LECTURE 1: OVERVIEW

1. Geometric Class Field Theory

Let K be a global field. Let $\mathbb{A}_K$ be the ring of adèles and $\mathsf{Gal}_{\overline{K}/K}$ be the Galois group. The global class field theory says there is an *almost*[1] isomorphism

$$\theta_K : K^\times \backslash \mathbb{A}_K^\times \to \mathsf{Gal}_{\overline{K}/K}^{\mathsf{ab}} \simeq \mathsf{Gal}_{K^{\mathsf{ab}}/K}.$$

Here K^{ab} is the maximal abelian extension of K inside $\overline{K}$.

The almost isomorphism θ_K is characterized by the formula

$$\theta_K(\pi_v) = \mathsf{Fr}_v,$$

where

- v is any finite place of K;
- π_v is the image of an uniformizer of the local field K_v, which is well-defined up to multiplication by an unit in its ring of integers $O_v \subset K_v$;
- Fr_v is a lift of the geometirc Frobenius automorphism of $\mathsf{Gal}_{\overline{\kappa}_v/\kappa_v}$, and is well-defined up to multiplication by an element in the inertia group I_v.

Hence we obtain the following representation-theoretic interpretation of unramified global class field theory.

Theorem 1. *Let ℓ be a prime not equal to $\mathsf{char}(K)$. There is an almost bijection*[2]

$$\{\textit{unramified } \xi : K^\times \backslash \mathbb{A}_K^\times \to \overline{\mathbb{Q}}_\ell^\times\} \longleftrightarrow \{\textit{unramified } \rho : \mathsf{Gal}_{\overline{K}/K} \to \overline{\mathbb{Q}}_\ell^\times\}.$$

Here a continuous character ξ (resp. ρ) is said to be unramifed *if it vanishes on O_v (resp. I_v) for any finite place v. Also, the above almost bijection is characterized by the formula*

$$\xi(\pi_v) = \rho(\mathsf{Fr}_v). \tag{1.1}$$

We restrict to the case when K is a function field over a finite field $k = \mathbb{F}_q$. One simplification in this case is that the topology on $K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_K^\times$ is discrete. Let X be the corresponding complete curve defined over k. Then both sides in the above theorem have geometric interpretation.

According to a theorem of Weil, we have

$$K^\times \backslash \mathbb{A}_K^\times / \mathbb{O}_K^\times \simeq \mathsf{Pic}(X),\ \pi_v \mapsto \mathcal{O}(v)$$

where $\mathsf{Pic}(X)$ is the Picard group of line bundles on X. Hence we have tautological bijection:

$$\{\text{unramified } \xi : K^\times \backslash \mathbb{A}_K^\times \to \overline{\mathbb{Q}}_\ell^\times\} \longleftrightarrow \{\xi : \mathsf{Pic}(X) \to \overline{\mathbb{Q}}_\ell^\times\}.$$

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[1] Do not worry about imprecise words in this lecture. We will give precise statements soon.

[2] If K is a number field, the bijection is as it is because $\overline{\mathbb{Q}}_\ell^\times$ is totally disconnected and therefore the kernel of θ_K, which is connected, is irrelevant.

If K is a function field over a finite field $k = \mathbb{F}_q$, we have to replace $\mathsf{Gal}_{\overline{K}/K}$ by the *Weil group* $\mathsf{W}_{\overline{K}/K}$, which will be explained in future lectures.

On the other hand, more or less by definition,

$$\mathsf{Gal}_{\overline{K}/K} \simeq \pi_1(\mathsf{Spec}(K), \overline{\eta}),$$

where $\pi_1(\mathsf{Spec}(K), \overline{\eta})$ is the *étale fundamental group* of $\mathsf{Spec}(K)$ with base geometric point given by $\overline{\eta} : \mathsf{Spec}(\overline{K}) \to \mathsf{Spec}(K)$. Also, a character ρ is unramified iff it factors through the surjection

$$\pi_1(\mathsf{Spec}(K), \overline{\eta}) \to \pi_1(X, \overline{\eta}).$$

Hence we obtain a bijection

$$\left\{\text{unramified } \rho : \mathsf{Gal}_{\overline{K}/K} \to \overline{\mathbb{Q}}_\ell^\times\right\} \longleftrightarrow \left\{\rho : \pi_1(X, \overline{\eta}) \to \overline{\mathbb{Q}}_\ell^\times\right\}.$$

More or less by definition, we have

$$\left\{\rho : \pi_1(X, \overline{\eta}) \to \overline{\mathbb{Q}}_\ell^\times\right\} \longleftrightarrow \left\{\text{rank one } \ell\text{-adic local system } \sigma \text{ on } X\right\}. \tag{1.2}$$

Hence the almost bijection in Theorem 1 can be rewritten as

$$\left\{\xi : \mathsf{Pic}(X) \to \overline{\mathbb{Q}}_\ell^\times\right\} \longleftrightarrow \left\{\text{rank one } \ell\text{-adic local system } \sigma \text{ on } X\right\}.$$

To finish the translation, we have to interpret the equation (1.1) geometrically. We need the following construction, known as *Grothendieck's sheaf-function correspondence.*

Construction 2. *Let Y be an algebraic variety over $k = \mathbb{F}_q$ and $\mathcal{F}$ be an ℓ-adic sheaf on Y. For any k-point y of Y, let $\overline{y}$ be the geometric $\overline{k}$-point at y. The stalk $\mathcal{F}_{\overline{y}}$ is a $\overline{\mathbb{Q}}_\ell$-vector space equipped with a continuous $\mathsf{Gal}_{\overline{k}/k}$-action. Let $\mathsf{Fr}_{\overline{y}}$ be the geometric Frobenius automorphis. Define*

$$\mathsf{f}_{\mathcal{F}}(y) := \mathsf{Tr}(\mathsf{Fr}_{\overline{y}}, \mathcal{F}_{\overline{y}})$$

which is a function on $Y(k)$.

It follows from the definition that if ρ corresponds to σ via (1.2), then

$$\rho(\mathsf{Fr}_v) = \mathsf{f}_\sigma(v) = \mathsf{Tr}(\mathsf{Fr}_{\overline{v}}, \sigma_{\overline{v}}),$$

at least when v is a k-point on X.

Therefore we obtain the following translation of Theorem 1.

Corollary 3. *There is an almost bijection*

$$\left\{\xi : \mathsf{Pic}(X) \to \overline{\mathbb{Q}}_\ell^\times\right\} \longleftrightarrow \left\{\textit{rank one } \ell\textit{-adic local system } \sigma \textit{ on } X\right\} \tag{1.3}$$

characterized by the formula

$$\xi(\mathcal{O}(v)) = \mathsf{f}_\sigma(v). \tag{1.4}$$

We may go one step further by replacing the *Picard group* by the *Picard scheme.* The (relative) Picard scheme $\mathsf{Pic}_{X/k}$ is defined such that for any affine test scheme S over k,

$$\mathsf{Pic}_{X/k}(S) := \mathsf{Pic}(X \underset{k}{\times} S)/\mathsf{Pic}(S)$$

and in particular

$$\mathsf{Pic}_{X/k}(k) = \mathsf{Pic}(X).$$

Hence we are attempt to ask for the following dotted arrow:

$$\begin{array}{ccc} \left\{\xi : \mathsf{Pic}(X) \to \overline{\mathbb{Q}}_\ell^\times\right\} & \longleftrightarrow & \left\{\text{rank one } \ell\text{-adic local system } \sigma \text{ on } X\right\} \\ \uparrow{\scriptstyle \text{Const. 2}} & \swarrow{\scriptstyle \cdots} & \\ \left\{\ell\text{-adic sheaf } \mathcal{F} \text{ on } \mathsf{Pic}_{X/k}\right\}. & & \end{array}$$

But before finding the dotted arrow, we need to answer the following two questions:

(a) What condition on $\mathcal{F}$ can guarantee $\xi := \mathsf{f}_{\mathcal{F}}$ to be a character?
(b) What condition on $\mathcal{F}$ can guarantee the equation (1.4)?

It turns out there is a nice geometric answer for both questions.

Definition 4. Let $\mathcal{F}$ be a *sheaf*[3] on $\mathsf{Pic}_{X/k}$ and σ be a rank one local system σ on X, we say $\mathcal{F}$ is a *Hecke eigensheaf with eigenvalue* σ if it is equipped with an isomorphism

$$\mathsf{add}^*(\mathcal{F}) \simeq \sigma \boxtimes \mathcal{F}$$

that satisfies a commutativity law (to be addressed in future lectures). Here add is the morphism

$$\mathsf{add} : X \underset{k}{\times} \mathsf{Pic}_{X/k} \to \mathsf{Pic}_{X/k},\ (v, \mathcal{L}) \mapsto \mathcal{L}(v).$$

We will prove the following lemma in the next lecture.

Lemma 5. *Let $\mathcal{F}$ be a Hecke eigensheaf with eigenvalue σ such that its restriction to the unit point $\mathcal{O} \in \mathsf{Pic}_{X/k}$ is equivalent to the constant sheaf*[4]*. Consider the function $\xi := \mathsf{f}_{\mathcal{F}}$ on $\mathsf{Pic}(X) = \mathsf{Pic}_{X/k}(k)$. Then:*

(a) *The function ξ is a character.*
(b) *For any place v, the equation (1.4) holds, i.e., $\mathsf{f}_{\mathcal{F}}(\mathcal{O}(v)) = \mathsf{f}_{\sigma}(v)$.*

The following theorem is due to Deligne.

Theorem 6. *For any rank one local system σ on X, there exists an essential unique*[5] *Hecke eigensheaf $\mathcal{A}ut_\sigma$ on $\mathsf{Pic}_{X/k}$, which is in fact a local system.*

Remark 7. Note that the statement of the above theorem is pure geometric, and, as we will see in the next lecture, so is Deligne's proof. Therefore it is fair to call it the *unramified global geometric class field theory*, or more ambitiously, the

Unramified Global Geometric Langlands for GL_1.

2. Unramified Global Geometric Langlands Correspondence

We want to generalize this story to GL_n or even to all split reductive groups[6] G over k. From now on, X is a smooth, complete, geometrically connected over a perfect field k. Let $K = K(X)$ be the field of rational functors on X. We will consider a well-behaved sheaf theory with an algebraic closed coeffiencient field e. For nice enough schemes Y over k, the abelian category of such sheaves on Y is denoted by $\mathsf{Shv}_c(Y)^\heartsuit$ and its derived category is denoted by $\mathsf{Shv}_c(Y)$.

Example 8. The author is only aware of the following choices of sheaf theories (or minor variants of them) in geometric Langlands theory.

- The *Betti context* where $k = \mathbb{C}$ and we consider preverse sheaves for the complex topology;
- The *ℓ-adic context* where k is finite and we consider perverse ℓ-adic Weil sheaves;
- The *de Rham context* where k is algebraic closed with $\mathsf{char}(k) = 0$ and we consider coherent right D-modules.

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[3]See Example 8.

[4]This assumption is to guarantee ξ preserves the unit.

[5]Note that the tensor product of a Hecke eigensheaf with any object in $\mathsf{Shv}(\mathsf{Spec}(k))$ is still a Hecke eigensheaf with the same Hecke eigenvalue. Here the essential uniqueness says such tensor products are the only source of ambiguity.

[6]The story for *non-constant* reductive group scheme over X is also interesting, but we will not discuss it in this course.

On the Galois side, the generalization is immediate:

GL_n	local system σ on X with $\mathsf{rank}(\sigma) = n$
G	G-local system σ on X.

On the automorphic side, as a first task, we need to complete the following chart.

GL_1	$K^\times\backslash\mathbb{A}_K^\times/\mathbb{O}_K^\times$	$\mathsf{Pic}(X)$	$\mathsf{Pic}_{X/k}$
G	$G(K)\backslash G(\mathbb{A}_K)/G(\mathbb{O}_K)$	?	?.

The following result is called *Weil's uniformization theorem*[7].

Theorem 9. *The set $G(K)\backslash G(\mathbb{A}_K)/G(\mathbb{O}_K)$ is naturally bijective to the set of isomorphic classes of principle G-bundles (a.k.a. G-torsors) on X.*

However, it is impossible to find a *scheme* that plays the same role for G as $\mathsf{Pic}_{X/k}$ did for GL_1: we have to step out of our comfort zone and embrace the world of *algebraic stacks*[8].

Proposition-Definition 10. *There exists a smooth Artin stack Bun_G defined over k that classifies families of G-torsors on X. In particular,*

$$\mathsf{Bun}_G(k) \simeq G(K)\backslash G(\mathbb{A}_K)/G(\mathbb{O}_K).$$

The precise definition of Bun_G will be given in the future lectures. Also, a better notation would be $\mathsf{Bun}_{G,X/k}$. But people are lazy.

We also need to generalize the Hecke eigen-property. Recall for $G = \mathsf{GL}_1$, it is defined using the morphism

$$\mathsf{add} : X \underset{k}{\times} \mathsf{Pic}_{X/k} \to \mathsf{Pic}_{X/k},\ (x, \mathcal{L}) \mapsto \mathcal{L}(x),$$

which can be viewed as modifying a line bundle at a point x in a canonical way. However, in the general case, there is no *canonical* modification of a G-torsor at a point, but many of them. Therefore we need to replace add by a multi-valued morphism, a.k.a., a correspondence,

$$\begin{array}{ccc} & \mathsf{Hecke}_{G,X} & \\ \overleftarrow{\mathfrak{h}} \swarrow & & \searrow \overrightarrow{\mathfrak{h}} \\ \mathsf{Bun}_G & & X \times_k \mathsf{Bun}_G. \end{array}$$

which will be defined in futher lectures.

A more significant point is: the eigenvalues for Hecke eigensheaves will be local systems for the *Langlands dual group* $\check{G}$ (defined over e). Of course, this is not surprise from Langlands' philosophy.

Let us assume the definition of Hecke eigensheaves and state the conjectural unramified geometric Langlands correspondence:

Conjecture 11. *For any geometrically irreducible*[9] *$\check{G}$-local system σ on X, there is an essentially unique Hecke eigensheaf $\mathcal{A}ut_\sigma$ on Bun_G with eigenvalue σ.*

For $G = \mathsf{GL}_n$, the existence of $\mathcal{A}ut_\sigma$ is a theorem due to Deligne ($n = 1$), Drinfeld ($n = 2$) and Frenkel–Gaitsgory–Vilonen (for general n).

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[7] We need some assumptions on G and X, which will be articulate in future lectures.

[8] Roughly speaking, an algebraic stack is a "space" whose "points" are equipped with trivial actions by certain algebraic groups. Therefore an algebraic stack can remember not only the set of (double) cosets like $G(K)\backslash G(\mathbb{A}_K)/G(\mathbb{O}_K)$, but also the stablizer group for each point. Unlike the case $G = \mathsf{GL}_1$, these stablizer groups are not constant and thereby not ignorable.

[9] This means σ is not induced from proper parabolic subgroup of $\check{G}$ after base-change to $\overline{k}$. If σ is not geometrically irreducible, it is more reasonable to consider Hecke eigenobject in $\mathsf{Shv}(\mathsf{Bun}_G)$ rather than $\mathsf{Shv}(\mathsf{Bun}_G)^{\heartsuit}$. Also, the essential uniqueness is false in that case.

Remark 12. In the ℓ-adic context, taking trace of Frobenius, the above conjecture will provide a *Galois to automorphic* construction for Langlands correspondence of function fields. Also, the automorphic function given by $\mathcal{A}ut_\sigma$ is cuspidal.

3. Unramified Global Geometric Langlands Equivalence

We might also want to consider the algebro-geometric object $\mathsf{LS}^{\mathsf{irre}}_{\check{G}}$ that classifies families of geometrically irreducible $\check{G}$-local systems on X, and update Conjecture 11 to a functor

$$\mathsf{Coh}(\mathsf{LS}^{\mathsf{irre}}_{\check{G}})^{\heartsuit} \to \mathsf{Shv}_c(\mathsf{Bun}_G)^{\heartsuit},\ \delta_\sigma \mapsto \mathcal{A}ut_\sigma$$

where the LHS is the abelian category of coherent sheaves on $\mathsf{LS}^{\mathsf{irre}}_{\check{G}}$, however it is defined. In fact, by Langlands philosephy, one even expect this functor induces an equivalence between $\mathsf{Coh}(\mathsf{LS}^{\mathsf{irre}}_{\check{G}})^{\heartsuit}$ and the cuspidal part of $\mathsf{Shv}_c(\mathsf{Bun}_G)^{\heartsuit}$, however the latter is defined.

For example, in the de Rham context, $\mathsf{LS}^{\mathsf{irre}}_{\check{G}}$ is a smooth Artin stack and we have the following *categorical* conjecture, which generalizes Conjecture 11.

Conjecture 13. *In the de Rham context, there is a canonical t-exact equivalence*

$$\mathbb{L}_G : \mathsf{DMod}_{\mathsf{coh}}(\mathsf{Bun}_G)^{\mathsf{cusp}} \simeq \mathsf{Coh}(\mathsf{LS}^{\mathsf{irre}}_{\check{G}})$$

such that $\mathbb{L}_G^{-1}$ *sends the skyscrapter sheaf at an irreducible* $\check{G}$*-local system* σ *to a Hecke eigensheaf* $\mathcal{A}ut_\sigma$ *with eigenvalue* σ*.*

We can even take the reducible local systems into consideration once we give up the t-exactness.

Conjecture 14. *In the de Rham context, there is a canonical equivalence*[10]

$$\mathbb{L}_G : \mathsf{DMod}_{\mathsf{coh}}(\mathsf{Bun}_G) \simeq \mathsf{Coh}(\mathsf{LS}_{\check{G}})$$

such that $\mathbb{L}_G^{-1}$ *sends the skyscrapter sheaf at a* $\check{G}$*-local system* σ *to a Hecke eigen-object* $\mathcal{A}ut_\sigma$ *with eigenvalue* σ*.*

Both conjectures are not far away from becoming a theorem. Note that just like the classical Langlands correspondence, Conjecture 14 has two sides, the automorphic side (RHS) and the Galois side (LHS), but it is *1-categorical higher* than its classical analogue.

Remark 15. The ramified story is beyond my current knowledge.

4. Local Geometric Langlands Equivalence

Let us focus on the de Rham context. Recall the local Langlands correspondence classifies representations of $G(K_v)$ for a local field K_v, which form a 1-category. It turns out the local *geometric* Langlands correspondence is a conjectural classification of *categorical representations* of $G(K_v)$, which form a 2-category.

The geometric incarnation of $G(K_v)$ is the *loop group* $\mathcal{L}G$, which is an group indscheme whose $\mathbb{C}$-valued points are given by K_v-valued points of G. We want to define the notion of categories equipped with an $\mathcal{L}G$-action such that it is a reasonable categorification of the notion of vector spaces equipped with a $G(K_v)$-action. In future lectures, we will explain the following definitions.

[10] The most reasonable way to define $\mathsf{DMod}_{\mathsf{coh}}(\mathsf{Bun}_G)$ is to define it as a DG category, realized as an ∞-category equipped with k-linear structure. At this stage, we have to embrace the world of *higher category theory.*

Definition 16. Let

$$\mathcal{L}G\text{–}\mathbf{Mod}$$

be the $(\infty,2)$-category of DG categories acted by the monoidal DG category $\mathsf{DMod}(\mathcal{L}G)$, where the monoidal structure is given by convolution.

Given any nice algebro-geometric object Y equipped with a (geometric) $\mathcal{L}G$-action, the DG category $\mathsf{DMod}(Y)$ is a $\mathsf{DMod}(\mathcal{L}G)$-module, and thereby an object in $\mathcal{L}G\text{–}\mathbf{Mod}$.

Example 17. Interesting examples for Y include:

$$\mathsf{Gr}_G,\ \mathsf{Fl}_G,\ \mathsf{Bun}_G^{\mathsf{level}_{\infty\cdot v}},$$

known as the *affine Grassmannian*, the *affine flag (ind-)variety*, and the *moduli (ind-)stack of G-torsors on X equipped with a trivialization on the formal neighborhood $\mathcal{D}_v$ at v*[11].

We can view $\mathsf{DMod}(\mathcal{L}G)$ as a bi-module of itself. Then any operation with respect to the right action will produce a DG category that inherits the left $\mathcal{L}G$-action. In fact, the objects

$$\mathsf{DMod}(\mathsf{Gr}_G),\ \mathsf{DMod}(\mathsf{Fl}_G) \in \mathcal{L}G\text{–}\mathbf{Mod}$$

defined above can be equivalently defined as taking $\mathcal{L}^+G$-invariance (resp. I-invariance) for the right $\mathcal{L}G$-action on $\mathsf{DMod}(\mathcal{L}G)$, where the *arc group* $\mathcal{L}^+G$ (resp. the *Iwahori subgroup* I) are the geometric incarnation of $G(O_v)$ (resp. $G(O_v)\times_{G(\kappa_v)} B(\kappa_v)$).

Example 18. A signification example is the *Whittaker model*

$$\mathsf{Whit}(\mathcal{L}G) \in \mathcal{L}G\text{–}\mathbf{Mod}$$

which is obtained by taking $\mathcal{L}N$-invariance (for the right action) against a generic character χ. Here N is the unipotent radical of the Borel subgroup B.

Example 19. Another striking object in $\mathcal{L}G\text{–}\mathbf{Mod}$ is given by $\mathsf{KM}_{\mathfrak{g}}$, the DG category of *representations of the affine Lie algebra $\widetilde{\mathcal{L}\mathfrak{g}}$ at the critical level.* We will explain it in future lectures.

On the Galois side, we should consider the moduli problem $\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ)$ of $\check{G}$-local systems on the punctured disk $\mathcal{D}^\circ := \mathsf{Spec}(K_v)$.

Definition 20. Let

$$\mathbf{QCohCat}(\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ))$$

be the $(\infty,2)$-category of $\mathsf{QCoh}(\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ))$-module categories, where the monoidal structure is given by tensor product.

Given any nice algebro-geometric object Z over $\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ)$, the DG category $\mathsf{QCoh}(Z)$ is a $\mathsf{QCoh}(\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ))$-module.

Example 21. Interesting examples for Z include[12]:

$$\mathsf{LS}_{\check{G}}(\mathcal{D}),\ \mathsf{LS}_{\check{G}}^{\mathsf{nilp}}(\mathcal{D}^\circ),\ \mathsf{LS}_{\check{G}}(X^\circ),$$

which classifies $\check{G}$-local system on the disk $\mathcal{D} := \mathsf{Spec}(O_v)$ (resp. on the punctured disk with nilpotent singulary, on the punctured curve $X^\circ := X \smallsetminus x$).

Example 22. Of course, we can also consider $\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ)$ itself.

[11]Its $\mathbb{C}$-valued points are bijective to $(\prod_{w\neq v} G(O_w))\backslash G(\mathbb{A}_K)/G(K)$.

[12] We have $\mathsf{LS}_G(\mathcal{D}) \simeq \mathrm{pt}/G$ and $\mathsf{LS}_G^{\mathsf{nilp}}(\mathcal{D}^\circ) \simeq \mathcal{N}/G$, where $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$.

Example 23. Another significant example is

$$\mathsf{Op}_{\check{G}}(\mathcal{D}^\circ)$$

which classifies $\check{G}$-opers on the puncture disk. We will study them in future lectures.

Examples 17-19 and Examples 21-23 are not listed arbitrarily. We have:

Conjecture 24. *There is an almost equivalence*[13]

$$\begin{array}{rrcl}
& \mathcal{L}G\text{–}\mathbf{Mod} & \simeq & \mathbf{QCohCat}(\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ)) \\
(4.1) & \mathsf{DMod}(\mathsf{Gr}_G) & \mapsto & \mathsf{QCoh}(\mathsf{LS}_{\check{G}}(\mathcal{D})) \\
(4.2) & \mathsf{DMod}(\mathsf{Fl}_G) & \mapsto & \mathsf{QCoh}(\mathsf{LS}^{\mathsf{nilp}}_{\check{G}}(\mathcal{D}^\circ)) \\
(4.3) & \mathsf{DMod}(\mathsf{Bun}_G^{\mathsf{level}_{\infty\cdot v}}) & \mapsto & \mathsf{QCoh}(\mathsf{LS}_{\check{G}}(X^\circ)) \\
(4.4) & \mathsf{Whit}(\mathcal{L}G) & \mapsto & \mathsf{QCoh}(\mathsf{LS}_{\check{G}}(\mathcal{D}^\circ)) \\
(4.5) & \mathsf{KM}_{\mathfrak{g}} & \mapsto & \mathsf{QCoh}(\mathsf{Op}_{\check{G}}(\mathcal{D}^\circ)).
\end{array}$$

One can pick any two lines from above and calculating the Hom-categories for both sides, then (after replacing QCoh by Coh and DMod by $\mathsf{DMod}_{\mathsf{coh}}$) they would obtain a conjecture equivalence, and sometimes it is a known theorem. For example:

- $\mathsf{Hom}((4.1),(4.1))$ is the *derived geometric Satake equivalence* proven by Bezrukavnikov–Finkelbery;
- $\mathsf{Hom}((4.1),(4.2))$ is a theorem by Arkhipov–Bezrukavnikov–Ginzburg;
- $\mathsf{Hom}((4.2),(4.2))$ is Bezrukavnikov's equivalence between two geometric realizations of the affine Hecke algebra;
- $\mathsf{Hom}((4.1),(4.3))$ is Conjecture 14 and $\mathsf{Hom}((4.2),(4.3))$ is a tamely ramified version of it;
- $\mathsf{Hom}((4.1),(4.4))$ is the *geometric Casselman–Shalika formula* proven by Frenkel–Gaitsgpry–Viloner;
- $\mathsf{Hom}((4.2),(4.4))$ is a theorem by Arkhipov–Bezrukavnikov;
- $\mathsf{Hom}((4.1),(4.5))$ is a theorem by Feigin–Frenkel.
- $\mathsf{Hom}((4.4),(4.5))$ is a theorem by Raskin.

5. Quantum Local Geometric Langlands Equivalence

Using the notion of *twisted D-modules*, we can obtain a deformation of Conjecture 24. We will explain it in the future lectures:

Conjecture 25. *There is an almost equivalence*

$$\begin{array}{rcl}
\mathcal{L}G\text{–}\mathbf{Mod}_\kappa & \simeq & \mathcal{L}\check{G}\text{–}\mathbf{Mod}_{\check{\kappa}} \\
\mathsf{DMod}_\kappa(\mathsf{Gr}_G) & \mapsto & \mathsf{DMod}_{\check{\kappa}}(\mathsf{Gr}_{\check{G}}) \\
\mathsf{DMod}_\kappa(\mathsf{Fl}_G) & \mapsto & \mathsf{DMod}_{\check{\kappa}}(\mathsf{Fl}_{\check{G}}) \\
\mathsf{DMod}_\kappa(\mathsf{Bun}_G^{\mathsf{level}_{\infty\cdot v}}) & \mapsto & \mathsf{DMod}_{\check{\kappa}}(\mathsf{Bun}_{\check{G}}^{\mathsf{level}_{\infty\cdot v}}) \\
\mathsf{Whit}_\kappa(\mathcal{L}G) & \mapsto & \mathsf{KM}_{\check{\mathfrak{g}},\check{\kappa}} \\
\mathsf{KM}_{\mathfrak{g},\kappa} & \mapsto & \mathsf{Whit}_{\check{\kappa}}(\mathcal{L}\check{G}).
\end{array}$$

Note that the statement is completely symmetric for G and $\check{G}$.

[13] The word "almost" has to do with the difference between QCoh and Coh. Unfortunately, at the current stage, there is no convenient definitions of "coherent" parts of both sides that allow us to conjecture a perfect equivalence.