes}} \mathbf{C}_i$$

for this object.

C.9.9. A detailed proof of the following result[36] will be provided in [CFZ].

Proposition C.9.10. *Let* $\mathbf{A}$ *be a unital factorization category and*

$$\mathbf{C}_i \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_i}, \;\; i \in I$$

with I *being finite. Then the external fusion product*

$$\overset{\mathrm{fact}}{\underset{\mathbf{A}}{\boxtimes}} \mathbf{C}_i \in \mathbf{A}\text{-}\mathbf{mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i}$$

exists, and the structural functor

$$\big((\boxtimes \underline{\mathbf{C}}_i)_{/S_0}\big)|_{\mathrm{disj}} \to \mathrm{union}^*_{(\underline{x}_i)}(\underline{\overset{\mathrm{fact}}{\underset{\mathbf{A}}{\boxtimes}} \mathbf{C}_i})|_{\mathrm{disj}}$$

is an equivalence.

Sketch. It is enough to treat the case $I = \{1, 2\}$. To simplify the notations, we asssume $S_0 = \mathrm{pt}$ (otherwise one just replace absolute products, tensor products below by relative ones).

Consider the Bar simplicial diagram

$$\cdots \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2} \rightrightarrows \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2}$$

associated to the $\mathrm{Ran}^{\mathrm{untl}}$-modules $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_i}$. We can restrict to the disjoint loci and obtain a simplicial diagram

$$\cdots (\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times \mathrm{Ran}^{\mathrm{untl}} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2})_{\mathrm{disj}} \rightrightarrows (\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2})_{\mathrm{disj}}. \tag{C.26}$$

For $n \geq 0$, let

$$u_n : (\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times (\mathrm{Ran}^{\mathrm{untl}})^{\times n} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2})_{\mathrm{disj}} \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1 \sqcup \underline{x}_2}$$

be the union map. Note that these maps provide an augmentation of the diagram (C.26). One can show this gives an etale[37] hypercover for $\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1 \sqcup \underline{x}_2}$.

The factorization structures on $\mathbf{A}$ and $\mathbf{C}_i$ implies the objects

$$(\underline{\mathbf{C}}_1 \boxtimes \underline{\mathbf{A}}^{\boxtimes n} \boxtimes \underline{\mathbf{C}}_2)|_{\mathrm{disj}} \in \mathbf{CrysCat}^{\mathrm{strict}}((\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times (\mathrm{Ran}^{\mathrm{untl}})^{\times n} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2})_{\mathrm{disj}}). \tag{C.27}$$

are compatible with (C.26) and pullback functors. One can prove crystals of categories on categorical prestacks satisfy etale hyperdescent. It follows that there is a unique object

$$\underline{\mathbf{C}} \in \mathbf{CrysCat}^{\mathrm{strict}}((\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_1} \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_2})_{\mathrm{disj}})$$

such that $u^*_\bullet(\underline{\mathbf{C}}) \simeq (\underline{\mathbf{C}}_1 \boxtimes \underline{\mathbf{A}}^{\boxtimes \bullet} \boxtimes \underline{\mathbf{C}}_2)|_{\mathrm{disj}}$. It is easy to see $\underline{\mathbf{C}}$ has a natural factorization $\mathbf{A}$-module structure and the equivalence

$$(\underline{\mathbf{C}}_1 \boxtimes \underline{\mathbf{C}}_2)|_{\mathrm{disj}} \xrightarrow{\simeq} u_0^* \underline{\mathbf{C}}$$

exhibits $\mathbf{C}$ as the fusion product of $\mathbf{C}_1$ and $\mathbf{C}_2$.

□

C.9.11. *Example.* It is easy to see (for example from the proof of the above proposition) that the $\mathbf{A}$-linear bifunctor

$$(\mathbf{A}^{\mathrm{fact}_{\underline{x}}}, \mathbf{C}) \to \mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}_0 \sqcup \underline{x}}(\mathbf{C})$$

constructed in Sect. C.9.7 induces an equivalence

$$\mathbf{A}^{\mathrm{fact}_{\underline{x}}} \overset{\mathrm{fact}}{\underset{\mathbf{A}}{\boxtimes}} \mathbf{C} \simeq \mathbf{prop}_{\underline{x}_0 \subseteq \underline{x}_0 \sqcup \underline{x}}(\mathbf{C}).$$

In particular,

$$\mathbf{A}^{\mathrm{fact}_{\underline{x}}} \overset{\mathrm{fact}}{\underset{\mathbf{A}}{\boxtimes}} \mathbf{A}^{\mathrm{fact}_{\underline{x}'}} \simeq \mathbf{A}^{\mathrm{fact}_{\underline{x} \sqcup \underline{x}'}}.$$

C.9.12. The details for the remaining part of this subsection will be provided in [CFZ].

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[36] In fact, we only need the example in Sect. C.9.11.

[37] A categorical $\mathcal{Z}$ is etale over $\mathcal{Y}$ if it is a Cartesian space over $\mathcal{Y}$ (see Sect. C.1.5) and the base-change $\mathcal{Z} \times_{\mathcal{Y}} S$ is an algebraic space etale over $\mathcal{Y}$.

C.9.13. Let $\mathbf{A}$ be a unital factorization category and $\mathcal{A}$ be a factorization algebra in $\mathbf{A}$. For a finite collection of disjoint points $\underline{x}_i \in \mathrm{Ran}^{\mathrm{untl}}(S_0)$, $i \in I$, there is a canonical functor

$$\prod \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}_i} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i},\ (\mathcal{C}_i)_{i\in I} \mapsto \overset{\mathrm{fact}}{\underset{\mathcal{A}}{\boxtimes}} \mathcal{C}_i \tag{C.28}$$

that sends $(\mathcal{C}_i)_{i\in I}$ to the lax-unital factorization $\mathcal{A}$-linear functor

$$\mathrm{Vect}^{\mathrm{fact}_{\sqcup \underline{x}_i}} \to \mathbf{A}^{\mathrm{fact}_{\sqcup \underline{x}_i}}$$

corresponding to the lax-unital factorization $\mathcal{A}$-linear *multi-functor*

$$(\mathrm{Vect}^{\mathrm{fact}_{\underline{x}_i}})_{i\in I} \xrightarrow{(\mathcal{C}_i)} (\mathbf{A}^{\mathrm{fact}_{\underline{x}_i}})_{i\in I} \to \mathbf{A}^{\mathrm{fact}_{\sqcup \underline{x}_i}}.$$

One can show the functor (C.28) is $\mathrm{D}(S_0)$-multilinear. Hence we obtain a functor

$$\bigotimes_{\mathrm{D}(S_0)} \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}_i} \to \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i},\ (\mathcal{C}_i)_{i\in I} \mapsto \overset{\mathrm{fact}}{\underset{\mathcal{A}}{\boxtimes}} \mathcal{C}_i, \tag{C.29}$$

which is called the *external fusion functor* for factorization $\mathcal{A}$-modules. By construction, we have a canonical commutative diagram:

$$\begin{tikzcd} \bigotimes_{\mathrm{D}(S_0)} \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\underline{x}_i} \arrow[r] \arrow[d] & \mathcal{A}\text{-mod}^{\mathrm{fact}}_{\sqcup \underline{x}_i} \arrow[d] \\ \bigotimes_{\mathrm{D}(S_0)} \mathbf{A}_{\underline{x}_i} \arrow[r, "\simeq"] & \mathbf{A}_{\sqcup \underline{x}_i}, \end{tikzcd} \tag{C.30}$$

where the vertical arrows are the forgetful functors, and the bottom equivalence is due to the factorization structure on $\mathbf{A}$.

C.9.14. Moreover, using the universal propery of external fusion, one can show the functors (C.29) are compatible with the change-of-base functors (C.17). In other words, for any finite set I, we obtain a strict morphism

$$\mathrm{mult}_I : (\underset{i\in I}{\boxtimes} \mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}})|_{\mathrm{disj}} \to \mathrm{union}_I^*(\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}})|_{\mathrm{disj}}$$

in

$$\mathbf{CrysCat}^{\mathrm{strict}}((\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}})_{\mathrm{disj}}).$$

Finally, using the universal propery of external fusion, one can supply a datum of associativity and commutativity for the functors mult_I. In other words, we obtain a structure of unital *lax-factorization* category (see Sect. C.4.5) on $\mathcal{A}\text{-}\underline{\mathrm{mod}}^{\mathrm{fact}}$. We denote it just by

$$\mathcal{A}\text{-mod}^{\mathrm{fact}} \in \mathbf{UntlFactCat}.$$

Moreover, the (C.21) provides a *lax-unital* factorization functor

$$\mathrm{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\mathrm{fact}} \to \mathbf{A}.$$

By construction, it sends the unit

$$\mathrm{unit}_{\mathcal{A}\text{-mod}^{\mathrm{fact}}}$$

of $\mathcal{A}$-mod$^{\mathrm{fact}}$ to $\mathcal{A} \in \mathrm{FactAlg}(\mathbf{A})$. Therefore we write

$$\mathcal{A}^{\mathrm{enh}} := \mathrm{unit}_{\mathcal{A}\text{-mod}^{\mathrm{fact}}} \in \mathrm{FactAlg}(\mathcal{A}\text{-mod}^{\mathrm{fact}}).$$

C.9.15. The construction $(\mathbf{A},\mathcal{A})\mapsto \mathcal{A}\text{-mod}^{\mathrm{fact}}$ is functorial in $\mathbf{A}$. In other words, for any lax-unital factorization functor $F:\mathbf{A}\to\mathbf{A}'$, we have a canonical *unital*[38] factorization functor

$$F^{\mathrm{enh}}:\mathcal{A}\text{-mod}^{\mathrm{fact}}\to F(\mathcal{A})\text{-mod}^{\mathrm{fact}}$$

compatible with $\mathrm{oblv}_{\mathcal{A}}$, $\mathrm{oblv}_{F(\mathcal{A})}$ and F.

Combining with Lemma C.7.13, we obtain a canonical unital factorization functor

$$F^{\mathrm{enh}}:\mathbf{A}\to F(\mathrm{unit}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}$$

such that $\mathrm{oblv}_{F(\mathrm{unit}_{\mathbf{A}})}\circ F^{\mathrm{enh}}\simeq F$.

C.9.16. *Variant.* Let $\mathbf{A}$ be a unital factorization category and $\mathcal{A}$ be a factorization algebra in $\mathbf{A}$. For any $\mathbf{C}\in\mathbf{A}\mathbf{-mod}^{\mathrm{fact}}_{\underline{x}_0}$, one can similarly construct a unital *lax-factorization* ($\mathcal{A}\text{-mod}^{\mathrm{fact}}$)-module category at $\underline{x}_0$, denoted by

$$\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C}),$$

such that its fiber at $\underline{x}_0$ is the DG category of factorization $\mathcal{A}$-modules in $\mathbf{C}$ (see Lemma C.8.10). It is equipped with a lax-unital $\mathrm{oblv}_{\mathcal{A}}$-linear factorization functor

$$\mathrm{oblv}_{\mathcal{A},\mathbf{C}}:\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})\to\mathbf{C}.$$

C.9.17. Moreover, for a lax-unital factorization functor $(F,G):(\mathbf{A},\mathbf{C})\to(\mathbf{A}',\mathbf{C}')$, we have a *unital* factorization F^{enh}-linear functor

$$G^{\mathrm{enh}}:\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{C})\to F(\mathcal{A})\text{-mod}^{\mathrm{fact}}(\mathbf{C}').$$

In particular, we have a unital factorization F^{enh}-linear functor

$$G^{\mathrm{enh}}:\mathbf{C}\to F(\mathrm{unit}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}(\mathbf{C}').$$

Conversely, given any G^{enh} as above, one can recover G as $\mathrm{oblv}_{F(\mathrm{unit}_{\mathbf{A}}),\mathbf{C}}\circ G^{\mathrm{enh}}$. One can check these two constructions are inverse to each other. In other words, for fixed $F:\mathbf{A}\to\mathbf{A}'$ and modules $\mathbf{C}$ and $\mathbf{C}'$, the following two data are equivalent:

- A unital factorization F^{enh}-linear functor
$$G^{\mathrm{enh}}:\mathbf{C}\to F(\mathrm{unit}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}(\mathbf{C}').$$
- A lax unital factorization F-linear functor
$$G:\mathbf{C}\to\mathbf{C}'.$$

C.10. **Restrictions of factorization modules.**

C.10.1. We say a functor $\mathbf{E}\to\mathbf{B}$ between 2-categories is a (1,2)-Cartesian fibration if

- There are enough Cartesian 1-morphisms. In other words, for any morphism $f:u\to v$ in $\mathbf{B}$ and a lifting $V\in\mathbf{E}$ of v, there exists a lifting $F:U\to V$ of f such that for any $W\in\mathbf{E}$ over $w\in\mathbf{B}$, the following square of categories is Cartesian:
$$\begin{tikzcd} \mathrm{Maps}_{\mathbf{E}}(W,U) \arrow[r, "F\circ-"] \arrow[d] & \mathrm{Maps}_{\mathbf{E}}(W,V) \arrow[d] \\ \mathrm{Maps}_{\mathbf{B}}(w,u) \arrow[r, "f\circ-"] & \mathrm{Maps}_{\mathbf{B}}(w,v) \end{tikzcd}$$
- There are enough Caresian 2-morphisms. In other words, for objects $U,V\in\mathbf{E}$ over $u,v\in\mathbf{B}$, the functor
$$\mathrm{Maps}_{\mathbf{E}}(U,V)\to\mathrm{Maps}_{\mathbf{B}}(u,v)$$
is a Cartesian fibration.
- The collection of Cartesian 2-morphisms are closed under horizontal compositions.

[38] The functor below is strictly unital by Lemma C.5.9, which is also true for unital lax-factorization categories (including $\mathcal{A}\text{-mod}^{\mathrm{fact}}$).

C.10.2. *Remark.* The theory of (1,2)-Cartesian fibrations is developed in [GHL], [AS1] and [AS2] under the name of *inner 2-Cartesian fibrations*. In particular, there is a Grothendieck cosntruction in [AS2] which says knowing a (1,2)-Cartesian fibration $\mathbf{E} \to \mathbf{B}$ is equivalent to knowing a functor $\mathbf{B}^{1\text{-op},2\text{-op}} \to 2-\mathbf{Cat}$.

C.10.3. The following result will be one of the main theorems for [CFZ].

Theorem C.10.4. *The forgetful functor*

$$\mathbf{UntlFactModCat}_{\underline{x}_0}^{\text{lax-untl}} \to \mathbf{UntlFactCat}^{\text{lax-untl}} \tag{C.31}$$

is a (1,2)-Cartesian fibration.

C.10.5. We will explain the main ideas of the proof in the next subsection. For now, we deduce some useful results from it.

C.10.6. The Grothendieck construction in [AS2] provides a functor

$$(\mathbf{UntlFactCat}^{\text{lax-untl}})^{2\text{-op},1\text{-op}} \to 2-\mathbf{Cat}. \tag{C.32}$$

In other words, for a *lax-unital* factorization functor $\Phi : \mathbf{A} \to \mathbf{B}$, we have a contravariant transport functor between the fibers of (C.31):

$$\mathbf{Res}_{\Phi}^{\text{untl}} : \mathbf{B}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}} \to \mathbf{A}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}}.$$

The construction $\Phi \mapsto \mathbf{Res}_{\Phi}^{\text{untl}}$ is contravariant. In other words, for a 2-morphism $\Phi \to \Phi'$, we have a natural transformation

$$\mathbf{Res}_{\Phi'}^{\text{untl}} \to \mathbf{Res}_{\Phi}^{\text{untl}}.$$

When Φ is unital, we also write $\mathbf{Res}_{\Phi} := \mathbf{Res}_{\Phi}^{\text{untl}}$.

C.10.7. Recall that inside any 2-category, there is a notion of *adjoint pair of 1-morphisms.* Moreover, a functor between 2-categories always sends adjoint pairs to adjoint pairs. Applying to the functor to (C.32), we obtain:

Proposition C.10.8. *Suppose* $\Phi : \mathbf{A} \rightleftarrows \mathbf{B} : \Psi$ *is an adjoint pair in* $\mathbf{UntlFactCat}^{\text{lax-untl}}$*, then we have an adjoint pair*

$$\mathbf{Res}_{\Phi}^{\text{untl}} : \mathbf{B}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}} \rightleftarrows \mathbf{A}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}} : \mathbf{Res}_{\Psi}^{\text{untl}}$$

in $2-\mathbf{Cat}$*. In particular, for* $\mathbf{M} \in \mathbf{A}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}}$ *and* $\mathbf{N} \in \mathbf{B}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}}$*, we have a canonical equivalence between the following categories:*

- *The category of unital factorization* $\mathbf{A}$*-linear functors* $\mathbf{Res}_{\Phi}^{\text{untl}}(\mathbf{N}) \to \mathbf{M}$*;*
- *The category of unital factorization* $\mathbf{B}$*-linear functors* $\mathbf{N} \to \mathbf{Res}_{\Psi}^{\text{untl}}(\mathbf{M})$*.*

C.10.9. For $\mathbf{N} \in \mathbf{B}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}}$, the object $\mathbf{Res}_{\Phi}^{\text{untl}}(\mathbf{N})$ is called the *restriction of* $\mathbf{D}$ *along* Φ. By definition, it is equipped with a lax-unital Φ-linear factorization functor

$$\mathbf{Res}_{\Phi}^{\text{untl}}(\mathbf{N}) \to \mathbf{N},$$

which is a Cartesian lifting of Φ in $\mathbf{UntlFactModCat}_{\underline{x}_0}^{\text{lax-untl}}$.

In other words, for any test lax-unital factorization functor $F : \mathbf{A}' \to \mathbf{A}$ and test object $\mathbf{M}' \in \mathbf{A}'\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}}$, pre-composing with $\mathbf{Res}_{F}^{\text{untl}}(\mathbf{N}) \to \mathbf{N}$ induces an equivalence between the following categories:

- The category of lax-unital $\Phi \circ F$-linear factorization functors $\mathbf{M}' \to \mathbf{N}$;
- The category of lax-unital F-linear factorization functors $\mathbf{M}' \to \mathbf{Res}_{F}^{\text{untl}}(\mathbf{N})$.

Taking $\mathbf{A}' = \text{Vect}$ and $\mathbf{M}' = \text{Vect}^{\text{fact}_{\underline{x}_0}}$, we obtain the following result.

Proposition C.10.10. *Let* $\Phi : \mathbf{A} \to \mathbf{B}$ *be a lax-unital factorization functor and* $\mathbf{N} \in \mathbf{B}\text{--}\mathbf{mod}_{\underline{x}_0}^{\text{fact}}$*. Then for any* $\mathcal{A} \in \text{FactAlg}(\mathbf{A})$*, pre-composing with* $\mathbf{Res}_{F}^{\text{untl}}(\mathbf{N}) \to \mathbf{N}$ *induces an equivalence*

$$\mathcal{A}\text{-mod}^{\text{fact}}(\mathbf{Res}_{F}^{\text{untl}}(\mathbf{N}))_{\underline{x}_0} \xrightarrow{\simeq} F(\mathcal{A})\text{-mod}^{\text{fact}}(\mathbf{N})_{\underline{x}_0}.$$

C.10.11. Combining with Lemma C.7.13, we obtain the following result.

Corollary C.10.12. *Let $\Phi : \mathbf{A} \to \mathbf{B}$ be a lax-unital factorization functor and $\mathbf{N} \in \mathbf{B}\mathrm{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0}$. There is a canonical dotted equivalence making the following diagram commute*

$$\begin{tikzcd}
(\mathbf{Res}^{\mathrm{untl}}_F(\mathbf{N}))_{\underline{x}_0} \arrow[r, dotted, "\simeq"] \arrow[d] & F(\mathrm{unit}_{\mathbf{A}})\text{-mod}^{\mathrm{fact}}(\mathbf{N})_{\underline{x}_0} \arrow[d, "\mathrm{oblv}_{\underline{x}_0}"] \\
\mathbf{N}_{\underline{x}_0} \arrow[r, equal] & \mathbf{N}_{\underline{x}_0}.
\end{tikzcd}$$

C.10.13. In particular, for $\mathbf{A} = \mathrm{Vect}$, we obtain:

Corollary C.10.14. *Let $\mathbf{B}$ be a unital factorization category and $\mathcal{B} \in \mathrm{FactAlg}(\mathbf{B})$. For any $\mathbf{N} \in \mathbf{B}\mathrm{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0}$, there is a canonical equivalence*

$$(\mathbf{Res}^{\mathrm{untl}}_{\mathcal{B}}(\mathbf{N}))_{\underline{x}_0} \simeq \mathcal{B}\text{-mod}^{\mathrm{fact}}(\mathbf{N})_{\underline{x}_0}.$$

C.10.15. *Remark.* Let $\Phi : \mathbf{A} \to \mathbf{B}$ be a *unital* factorization functor and $\mathbf{M} \in \mathbf{B}\mathrm{-}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0}$. Then the underlying crystal of categories for $\mathbf{Res}_\Phi(\mathbf{M})$ can be explicitly calculated as a limit (see Sect. C.11.7 and Sect. C.11.11). For example, the restriction of $\mathbf{Res}_\Phi(\mathbf{M})$ along the map

$$X \times S_0 \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0},\ y \mapsto y \cup \underline{x}_0$$

fits into the following Cartesian square (Sect. C.11.18):

$$\begin{tikzcd}
\mathbf{Res}_\Phi(\mathbf{M})|_{X\times S_0} \arrow[r] \arrow[d] & \mathbf{M}|_{X\times S_0} \arrow[d] \\
j_*j^*(\mathbf{A}|_X \boxtimes \mathbf{M}|_{\underline{x}_0}) \arrow[r] & j_*j^*(\mathbf{M}|_{X\times S_0}),
\end{tikzcd}$$

where

- $j : (X \times S_0) \setminus \mathrm{graph}_{\underline{x}_0} \to X \times S_0$ is the complement of the union of the graphs for elements in $\underline{x}_0 \subseteq X(S_0)$;
- the bottom horizontal functor is provided by the factorization $\mathbf{B}$-module structure on $\underline{\mathbf{M}}$ and the functor Φ.

C.10.16. Let $g : \mathcal{B} \to \mathcal{B}'$ be a morphism in $\mathrm{FactAlg}(\mathbf{B})$. The natural transformation $\mathbf{Res}^{\mathrm{untl}}_g : \mathbf{Res}^{\mathrm{untl}}_{\mathcal{B}'} \to \mathbf{Res}^{\mathrm{untl}}_{\mathcal{B}}$ provides a functor

$$(\mathbf{Res}^{\mathrm{untl}}_{\mathcal{B}'}(\mathbf{N}))_{\underline{x}_0} \to (\mathbf{Res}^{\mathrm{untl}}_{\mathcal{B}}(\mathbf{N}))_{\underline{x}_0}.$$

By the above corollary, we obtain a canonical functor

$$\mathrm{Res}_g : \mathcal{B}'\text{-mod}^{\mathrm{fact}}(\mathbf{N})_{\underline{x}_0} \to \mathcal{B}\text{-mod}^{\mathrm{fact}}(\mathbf{N})_{\underline{x}_0}.$$

For $\mathcal{N} \in \mathcal{B}'\text{-mod}^{\mathrm{fact}}(\mathbf{N})_{\underline{x}_0}$, its image $\mathrm{Res}_g(\mathcal{N})$ is called the *restriction of* $\mathcal{N}$ *along* g.

C.10.17. *Remark.* By definition, a (1,2)-Cartesian fibration $\pi : \mathbf{E} \to \mathbf{F}$ induces a Cartesian fibrations

$$\mathrm{Maps}_{\mathbf{E}}(u, v) \to \mathrm{Maps}_{\mathbf{F}}(\pi(u), \pi(v))$$

of 1-categories. Hence Theorem C.31 implies the forgetful functor

$$\mathrm{FactMod}(\mathbf{B}, \mathbf{N})_{\underline{x}_0} \to \mathrm{FactAlg}(\mathbf{B})$$

is a Cartesian fibration between 1-categories. It follows from construction that the functor Res_g is the contravariant transport functor for this Cartesian fibration.

C.10.18. We now provide a useful criterion to check whether a factorization module category is obtained via restriction. We need the following lemma.

Lemma C.10.19. *The forgetful functor*

$$\mathbf{A}\text{–}\mathbf{mod}^{\mathrm{fact}}_{\underline{x}_0} \to \mathbf{CrysCat}(S_0),\ \mathbf{M} \mapsto \mathbf{M}_{\underline{x}_0}$$

is conservative on 2-morphisms.

Sketch. Let $G_1, G_2 : \mathbf{M} \to \mathbf{M}'$ be factorization $\mathbf{A}$-linear functors and $\alpha : G_1 \to G_2$ be a 2-morphism between them such that $\alpha_{\underline{x}_0}$ is invetible. For any affine test scheme $\underline{x} : S \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}$, we need to show $\alpha_{\underline{x}}$ is also invertible.

Suppose $\underline{x} = \underline{x}_0|_S \sqcup \underline{y}$ can be written as a disjoint union of subsets. Then the factorization structure implies

$$(G_i)_{\underline{x}} : \mathbf{M}_{\underline{x}} \to \mathbf{M}'_{\underline{x}}$$

can be identified with

$$\mathrm{Id} \otimes (G_i)_{\underline{x}_0} : \mathbf{A}_{\underline{y}} \otimes_{\mathrm{D}(S)} \mathbf{M}_{\underline{x}_0|_S} \to \mathbf{A}_{\underline{y}} \otimes_{\mathrm{D}(S)} \mathbf{M}'_{\underline{x}_0|_S},$$

and the 2-morphism $\alpha_{\underline{x}}$ can be identified with $\mathrm{Id} \otimes \alpha_{\underline{x}_0|_S}$. This implies $\alpha_{\underline{x}}$ is invertible because $\alpha_{\underline{x}_0}$ is so.

For the general case, we can replace S with a covering of locally closed subschemes such that the above property holds on each subscheme. This reduces the general case to the above case.

□

Proposition C.10.20. *Let*

$$(\Phi, \Phi^m) : (\mathbf{A}, \mathbf{M}) \rightleftarrows (\mathbf{B}, \mathbf{N}) : (\Psi, \Psi^m)$$

be an adjoint pair in $\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0}$ *such that:*

(i) *The left adjoint Φ is unital*[39]*;*
(ii) *It induces an equivalence*

$$\Phi^m_{\underline{x}_0} : \mathbf{M}_{\underline{x}_0} \rightleftarrows \mathbf{N}_{\underline{x}_0} : \Psi^m_{\underline{x}_0}$$

in $\mathbf{CrysCat}(S_0)$.

Then the canonical factorization $\mathbf{A}$*-linear functor*

$$\mathbf{M} \to \mathbf{Res}^{\mathrm{untl}}_{\Phi}(\mathbf{N})$$

is an equivalence.

Sketch. Using the universal property of $\mathbf{Res}^{\mathrm{untl}}_{\Phi}$, it is easy to show the given adjoint pair can be written as the composition of[40]

$$(\mathbf{A}, \mathbf{M}) \rightleftarrows (\mathbf{A}, \mathbf{Res}^{\mathrm{untl}}_{\Phi}(\mathbf{N}))$$

and

$$(\mathbf{A}, \mathbf{Res}^{\mathrm{untl}}_{\Phi}(\mathbf{N})) \rightleftarrows (\mathbf{B}, \mathbf{N}).$$

Moreover, by Corollary C.10.12 and assumption (i), the second pair induces an equivalence

$$\mathbf{Res}^{\mathrm{untl}}_{\Phi}(\mathbf{N})_{\underline{x}_0} \rightleftarrows \mathbf{N}_{\underline{x}_0}.$$

Hence by assumption (ii), the first pair also induces an equivalence

$$\mathbf{M}_{\underline{x}_0} \rightleftarrows \mathbf{Res}^{\mathrm{untl}}_{\Phi}(\mathbf{N})_{\underline{x}_0}.$$

Now the claim follows formally from Lemma C.10.19.

□

C.11. **Sketch of Theorem C.10.4.** In this subsection, we explain the main ideas in the proof of Theorem C.10.4. A detailed proof will be provided in [CFZ].

[39] In fact, a standard argument shows that this is automatic.
[40] Similar claim is true for any (1,2)-Cartesian fibration.

C.11.1. Let $\mathbf{E} \to \mathbf{B}$ be a functor between 2-categories. To show it is a (1,2)-Cartesian fibration, one only needs to show:

(a) There are enough *locally Cartesian 1-morphisms*. By definition, this means for any arrow $\Delta^1 \to \mathbf{B}$, the base-change $\mathbf{E} \times_{\mathbf{B}} \Delta^1 \to \Delta^1$ has enough Cartesian 1-morphisms.
(b) There are enough *locally Cartesian 2-morphisms*.
(c) The collection of locally Cartesian 1-morphisms are closed under compositions.
(d) The collection of locally Cartesian 2-morphisms are closed under both horizontal and vertical compositions.

In [CFZ], we will provide a *constructive* proof for (a) and (b), and use these explicit constructions to verify (c) and (d). In this subsection, we only explain the construction of locally Cartsian 1-morphisms in

$$\mathbf{UntlFactModCat}^{\text{lax-untl}}_{\underline{x}_0} \to \mathbf{UntlFactCat}^{\text{lax-untl}} \tag{C.33}$$

The construction for locally Cartesian 2-morphisms is similar but much simpler.

C.11.2. Let $\Phi : \mathbf{A} \to \mathbf{B}$ be a lax-unital factorization functor, i.e., a moprhism in the base of (C.33). Let $\mathbf{N}$ be a unital factorization $\mathbf{B}$-module at $\underline{x}_0$, i.e., an object in the fiber of (C.33) over $\mathbf{B}$. We will construct a locally coCartesian arrow $\Phi_m : \mathbf{M} \to \mathbf{N}$ lying over Φ.

We can enlarge (C.33) to allow *lax-factorization* (module)-cateogires:

$$\mathbf{UntlLaxFactModCat}^{\text{lax-untl}}_{\underline{x}_0} \to \mathbf{UntlLaxFactCat}^{\text{lax-untl}}. \tag{C.34}$$

By Sect. C.9.15 and Sect. C.9.16, the morphism Φ factors as

$$\mathbf{A} \xrightarrow{\Phi^{\text{enh}}} \Phi(\text{unit}_{\mathbf{A}})\text{-mod}^{\text{fact}} \xrightarrow{\text{oblv}_{\text{unit}_{\mathbf{A}}}} \mathbf{B}$$

in the base of (C.34), and there is a canonical morphism in the source of (C.34)

$$\text{oblv}_{\text{unit}_{\mathbf{A}},\mathbf{N}} : \Phi(\text{unit}_{\mathbf{A}})\text{-mod}^{\text{fact}}(\mathbf{N}) \to \mathbf{N}$$

that lifts the morphism $\text{oblv}_{\text{unit}_{\mathbf{A}}}$.

By the universal property in Sect. C.9.17, we only need to show there exists an arrow in (C.34)

$$\mathbf{M} \to \Phi(\text{unit}_{\mathbf{A}})\text{-mod}^{\text{fact}}(\mathbf{N}) \tag{C.35}$$

that lifts Φ^{enh} such that for any test object $\mathbf{M}' \in \mathbf{A}\text{-}\mathbf{mod}^{\text{fact}}_{\underline{x}_0}$, it induces an equivalence between:

- The category of unital factorization $\mathbf{A}$-linear functors $\mathbf{M}' \to \mathbf{M}$;
- The category of unital factorization Φ^{enh}-linear functors $\mathbf{M}' \to \Phi(\text{unit}_{\mathbf{A}})\text{-mod}^{\text{fact}}(\mathbf{N})$.

C.11.3. Roughly speaking, the above reduction allows us to get rid of lax-unital functors, with the caveat that $\mathbf{B}$ (and $\mathbf{N}$) is allowed to be *lax-factorization*.

C.11.4. Now the desired claim (about existence of (C.35)) follows formally from the following two claims:

(i) The functor

$$\mathbf{UntlLaxFactModCat}_{\underline{x}_0} \to \mathbf{UntlLaxFactCat} \tag{C.36}$$

has enough locally Cartesian 1-morphisms.

(ii) Consider the embedding

$$\mathbf{UntlFactModCat}_{\underline{x}_0} \to \mathbf{UntlLaxFactModCat}_{\underline{x}_0}$$

and its fiber at an object $\mathbf{A} \in \mathbf{UntlFactCat}$:

$$\mathbf{A}\text{-}\mathbf{mod}^{\text{fact}}_{\underline{x}_0} \to \mathbf{A}\text{-}\mathbf{mod}^{\text{laxfact}}_{\underline{x}_0}.$$

The latter functor admits a right adjoint.

C.11.5. Claim (i) is obvious modulo homotopy-coherent issues.

Namely, for any morphism $\Phi : \mathbf{A} \to \mathbf{B}$ in **UntlLaxFactCat** and an object $\mathbf{N} \in \mathbf{B}\text{−}\mathbf{mod}^{\text{laxfact}}_{\underline{x}_0}$, the underlying crystal of categories $\underline{\mathbf{N}}$ has a unital *lax*-factorization $\mathbf{A}$-module structure given by

$$\big((\underset{i\in I^\circ}{\boxtimes} \underline{\mathbf{A}}) \boxtimes \underline{\mathbf{N}}\big)|_{\text{disj}} \to \big((\underset{i\in I^\circ}{\boxtimes} \underline{\mathbf{B}}) \boxtimes \underline{\mathbf{N}}\big)|_{\text{disj}} \to \text{union}^*_I(\underline{\mathbf{N}})|_{\text{disj}},$$

where the first functor is given by Φ, and the second functor is the lax-factorization $\mathbf{B}$-module structure on $\underline{\mathbf{N}}$ (see Sect. C.6.8). In other words, we obtain a canonical object

$$\mathbf{Res}^{\text{lax-fact}}_{\Phi}(\mathbf{N}) \in \mathbf{A}\text{−}\mathbf{mod}^{\text{laxfact}}_{\underline{x}_0}$$

equipped with a unital factorization Φ-linear functor

$$\mathbf{Res}^{\text{lax-fact}}_{\Phi}(\mathbf{N}) \to \mathbf{N}.$$

One can check this is a locally Cartesian 1-morphism in (C.36). A homotopy-coherent proof using the language of (generalized) operads will be provided in [CFZ].

C.11.6. Claim (ii) is proved via an explicit construction of the desired right adjoint *strictening* functor

$$\mathbf{Str}_{\mathbf{A}} : \mathbf{A}\text{−}\mathbf{mod}^{\text{laxfact}}_{\underline{x}_0} \to \mathbf{A}\text{−}\mathbf{mod}^{\text{fact}}_{\underline{x}_0}.$$

In fact, for future reference, we will show the diagram

$$\begin{tikzcd}
\mathbf{UntlFactModCat}_{\underline{x}_0} \arrow[r, "\subseteq"] \arrow[d] & \mathbf{UntlLaxFactModCat}_{\underline{x}_0} \arrow[d] \\
\mathbf{UntlFactCat} \arrow[r, "\subseteq"] & \mathbf{UntlLaxFactCat}
\end{tikzcd}$$

is right adjiontable along the horizontal directions. In other words, the horizontal functors admit right adjoints, and the Beck–Chevalley transformation is invertible:

$$\begin{tikzcd}
\mathbf{UntlFactModCat}_{\underline{x}_0} \arrow[d] & \mathbf{UntlLaxFactModCat}_{\underline{x}_0} \arrow[l, "\mathbf{Str}"'] \arrow[d] \\
\mathbf{UntlFactCat} & \mathbf{UntlLaxFactCat}. \arrow[l, "\mathbf{Str}"']
\end{tikzcd}$$

Note that the desired functor $\mathbf{Str}_{\mathbf{A}}$ can be given by the restriction of the top horizontal functor on the fiber over $\mathbf{A}$.

C.11.7. We will construct an endo-functor

$$\sharp : \mathbf{UntlLaxFactModCat}_{\underline{x}_0} \to \mathbf{UntlLaxFactModCat}_{\underline{x}_0}$$

equipped with a natural transformation $\mu : \sharp \to \text{Id}$ and define

$$\mathbf{Str}(\mathbf{A}, \mathbf{M}) := \lim\big(\cdots \to (\mathbf{A}^{\sharp\sharp}, \mathbf{M}^{\sharp\sharp}) \xrightarrow{\mu(\mathbf{A}^\sharp, \mathbf{M}^\sharp)} (\mathbf{A}^\sharp, \mathbf{M}^\sharp) \to (\mathbf{A}, \mathbf{M})\big)$$

to be the sequential limit of the $\sharp$-construction. We will show

(1) The objects $\mathbf{Str}(\mathbf{A}, \mathbf{M})$ is contained in $\mathbf{UntlFactModCat}_{\underline{x}_0}$;
(2) The functor
$$\text{Fun}(-, \mathbf{A}^\sharp) \to \text{Fun}(-, \mathbf{A})$$
is invertible when restricted to **UntlFactCat**;
(3) The functor
$$\text{Fun}(-, (\mathbf{A}^\sharp, \mathbf{M}^\sharp)) \to \text{Fun}(-, (\mathbf{A}, \mathbf{M}))$$
is invertible when restricted to $\mathbf{UntlFactModCat}_{\underline{x}_0}$.

It is clear that these properties imply the claim in Sect. C.11.6.

C.11.8. *Remark.* The definition of $\sharp$ below might look mysterious, but in fact, it comes from a general construction about operads once we reformulate factorization structures using the language in Sect. C.13.

C.11.9. Let $(\mathbf{A},\mathbf{M}) \in \mathbf{UntlLaxFactModCat}_{\underline{x}_0}$ be a pair. To define its image under $\sharp$, we need some notations.

For any finite set $I \in \mathrm{Fin}$, we write

$$\mathcal{R}_I := \big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}\big)_{\mathrm{disj}}$$

and

$$\underline{\mathbf{A}}_I := (\underset{i\in I}{\boxtimes} \underline{\mathbf{A}})|_{\mathrm{disj}} \in \mathbf{CrysCat}(\mathcal{R}_I).$$

For any *marked* finite set $I = I^\circ \sqcup \{0\} \in \mathrm{Fin}_*$, we write

$$\mathcal{R}_I := \big((\prod_{i\in I^\circ} \mathrm{Ran}^{\mathrm{untl}}) \times \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}\big)_{\mathrm{disj}}$$

and

$$\underline{\mathbf{M}}_I := \big((\underset{i\in I^\circ}{\boxtimes} \underline{\mathbf{A}}) \boxtimes \underline{\mathbf{M}}\big)|_{\mathrm{disj}} \in \mathbf{CrysCat}(\mathcal{R}_I).$$

The readers should be able to distinguish the marked and non-marked notations based on the context.

For a morphism $\phi : I \to J$ in either Fin or Fin_*, we have an etale map

$$\mathrm{union}_{I\to J} : \mathcal{R}_I \to \mathcal{R}_J,\ (\underline{y}_i)_{i\in I} \to (\underline{z}_j)_{j\in J}$$

given by $\underline{z}_j := \bigsqcup_{i\in\phi^{-1}(j)} \underline{y}_i$ (see Footnote 37 for the definition of etale maps). One can show the functor

$$T_{I\to J} := \mathrm{union}^*_{I\to J} : \mathbf{CrysCat}(\mathcal{R}_J) \to \mathbf{CrysCat}(\mathcal{R}_I)$$

admits a right adjoint

$$T_{J\leftarrow I} := \mathrm{union}_{I\to J,*} : \mathbf{CrysCat}(\mathcal{R}_I) \to \mathbf{CrysCat}(\mathcal{R}_J),$$

and there are base-change isomorphisms between them. In particular, one can prove

$$T_{J\to K} \circ T_{K\leftarrow I} \simeq T_{J\leftarrow I\times_K J} \circ T_{I\times_K J\to I} =: T_{J\leftarrow I\times_K J\to I} \tag{C.37}$$

C.11.10. Note that the lax-factorization structure on $(\underline{\mathbf{A}},\underline{\mathbf{M}})$ provides canonical morphisms

$$\begin{aligned} \theta_{I\to J} : \underline{\mathbf{A}}_I \to T_{I\to J}(\underline{\mathbf{A}}_J) \quad &\text{for} \quad I, J \in \mathrm{Fin} \\ \theta_{I\to J} : \underline{\mathbf{M}}_I \to T_{I\to J}(\underline{\mathbf{M}}_J) \quad &\text{for} \quad I, J \in \mathrm{Fin}_*\,. \end{aligned} \tag{C.38}$$

We have a functor

$$\begin{aligned} \mathrm{TwArr}(\mathrm{Fin}) &\to \mathbf{CryCat}(\mathrm{Ran}^{\mathrm{untl}}) \\ (I \xrightarrow{\phi} J) &\mapsto T_{\{1\}\leftarrow I\to J}(\underline{\mathbf{A}}_J), \end{aligned} \tag{C.39}$$

where

- $\mathrm{TwArr}(\mathrm{Fin})$ is the category of twisted arrows in Fin. In other words, an object in $\mathrm{TwArr}(\mathrm{Fin})$ is a morphism $\phi : I \to J$ in Fin, while a morphism in $\mathrm{TwArr}(\mathrm{Fin})$ is a commutative diagram

$$\begin{tikzcd} I \arrow[r, "\phi"] & J \arrow[d, "\beta"] \\ I' \arrow[u, "\alpha"] \arrow[r, "\phi'"] & J'; \end{tikzcd}$$

- The functor (C.39) sends the above commutative diagram to the composition

$$T_{\{1\}\leftarrow I\to J}(\underline{\mathbf{A}}_J) \xrightarrow{T_{\{1\}\leftarrow I\to J}(\theta_{J\to J'})} T_{\{1\}\leftarrow I\to J'}(\underline{\mathbf{A}}_{J'}) \to T_{\{1\}\leftarrow I'\to J'}(\underline{\mathbf{A}}_{J'}),$$

  where the last morphism is induced by the adjunction $(T_{I'\to I}, T_{I\leftarrow I'})$.

Similarly we have a functor

$$\begin{aligned} \mathrm{TwArr}(\mathrm{Fin}_*) &\to \mathbf{CryCat}(\mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0}) \\ (I \xrightarrow{\phi} J) &\mapsto T_{\{0\}\leftarrow I\to J}(\underline{\mathbf{M}}_J), \end{aligned} \tag{C.40}$$

C.11.11. We now define

$$\underline{\mathbf{A}}^\sharp := \lim_{I\to J} T_{\{1\}\leftarrow I\to J}(\underline{\mathbf{A}}_J) \quad \text{indexed by} \quad \mathrm{TwArr}(\mathrm{Fin});$$

$$\underline{\mathbf{M}}^\sharp := \lim_{I\to J} T_{\{0\}\leftarrow I\to J}(\underline{\mathbf{M}}_J) \quad \text{indexed by} \quad \mathrm{TwArr}(\mathrm{Fin}_*).$$

Note that there are obvious morphisms

$$\underline{\mathbf{A}}^\sharp \to \underline{\mathbf{A}},\ \underline{\mathbf{M}}^\sharp \to \underline{\mathbf{M}} \tag{C.41}$$

given by evaluations at $\mathrm{id}_{\{1\}}$ and $\mathrm{id}_{\{0\}}$ respectively.

C.11.12. We now explain that $(\underline{\mathbf{A}}^\sharp, \underline{\mathbf{M}}^\sharp)$ has a canonical unital lax-factorization structure. We will only do this for $\underline{\mathbf{A}}^\sharp$. The module part can be constructed by replacing non-marked sets with marked ones.

We will only construct the structure morphisms (for any finite set $K \in \mathrm{Fin}$)

$$(\boxtimes_{k\in K} \underline{\mathbf{A}}^\sharp)|_{\mathrm{disj}} \to \mathrm{union}_K^*(\underline{\mathbf{A}}^\sharp)|_{\mathrm{disj}} = T_{K\to\{1\}}(\mathbf{A}^\sharp), \tag{C.42}$$

and leave the higher compatibilities to [CFZ].

We have a canonical morphism

$$(\boxtimes_{k\in K} \underline{\mathbf{A}}^\sharp)|_{\mathrm{disj}} \to \lim_{(I_k\to J_k)_{k\in K}} \big(\boxtimes_{k\in K} T_{\{1\}\leftarrow I_k\to J_k}(\underline{\mathbf{A}}_{J_k})\big)|_{\mathrm{disj}} \tag{C.43}$$

by exchanging limits with external products and restrictions[41]. Unwinding the definitions, we have

$$\big(\boxtimes_{k\in K} T_{\{1\}\leftarrow I_k\to J_k}(\underline{\mathbf{A}}_{J_k})\big)|_{\mathrm{disj}} \simeq T_{K\leftarrow \sqcup I_k\to \sqcup J_k} \underline{\mathbf{A}}_{\sqcup J_k}$$

Hence we obtain a morphism

$$(\boxtimes_{k\in K} \underline{\mathbf{A}}^\sharp)|_{\mathrm{disj}} \to \lim_{(I_k\to J_k)_{k\in K}} T_{K\leftarrow \sqcup I_k\to \sqcup J_k} \underline{\mathbf{A}}_{\sqcup J_k} \simeq \lim_{I\to J\to K} T_{K\leftarrow I\to J} \underline{\mathbf{A}}_J, \tag{C.44}$$

where the last limit is indexed by $\mathrm{TwArr}(\mathrm{Fin}_{/K})$, which is equivalence to the category of twisted arrows $I \to J$ equipped with a map $J \to K$.

On the other hand, one can show $T_{K\to\{1\}}$ commutes with limits (see Footnote 41). Hence we have

$$T_{K\to\{1\}}(\mathbf{A}^\sharp) \simeq \lim_{I\to J} T_{K\to\{1\}} \circ T_{\{1\}\leftarrow I\to J}(\underline{\mathbf{A}}_J) \simeq \lim_{I\to J} T_{K\leftarrow K\times I\to J}(\underline{\mathbf{A}}_J), \tag{C.45}$$

where the last equivalence is due to the base-change isomorphism (C.37). Let $\mathrm{TwArr}(\mathrm{Fin})_K$ be the category of twisted arrows $I \to J$ in Fin equipped with a map $I \to K$. Note that

$$\mathrm{TwArr}(\mathrm{Fin}) \to \mathrm{TwArr}(\mathrm{Fin})_K,\ (I\to J) \mapsto (K \leftarrow K\times I \to J)$$

is left adjoint to the forgetful functor. It follows that we have a canonical equivalence

$$\lim_{K\leftarrow I\to J} T_{K\leftarrow I\to J}(\underline{\mathbf{A}}_J) \xrightarrow{\simeq} \lim_{I\to J} T_{K\leftarrow K\times I\to J}(\underline{\mathbf{A}}_J),$$

where the first limit is indexed by $\mathrm{TwArr}(\mathrm{Fin})_K$. Combining with (C.45), we obtain an equivalence

$$T_{K\to\{1\}}(\mathbf{A}^\sharp) \simeq \lim_{K\leftarrow I\to J} T_{K\leftarrow I\to J}(\underline{\mathbf{A}}_J). \tag{C.46}$$

Finally, the forgetful functor $\mathrm{TwArr}(\mathrm{Fin}_{/K}) \to \mathrm{TwArr}(\mathrm{Fin})_K$ admits a right adjoint

$$(K \leftarrow I \to J) \mapsto (I \to K\times J \to K).$$

This implies we have a canonical equivalence

$$\lim_{K\leftarrow I\to J} T_{K\leftarrow I\to J}(\underline{\mathbf{A}}_J) \simeq \lim_{I\to J\to K} T_{K\leftarrow I\to J} \underline{\mathbf{A}}_J. \tag{C.47}$$

[41]In fact, the restriction functor $f^* : \mathbf{CrysCat}(\mathcal{Z}) \to \mathbf{CrysCat}(\mathcal{Y})$ commutes with limits for any map $f : \mathcal{Y} \to \mathcal{Z}$ between categorical prestacks. This follows from the fact that $- \otimes_{\mathrm{D}(S_1)} \mathrm{D}(S_2)$ commutes with limits for affine schemes S_1 and S_2.

Now the desired morphism (C.42) is defined to be the composition

$$(\text{C.46})^{-1} \circ (\text{C.47})^{-1} \circ (\text{C.44}).$$

In [CFZ], we will show these morphisms (when K varies) indeed define an object

$$\mathbf{A}^\sharp \in \mathbf{UntlLaxFactCat}.$$

Similarly, we have

$$(\mathbf{A}^\sharp, \mathbf{M}^\sharp) \in \mathbf{UntlLaxFactModCat}_{\underline{x}_0}.$$

Moreover, (C.41) can be upgraded to a morphism

$$(\mathbf{A}^\sharp, \mathbf{M}^\sharp) \to (\mathbf{A}, \mathbf{M}).$$

C.11.13. We now explain claim (1) in Sect. C.11.7. We will only do this for

$$\mathbf{Str}(\mathbf{A}) := \lim \big(\cdots \to \mathbf{A}^{\sharp\sharp} \to \mathbf{A}^\sharp \to \mathbf{A} \big)$$

The module part can be constructed by replacing non-marked sets with marked ones.

For any integer m, we say a unital lax-factorization category $\mathbf{A}$ is *m-strict* if for any collection of disjoint affine points $\underline{x}_k \in \mathrm{Ran}^{\mathrm{untl}}(S)$, $k \in K$ satisfying $|\sqcup \underline{x}_k| \leq m$, the structural functor

$$\mathrm{mult}_{(\underline{x}_k)} : \otimes_{\mathrm{D}(S)} \mathbf{A}_{\underline{x}_k} \to \mathbf{A}_{\sqcup \underline{x}_k}$$

is invertible. Note that $\mathbf{A}$ is always (-1)-strict. Also note that $\mathbf{A}$ being 0-strict is equivalent to $\mathbf{A}_\emptyset \simeq \mathrm{Vect}$.

To prove claim (1), we only need to show $\mathbf{A}^\sharp$ is m-strict whenever $\mathbf{A}$ is $(m-1)$-strict.

We first consider the case $m = 0$. Unwinding the definitions, we have

$$\mathbf{A}^\sharp_\emptyset \simeq \lim_{I \to J} \big(T_{\{1\} \leftarrow I \to J}(\underline{\mathbf{A}}_J) \big)_\emptyset \simeq \lim_{I \to J} (\underline{\mathbf{A}}_J)_{(\emptyset)_{j \in J}} \simeq \lim_{I \to J} (\mathbf{A}_\emptyset)^{\otimes J}$$

where recall the limit is indexed by TwArr(Fin). Note that the forgetful functor

$$\mathrm{TwArr}(\mathrm{Fin}) \to \mathrm{Fin}, \ (I \to J) \mapsto J$$

is a weak homotopy equivalence. It follows that

$$\lim_{I \to J} (\mathbf{A}_\emptyset)^{\otimes J} \simeq \lim_{J \in \mathrm{Fin}} (\mathbf{A}_\emptyset)^{\otimes J} \simeq (\mathbf{A}_\emptyset)^{\otimes \emptyset} \simeq \mathrm{Vect}\,.$$

Hence $\mathbf{A}^\sharp_\emptyset \simeq \mathrm{Vect}$ as desired.

We now prove the general case when $m > 0$. We need to show the structural functor

$$\mathrm{mult}_{(\underline{x}_k)} : \otimes_{\mathrm{D}(S)} \mathbf{A}^\sharp_{\underline{x}_k} \to \mathbf{A}^\sharp_{\sqcup \underline{x}_k}$$

is inverible when $|\sqcup \underline{x}_k| \leq m$. Since the $m = 0$ case is known, we can assume $|K| \geq 2$ and each $\underline{x}_k$ is non-empty. This implies $|\underline{x}_k| < m$.

By construction in Sect. C.11.12, we only need to show the fiber of (C.43) at

$$(\underline{x}_k)_{k \in K} \in \big(\prod_{k \in K} \mathrm{Ran}^{\mathrm{untl}} \big)_{\mathrm{disj}}(S) = \mathcal{R}_K(S)$$

is invertible. In other words, we need to show we can exchange limits with tensor products in the following expression:

$$\bigotimes_{k \in K} \big(\lim_{I_k \to J_k} \big(T_{\{k\} \leftarrow I_k \to J_k} \underline{\mathbf{A}}_{J_k} \big)_{\underline{x}_k} \big)_{/\,\mathrm{D}(S)}$$

For this purpose, we prove the following stronger claim: for fixed $k \in K$ and *any* $\mathrm{D}(S)$-module category $\mathcal{C}$, the functor

$$\mathcal{C} \otimes_{\mathrm{D}(S)} \lim_{I_k \to J_k} \big(T_{\{k\} \leftarrow I_k \to J_k} \underline{\mathbf{A}}_{J_k} \big)_{\underline{x}_k} \to \lim_{I_k \to J_k} \mathcal{C} \otimes_{\mathrm{D}(S)} \big(T_{\{k\} \leftarrow I_k \to J_k} \underline{\mathbf{A}}_{J_k} \big)_{\underline{x}_k}$$

is invertible. For this purpose, we prove the following stronger claim: for $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}(S)$ such that $|\underline{x}| < m$, the functor

$$\lim_{I\to J} \mathcal{C} \otimes_{\mathrm{D}(S)} \left(T_{\{1\}\leftarrow I\to J}\underline{\mathbf{A}}_J\right)_{\underline{x}} \to \mathcal{C} \otimes_{\mathrm{D}(S)} \mathbf{A}_{\underline{x}} \tag{C.48}$$

is an equivalence.

Using the assumption that $\mathbf{A}$ is $(m-1)$-strict, we have

$$\left(T_{\{1\}\leftarrow I\to J}\underline{\mathbf{A}}_J\right)_{\underline{x}} \simeq \left(T_{\{1\}\leftarrow I}\underline{\mathbf{A}}_I\right)_{\underline{x}}.$$

Since the forgetful functor $\mathrm{TwArr}(\mathrm{Fin}) \to \mathrm{Fin}^{\mathrm{op}}$, $(I \to J) \mapsto I$ is a weak equivalence, we obtain

$$\lim_{I\to J} \mathcal{C}\otimes_{\mathrm{D}(S)} \left(T_{\{1\}\leftarrow I\to J}\underline{\mathbf{A}}_J\right)_{\underline{x}} \simeq \lim_{I\in\mathrm{Fin}^{\mathrm{op}}} \mathcal{C}\otimes_{\mathrm{D}(S)} \left(T_{\{1\}\leftarrow I}\underline{\mathbf{A}}_I\right)_{\underline{x}} \simeq \mathcal{C}\otimes_{\mathrm{D}(S)} \left(T_{\{1\}\leftarrow \{1\}}\underline{\mathbf{A}}_{\{1\}}\right)_{\underline{x}} \simeq \mathcal{C}\otimes_{\mathrm{D}(S)} \mathbf{A}_{\underline{x}}$$

as desired.

C.11.14. *Remark.* The above argument is closely related to the notion of *pro-nilpotent operads* in [FG]. We will explain this in [CFZ].

C.11.15. Note that the equivalence (C.48) says:

Lemma C.11.16. *Let $\mathbf{A}$ be a unital $(m-1)$-strict factorization category. Then the functor $\mathbf{A}^\sharp \to \mathbf{A}$ induces an equivalence*

$$\mathbf{A}^\sharp_{\underline{x}} \to \mathbf{A}_{\underline{x}}$$

for any affine point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}(S)$ with $|\underline{x}| < m$.

C.11.17. In particular, we have shown the restriction of the sequence

$$\mathbb{Z}_{\leq 0} \to \mathbf{UntlLaxFactCat},\ -n \mapsto \mathbf{A}^{n\sharp}$$

at a point $\underline{x} \in \mathrm{Ran}^{\mathrm{untl}}(S)$ becomes stable for $-n < -|\underline{x}|$. In particular,

$$\mathbf{Str}(\mathbf{A})_{\underline{x}} \simeq (\mathbf{A}^{n\sharp})_{\underline{x}} \text{ for } n > |\underline{x}|.$$

C.11.18. *Remark.* In fact, a more elaborate analysis gives

$$\mathbf{Str}(\mathbf{A})_{\underline{x}} \simeq (\mathbf{A}^{n\sharp})_{\underline{x}} \text{ for } n \geq |\underline{x}| - 1 \geq 0.$$

For example, a direct calculation shows the restriction $\mathbf{Str}(\mathbf{A})$ along $X^2 \to \mathrm{Ran}^{\mathrm{untl}}$ fits into the following Cartesian square

$$\begin{tikzcd}
\mathbf{Str}(\mathbf{A})|_{X^2} \arrow[r] \arrow[d] & \mathbf{A}|_{X^2} \arrow[d] \\
j_*j^*(\mathbf{A}|_X \boxtimes \mathbf{A}|_X) \arrow[r] & j_*j^*(\mathbf{A}|_{X^2}),
\end{tikzcd}$$

where

- $j : X^2 \setminus X \to X^2$ is the complement of the diagonal embedding;
- the bottom horizontal functor is provided by the lax-factorization structure on $\underline{\mathbf{A}}$.

Similarly, one can show

$$\mathbf{Str}(\mathbf{M})_{\underline{x}} \simeq (\mathbf{M}^{n\sharp})_{\underline{x}} \text{ for } n \geq |\underline{x}| - |\underline{x}_0|.$$

For example, the restriction of $\mathbf{Str}(\mathbf{M})$ along the map

$$X \times S_0 \to \mathrm{Ran}^{\mathrm{untl}}_{\underline{x}_0},\ y \mapsto y \cup \underline{x}_0$$

fits into the following Cartesian square

$$\begin{tikzcd}
\mathbf{Str}(\mathbf{M})|_{X\times S_0} \arrow[r] \arrow[d] & \mathbf{M}|_{X\times S_0} \arrow[d] \\
j_*j^*(\mathbf{A}|_X \boxtimes \mathbf{M}|_{\underline{x}_0}) \arrow[r] & j_*j^*(\mathbf{M}|_{X\times S_0}),
\end{tikzcd}$$

where

- $j : (X \times S_0) \setminus \mathrm{graph}_{\underline{x}_0} \to X \times S_0$ is the complement of the union of the graphs for elements in $\underline{x}_0 \subseteq X(S_0)$;
- the bottom horizontal functor is provided by the lax-factorization $\mathbf{A}$-module structure on $\underline{\mathbf{M}}$.

C.11.19. Finally, we explain claim (2) in Sect. C.11.7. Claim (3) can be proved similarly by replacing non-marked sets with marked ones.

Let $\mathbf{B}$ be any test unital factorization category. We want to show

$$\mathrm{Fun}(\mathbf{B}, \mathbf{A}^\sharp) \to \mathrm{Fun}(\mathbf{B}, \mathbf{A})$$

is an equivalence. We only explain how to construct the desired inverse functor

$$\mathrm{Fun}(\mathbf{B}, \mathbf{A}) \to \mathrm{Fun}(\mathbf{B}, \mathbf{A}^\sharp), \tag{C.49}$$

and leave the verification to [CFZ].

Let $\Phi : \mathbf{B} \to \mathbf{A}$ be a unital factorization functor. The $\sharp$-construction is functorial, hence we have a functor

$$\Phi^\sharp : \mathbf{B}^\sharp \to \mathbf{A}^\sharp.$$

Since $\mathbf{B}$ is ∞-strict by assumption, Lemma implies $\mu_{\mathbf{B}} : \mathbf{B}^\sharp \to \mathbf{B}$ is an equivalence. We now define the functor (C.49) to be $\Phi \mapsto \mu_{\mathbf{B}}^{-1} \circ \Phi^\sharp$.

□[Sketch of Theorem C.10.4]

C.12. **Induced modules.** In this subsection, we fix a unital factorization category $\mathbf{A}$ and a unital factorization $\mathbf{A}$-module category $\mathbf{M}$ at $\underline{x}_0$. For simplicity, we assume $\underline{x}_0 = x_0$ is a single k-point on X.

Let $\mathcal{A} \in \mathrm{FactAlg}(\mathbf{A})$ be a factorization algebra in $\mathbf{A}$ and

$$\mathrm{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})_{x_0} \to \mathbf{M}_{x_0}$$

be the forgetful functor. We will study the *partially defined* left adjoint $\mathrm{ind}_{\mathcal{A}}$ of this functor.

C.12.1. *Warning.* For general $\mathcal{A}$, the functor $\mathrm{oblv}_{\mathcal{A}}$ does not preserve limits, because general external tensor products and !-pullback functors do not preserve limits. In fact, we do not know how to calculate limits in $\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})_{x_0}$ (although we know they exist by presentability). As a consequence, $\mathrm{oblv}_{\mathcal{A}}$ does not admit a left adjoint.

C.12.2. We are going to provide a sufficient condition on $\mathcal{A}$ and V such that there exists an *object* $\mathrm{ind}_{\mathcal{A}}(V)$ such that

$$\mathrm{Hom}_{\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})_{x_0}}(\mathrm{ind}_{\mathcal{A}}(V), \mathcal{M}) \simeq \mathrm{Hom}_{\mathbf{M}_{x_0}}(V, \mathcal{M}_{x_0}).$$

Such an object $\mathrm{ind}_{\mathcal{A}}(V)$ is called the *induced* (a.k.a. *free*) factorization $\mathcal{A}$-module in $\mathbf{M}$.

C.12.3. *Remark.* More generally, one can ask what are the coCartesian arrows in the Cartesian fibration (see Sect. C.10.17)

$$\mathrm{FactMod}(\mathbf{A}, \mathbf{M})_{x_0} \to \mathrm{FactAlg}(\mathbf{A}).$$

We will treat this problem in [CFZ].

C.12.4. To construct $\mathrm{ind}_{\mathcal{A}}(V)$, we need some notations.

Let $X_\circ := X \setminus x_0$ be the punctured curve and $\mathrm{Ran}_\circ^{\mathrm{untl}}$ be the unital Ran space for $X_\circ$. Consider the map

$$\mathfrak{j} : \mathrm{Ran}_\circ^{\mathrm{untl}} \times x_0 \to \mathrm{Ran}_{x_0}^{\mathrm{untl}}, (\underline{y}, x_0) \mapsto \underline{y} \sqcup \{x_0\}.$$

Note that $\mathfrak{j}$ induces a bijection between k-points, but is not an isomorphism. In fact, one can check

- $\mathrm{Ran}_\circ^{\mathrm{untl}} \times x_0$ is a Cartesian space over $\mathrm{Ran}_{x_0}^{\mathrm{untl}}$ (see Sect. C.1.5);
- For any affine points $S \to \mathrm{Ran}_{x_0}^{\mathrm{untl}}$, the base-change of $\mathfrak{j}$ is a finite disjoint union of locally closed immersions.

Similarly, for any marked finite set $I = I^\circ \sqcup \{0\}$, we have a map

$$\mathrm{j}_I : \big(\prod_{i\in I} \mathrm{Ran}^{\mathrm{untl}}_\circ\big)_{\mathrm{disj}} \times x_0 \to \big(\big(\prod_{i\in I^\circ} \mathrm{Ran}^{\mathrm{untl}}_\circ\big) \times \mathrm{Ran}^{\mathrm{untl}}_{x_0}\big)_{\mathrm{disj}} \simeq \big(\big(\prod_{i\in I^\circ} \mathrm{Ran}^{\mathrm{untl}}\big) \times \mathrm{Ran}^{\mathrm{untl}}_{x_0}\big)_{\mathrm{disj}} \tag{C.50}$$

given by $((\mathrm{id})_{i\in I^\circ}, \mathrm{j})$. To simplify the notations, we write it as

$$\mathrm{j}_I : \mathcal{R}_{I,\circ} \times x_0 \to \mathcal{R}_I.$$

Note that we have a functor

$$\begin{aligned} \mathrm{j}_I^! : \Gamma^{\mathrm{lax}}(\mathcal{R}_I, \underline{\mathbf{M}}|_{\mathcal{R}_I}) \to \Gamma^{\mathrm{lax}}(\mathcal{R}_{I,\circ} \times x_0, \underline{\mathbf{M}}|_{\mathcal{R}_{I,\circ}\times x_0}) \simeq \\ \simeq \Gamma^{\mathrm{lax}}(\mathcal{R}_{I,\circ} \times x_0, \underline{\mathbf{A}}|_{\mathcal{R}_{I,\circ}} \boxtimes \mathbf{M}_{x_0}) \simeq \Gamma^{\mathrm{lax}}(\mathcal{R}_{I,\circ}, \underline{\mathbf{A}}|_{\mathcal{R}_{I,\circ}}) \otimes \mathbf{M}_{x_0}, \end{aligned} \tag{C.51}$$

where the first equivalence is given by the factorization $\mathbf{A}$-module structure on $\mathbf{M}$.

C.12.5. Note that $\underline{\mathbf{A}}|_{\mathrm{Ran}^{\mathrm{untl}}_\circ}$ is a unital factorization category on the punctured curve $X_\circ$. We denote this object by

$$\mathbf{A}_\circ \in \mathbf{UntlFactCat}(X_\circ),$$

to distinguish it from the object $\mathbf{A} \in \mathbf{UntlFactCat}(X)$. Note that the latter category is denoted just by $\mathbf{UntlFactCat}$ in the rest of this appendix.

By restriction, we also obtain an object

$$\mathcal{A}_\circ \in \mathrm{FactAlg}(\mathbf{A}_\circ).$$

C.12.6. Similarly, $\underline{\mathbf{M}}|_{\mathrm{Ran}^{\mathrm{untl}}_\circ \times x_0}$ is a unital factorization $\mathbf{A}_\circ$-module category *at the point* $\emptyset \in \mathrm{Ran}^{\mathrm{untl}}_\circ$. Here we use the identification

$$(\mathrm{Ran}^{\mathrm{untl}}_\circ)_\emptyset \simeq \mathrm{Ran}^{\mathrm{untl}}_\circ \times x_0,\ \underline{y} \mapsto (\underline{y}, x_0).$$

We denote this object by

$$\mathbf{M}_\circ \in \mathbf{A}_\circ\text{–}\mathbf{mod}^{\mathrm{fact}}_\emptyset.$$

Via the correspondence in Sect. C.6.6, $\mathbf{M}_\circ$ is given by the DG category $\mathbf{M}_{x_0}$. It follows that

$$\mathcal{A}_\circ\text{-mod}^{\mathrm{fact}}(\mathbf{M}_\circ)_\emptyset := \mathrm{Fun}_{\mathcal{A}}(\mathrm{Vect}^{\mathrm{fact}_\emptyset}, \mathbf{M}_\circ) \simeq \mathrm{Fun}(\mathrm{Vect}, \mathbf{M}_{x_0}) \simeq \mathbf{M}_{x_0}. \tag{C.52}$$

By construction, this is just the forgetful functor $\mathrm{oblv}_{\mathcal{A}_\circ}$

C.12.7. The above equivalence can be proved in a more explicit way.

Given $\mathcal{M} \in \mathcal{A}_\circ\text{-mod}^{\mathrm{fact}}(\mathbf{M}_\circ)_\emptyset$, its fiber $\mathcal{M}_\emptyset$ is an object in $(\mathbf{M}_\circ)_\emptyset \simeq \mathbf{M}_{x_0}$; conversely, given an object $V \in \mathbf{M}_{x_0}$, the tensor product

$$\underline{\mathcal{A}}_\circ \boxtimes V \in \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_\circ, \underline{\mathbf{A}}_\circ) \otimes \mathbf{M}_{x_0} \simeq \Gamma^{\mathrm{lax}}((\mathrm{Ran}^{\mathrm{untl}}_\circ)_\emptyset, \underline{\mathbf{M}}_\circ)$$

has a canonical factorization $\mathbf{A}_\circ$-module structure. One can check these two constructions are inverse to each other.

C.12.8. Given an object $V \in \mathbf{M}_{x_0}$, we say it is *adapted to* $\mathcal{A}$*-induction* if it satisfies the following conditions:

- For any marked finite set $I = I^\circ \sqcup \{0\}$, the partially defined left $\mathrm{j}_{I,!}$ to the functor (C.51) is well-defined at the object $\underline{\mathcal{A}}|_{\mathcal{R}_{I,\circ}} \boxtimes V$, i.e., the following object exists:

  $$\mathrm{j}_{I,!}\big((\underset{i\in I}{\boxtimes} \underline{\mathcal{A}}_\circ)_{\mathrm{disj}} \boxtimes V\big) \in \Gamma^{\mathrm{lax}}(\mathcal{R}_I, \underline{\mathbf{M}}|_{\mathcal{R}_I}).$$

  In particular, we have an object

  $$\mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V) \in \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}, \underline{\mathbf{M}}).$$

- For any marked finite set $I = I^\circ \sqcup \{0\}$, the canonical (Beck–Chevalley) morphism

  $$\mathrm{j}_{I,!}\big((\underset{i\in I}{\boxtimes} \underline{\mathcal{A}}_\circ)_{\mathrm{disj}} \boxtimes V\big) \to \big((\underset{i\in I^\circ}{\boxtimes} \underline{\mathcal{A}}_\circ) \boxtimes \mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V)\big)|_{\mathrm{disj}}$$

  is invertible.

C.12.9. Let $V \in \mathbf{M}_{x_0}$ be an object adapted to $\mathcal{A}$-induction. We claim $\mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V)$ can be canonically upgraded to an object

$$\mathrm{j}_!(\mathcal{A}_\circ \boxtimes V) \in \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})_{x_0}.$$

We will only construct the structural isomorphisms (see Sect. C.7.8)

$$\mathrm{act}_I\left(\left(\left(\underset{i\in I^\circ}{\boxtimes} \underline{\mathcal{A}}\right) \boxtimes \mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V)\right)|_{\mathrm{disj}}\right) \overset{\simeq}{\longrightarrow} \mathrm{union}_I^!(\mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V))|_{\mathrm{disj}}, \tag{C.53}$$

and leave the higher compatibilities to [CFZ].

By assumption, we have

$$\left(\left(\underset{i\in I^\circ}{\boxtimes} \underline{\mathcal{A}}\right) \boxtimes \mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V)\right)|_{\mathrm{disj}} \simeq \left(\left(\underset{i\in I^\circ}{\boxtimes} \underline{\mathcal{A}}_\circ\right) \boxtimes \mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V)\right)|_{\mathrm{disj}} \simeq \mathrm{j}_{I,!}\left(\left(\underset{i\in I}{\boxtimes} \underline{\mathcal{A}}_\circ\right)_{\mathrm{disj}} \boxtimes V\right). \tag{C.54}$$

On the other hand, we have a Cartesian square

$$\begin{tikzcd}
\mathcal{R}_{I,\circ} \times x_0 \arrow[r, "\mathrm{j}_I"] \arrow[d, "\mathrm{union}_{I,\circ}"'] & \mathcal{R}_I \arrow[d, "\mathrm{union}_I"] \\
\mathrm{Ran}^{\mathrm{untl}}_\circ \times x_0 \arrow[r, "\mathrm{j}"'] & \mathrm{Ran}^{\mathrm{untl}}_{x_0}
\end{tikzcd}$$

such that the vertical arrows are etale (see 37). This implies the Beck–Chevalley natural transformation

$$\begin{tikzcd}
\Gamma^{\mathrm{lax}}(\mathcal{R}_{I,\circ} \times x_0, \underline{\mathbf{M}}|_{\mathcal{R}_{I,\circ}\times x_0}) \arrow[d, "\mathrm{union}_{I,\circ,*}"'] & \Gamma^{\mathrm{lax}}(\mathcal{R}_I, \underline{\mathbf{M}}|_{\mathcal{R}_I}) \arrow[l, "\mathrm{j}_I^!"'] \arrow[d, "\mathrm{union}_{I,*}"] \\
\Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_\circ \times x_0, \underline{\mathbf{M}}|_{\mathrm{Ran}^{\mathrm{untl}}_\circ \times x_0}) & \Gamma^{\mathrm{lax}}(\mathrm{Ran}^{\mathrm{untl}}_{x_0}, \underline{\mathbf{M}}) \arrow[l, "\mathrm{j}^!"]
\end{tikzcd}$$

is invertible, where the vertical functors are right adjoint to the !-pullback functors. Passing to partially defined left adjoints, we obtain

$$\mathrm{union}_I^!(\mathrm{j}_!(\underline{\mathcal{A}}_\circ \boxtimes V))|_{\mathrm{disj}} \simeq \mathrm{j}_{I,!}\left(\mathrm{union}_{I,\circ}^!(\underline{\mathcal{A}}_\circ \boxtimes V)|_{\mathrm{disj}}\right) \tag{C.55}$$

Via the isomorphisms (C.54) and (C.55), the desired isomorphism (C.53) is given by applying $\mathrm{j}_{I,!}$ to

$$\mathrm{act}_{I,\circ}\left(\left(\underset{i\in I}{\boxtimes} \underline{\mathcal{A}}_\circ\right)_{\mathrm{disj}} \boxtimes V\right) \overset{\simeq}{\longrightarrow} \mathrm{union}_{I,\circ}^!(\underline{\mathcal{A}}_\circ \boxtimes V)|_{\mathrm{disj}},$$

which is given by the factorizaiton $\underline{\mathcal{A}}_\circ$-module structure on $\underline{\mathcal{A}}_\circ \boxtimes V$ (see Sect. C.12.7).

C.12.10. Modulo issues about homotopy coherence, it is clear

$$\mathrm{Maps}_{\mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})_{x_0}}(\mathrm{j}_!(\mathcal{A}_\circ \boxtimes V), \mathcal{M}) \simeq \mathrm{Maps}_{\mathcal{A}_\circ\text{-mod}^{\mathrm{fact}}(\mathbf{M}_\circ)_\emptyset}(\mathcal{A}_\circ \boxtimes V, \mathcal{M}|_{\mathrm{Ran}^{\mathrm{untl}}_\circ \times x_0}).$$

By (C.52), the RHS can be identified with

$$\mathrm{Maps}_{\mathbf{M}_{x_0}}(V, \mathcal{M}_{x_0}).$$

Hence we obtain the following result.

Proposition C.12.11. *Let* $\mathbf{A}$, $\mathbf{M}$, $\mathcal{A}$ *be as in Sect. C.12. Suppose* $V \in \mathbf{M}_{x_0}$ *is adapted to* $\mathcal{A}$*-induction (see Sect. C.12.8), then the partially defined left adjoint* $\mathrm{ind}_{\mathcal{A}}$ *to the forgetful functor*

$$\mathrm{oblv}_{\mathcal{A}} : \mathcal{A}\text{-mod}^{\mathrm{fact}}(\mathbf{M})_{x_0} \to \mathbf{M}_{x_0}$$

is defined on V*, and we have*

$$\mathrm{ind}_{\mathcal{A}}(V) \simeq \mathrm{j}_!(\mathcal{A}_\circ \boxtimes V),$$

where the RHS is defined in Sect. C.12.9.

C.13. **The Ran operad.** This subsection serves as an advertisement for [CFZ], where all the homotopy-coherent difficulties in this appendix (as well as those ignored in [GLC2, Appendix B, C]) will be treated by the methods developed in Lurie's *Higher Algebra* [Lu].

C.13.1. In [CFZ], we will rewrite the foundations of factorization structures[42] using the language of *generalized operads* developed in [Lu].

C.13.2. In [Lu, Chapter 2], Lurie defined an ∞-operad as an ∞-cateogry $\mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ over the category of marked finite sets that satisfies certain conditions. Roughly speaking, an ∞-operad is a colored operad (introduced by May in [Ma]) enriched over the ∞-category of spaces, except that the collection of colors is allowed to form a category[43] rather than a space/set. Equivalently, an ∞-operad is a pseudo-tensor category (introduced by Beilinson–Drinfeld in [BD1]), but the underlying category is allowed to be an ∞-category.

C.13.3. A generalized ∞-operad should be viewed as a *family of ∞-operads* parameterized by some base category $\mathcal{C}$ (see [Lu, Sect. 2.3]). Just like the usual theory of various types of algebras and monoidal categories can be developed using the corresponding operads, for any generalized ∞-operad $\mathcal{O}^{\otimes} \to \mathrm{Fin}_* \times \mathcal{C}$, one can develop the notion of $\mathcal{O}$-monoidal categories and $\mathcal{O}$-algebras in them.

The main idea behind [CFZ] is: there should exist a (classical) generalized operad $\mathrm{Ran}^{\otimes}$ parameterized by the category Aff of affine schemes, such that

$$\text{lax-factorization objects in } \mathcal{D} = \text{Ran-algebras in } \mathcal{D},$$

where $\mathcal{D}$ is any symmetric monoidal $(\infty, 2)$-category. For instance,

$$\text{lax-factorization DG categories} = \text{Ran-algebras in } \mathbf{DGCat}.$$

C.13.4. The construction of the generalized operad $\mathrm{Ran}^{\otimes}$ is easy. For any affine scheme S, we have a symmetric monoidal category $\mathrm{Ran}^{\mathrm{untl}}(S)$ with tensor products given by unions of finite sets. In particular, it corresponds to a (classical) ∞-operad $\mathrm{Ran}^{\mathrm{untl}}(S)^{\cup}$. We now define

$$\mathrm{Ran}(S)^{\otimes} \subseteq \mathrm{Ran}^{\mathrm{untl}}(S)^{\cup}$$

to be the 1-full subcategory containing of those morphisms that corrspond to *disjoint* unions. Alternatively, we equip $\mathrm{Ran}^{\mathrm{untl}}(S)$ with a structure of pseudo-tensor categories, where a multi-map $\{x_i\}_{i\in I} \to y$ exists iff the points $\{x_i\}_{i\in I}$ are disjoint and $\sqcup x_i \subseteq y$.

The above construction is contravariantly functorial in S. Hence we have a functor from $\mathrm{Aff}^{\mathrm{op}}$ to the category of (classical) ∞-operads. Now the generalized operad $\mathrm{Ran}^{\otimes}$ is defined to be the corresponding coCartesian fibration

$$\mathrm{Ran}^{\otimes} \to \mathrm{Aff}^{\mathrm{op}}.$$

C.13.5. The main advantage of $\mathrm{Ran}^{\otimes}$ is that it provides a natural way to deal with factorization *module* structures.

In [Lu, Chapter 3], for any *coherent* ∞-operad $\mathcal{O}^{\otimes}$ and a color $\mathfrak{m} \in \mathcal{O}$, Lurie defined the notion of $\mathfrak{m}$-type $\mathcal{O}$-modules for $\mathcal{O}$-algebras, and provided a framework to deal with restrictions and relative tensor products of such modules, where the higher compatibilities for these constructions are encoded as certain fibrations of ∞-categories.

In [CFZ], we will generalize the notion of coherence even to generalized $(\infty, 2)$-operads, and prove:

Theorem C.13.6. $\mathrm{Ran}^{\otimes}$ *is coherent as a generalized* $(\infty, 2)$*-operads.*

[42] It is fair to say [Ra] is the only homotopy-coherent foundation of factorization categories that exists in the literature. However, there are several disadvantages in Raskin's approach which makes it hard to prove results claimed in Sect. C.9.

[43] It is the category $\mathcal{O} := \mathcal{O}^{\otimes} \times_{\mathrm{Fin}_*} \{0, 1\}$.

C.13.7. As a result, we can deal with restrictions and relative tensor products of $\underline{x}$-type Ran-modules internal to any symmetric monoidal $(\infty, 2)$-category, such as **DGCat**, in a way similar to [Lu, Chapter 3].

Note that a color $\underline{x}$ in $\mathrm{Ran}^{\otimes}$ is exactly an affine point $\underline{x} : S \to \mathrm{Ran}^{\mathrm{untl}}$ for some affine scheme S. We will show, for example, the following two notions are the same:

- A pair $(\mathbf{A}, \mathbf{M})$ of a (unital) lax-factorization category $\mathbf{A}$ and its module $\mathbf{M}$ at $\underline{x}$, as defined in this appendix;
- A pair $(\mathbf{A}, \mathbf{M})$ of a Ran-algebra in **DGCat** and its $\underline{x}$-type module $\mathbf{M}$, as defined in [CFZ].

C.13.8. Note that in above, Ran-algebras/modules correspond to *lax*-factorization algebras/modules. In [CFZ], we will explain there is an analog between

- A (unital) *strict* factorization algebra object in a symmetric monoidal category $\mathbf{D}$;
- A (properly defined verison of) *monoidal* functor $\mathrm{Ran} \to \mathbf{D}$.

As a consequence, the construction of the strictening functor (C.11.6) will be interpreted via *monoidal envelopes* and *operadic Kan extensions* (see [Lu, Sect. 2.2.4 and Sect. 3.1.2]).

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