Cleanese

Last time:

Fourier transformation:

$$\begin{tikzcd}[column sep=small]
\overline{Coh'_n} \arrow[d] & \overset{\mathcal{F}_n}{\overset{j_n}{\supset}} & Coh'_n \arrow[dr] & & \overline{Coh'_{n-1}} \arrow[dl] & \overset{\mathcal{F}_{n-1}}{\overset{j_{n-1}}{\supset}} & Coh'_{n-1} \arrow[d] & \cdots \arrow[dr] & & \overline{Coh'_2} \arrow[dl] & \overset{\mathcal{F}_2}{\overset{j_2}{\supset}} & Coh'_2 \arrow[dr] & & \overline{Coh'_1} \arrow[dl] & \overset{\mathcal{F}_1}{\overset{j_1}{\supset}} & Coh'_1 \arrow[dr] & \\
Coh_n & & & Coh_{n-1} & & & {} & & Coh_2 & & & & Coh_1 & & & & Tor
\end{tikzcd}$$

where
$$Coh_k = \{ M_k \in Coh(X),\ rank(M_k) = k \}$$
$$Coh'_k = \{ \Omega_X^{k-1} \hookrightarrow M_k \}$$
$$\overline{Coh'_k} = \{ \Omega_X^{k-1} \longrightarrow M_k \}.$$

Over an (undetermined) open $\mathcal{C}_k \subset Coh_k$ such that

① $Ext^1(\Omega_X^{k-1}, M_k) = 0 \qquad M_k \in \mathcal{C}_k$

② $\deg(M_k^{tf}) > nk(2g-2) \qquad$ (TBC …)

$$\begin{tikzcd}[column sep=small]
\overline{C'_{k+1}} \supset C'_{k+1} \subset \overline{C'_k}^{\vee} := Coh'_{k+1}|_{C_k} \arrow[dr] & & \overline{Coh'_k}|_{C_k} =: \overline{C'_k} \supset C'_k \overset{③}{\subset} \overline{C'_{k-1}}^{\vee} \arrow[dl] \\
& C_k &
\end{tikzcd}$$

dual vector bundle

③ $M_k \in \mathcal{C}_k \Rightarrow$ any $M_k/\Omega_X^{k-1} \in \mathcal{C}_{k-1}$

Def: Define $\mathcal{F}_k \in D(C'_k)$ by induct.

$$\mathcal{F}_{k+1} := \mathrm{Four} \circ j_{k,!}(\mathcal{F}_k)\big|_{C'_{k+1}}.$$

$$\mathcal{F}_1 := \mathcal{L}_E\big|_{C'_1}\,[\text{shift}](\text{twist})$$

Rmk:

$$\begin{tikzcd} \boxed{Bun'_n \cap C'_n} \arrow[r] \arrow[d] & C'_n \\ Bun'_n \arrow[r, phantom, "\subset"] & Coh'_n \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$$\mathcal{F}_n\big|_{Bun'_n \cap C'_n} = \mathrm{Aut}'_E\big|_{Bun'_n \cap C'_n}$$

where

$$\overbrace{\boxed{Gr_G^{reg} \tilde{\times} Bun_N}}^{Bun_N^{\subset}} \xrightarrow{\pi} Bun'_n$$

$$\mathrm{Aut}'_E = \pi_!\left(\mathcal{L}_E\big|_{Gr_G^{reg}} \tilde{\boxtimes} \exp_\psi\big|_{Bun_N}\right)[\cdot](\cdot)$$

More generally, replace $Gr_G^{reg} \tilde{\times} Bun_N$ by

$$\left\{ \begin{array}{l} (0 \subset M_1 \subset \cdots \subset M_n) \in Bun_N \\ M_n \hookrightarrow M \in Coh_n \end{array} \right\} \text{ gives } \mathcal{F}_n$$

rather than Bun_G

$$:= Bun_N \times_{Coh_n} Coh_n^{\subset} =: Bun_N^{\subset, coh}$$

($\widetilde{G}$ in the paper)

$\underline{\text{Thm (C)}}$: $\underline{j_! \mathcal{F}_k}$ is a clean extension.

$\underline{\text{Proof}}$: $\mathrm{Coh}_k = \bigcup_{\ell} \mathrm{Coh}_k^{\mathrm{tor} \le \ell}$

$\mathrm{Coh}_k^{\mathrm{tor}\le \ell} = \left\{ M_k \in \mathrm{Coh}_k \,.\, \begin{array}{l}\text{torsion part of } M_k \\ \text{has length } \le \ell \end{array} \right\}.$

$\mathrm{Coh}_k^{\mathrm{tor}\le 0} = \mathrm{Bun}_k.$

$C_k = \bigcup_{\ell} C_k^{\mathrm{tor}\le \ell}.$

We will prove by induction that $j_! \mathcal{F}_k$ is clean over $C_k^{\mathrm{tor}\le \ell}$.

$\boxed{\ell = 0}$ $\Omega_X^{k-1} \longrightarrow M_k$ is either injective or 0.

$$\begin{tikzcd}
C_k^{\mathrm{tor}=0} \arrow[r, "i"] & \overline{C}_k^{\,\prime, \mathrm{tor}=0} \arrow[d, "f"] & \underline{C_k^{\prime, \mathrm{tor}=0}} \arrow[l, "j"'] \arrow[dl] \\
& \underline{C_k^{\mathrm{tor}=0}} &
\end{tikzcd}$$

$\boxed{i \text{ given by } \Omega_X^{k-1} \xrightarrow{0} M_k}$

By contraction principle, need

$$i^! j_!(\mathcal{F}_k^{(\text{tor'd})}) = \underbrace{q_! \circ j_!(\mathcal{F}_k)}= 0.$$

$$\mathrm{Gr}_k^{\mathrm{reg}} \tilde{\times} \mathrm{Bun}_{N_k} \xrightarrow{\pi} \mathrm{Bun}'_k \xrightarrow{r} \mathrm{Bun}_k$$

$$\mathrm{Bun}'_k \supset C_k^{',\text{torsio}} \qquad \mathrm{Bun}_k \supset C_k^{\text{torsio}}$$

$$q_! \circ j_!(\mathcal{F}_k) = (r\circ\pi)_! \left(\mathcal{L}_E|_{\mathrm{Gr}_k^{\mathrm{reg}}} \tilde{\boxtimes} \exp_\psi|_{\mathrm{Bun}_{N_k}} \right)[\text{shifts}](\text{twist}) \Big|_{C_k^{\text{torsio}}}$$

$$\begin{tikzcd}
\mathrm{Bun}_{N_k} \arrow[d] & \mathrm{Gr}_k^{\mathrm{reg}} \tilde{\times} \mathrm{Bun}_{N_k} \arrow[l] \arrow[d] \arrow[l, phantom, "\perp"] \arrow[drr, "r\circ\pi"] & & \\
\mathrm{Bun}_k & \underset{\text{“}\mathrm{Mod}_k\text{”}}{\mathrm{Gr}_k^{\mathrm{reg}} \tilde{\times} \mathrm{Bun}_k} \arrow[l, "\overleftarrow{h}"'] \arrow[rr, "\overrightarrow{h}"] & & \mathrm{Bun}_k
\end{tikzcd}$$

$$\overrightarrow{h}_!(\mathcal{L}_E \tilde{\boxtimes} W_!)$$

Def: $W_!$ is the basic Whittaker, direct image of $\exp_\psi|_{\mathrm{Bun}_{N}}$. It is supported on Bun_k°

(Recall $\deg^\circ$ is normalized s.t.

$$\mathrm{Bun}_{N_k} \to \mathrm{Bun}_k^\circ \;)$$

Def $\mathrm{Av}_E^{(d)}(-) : D(\mathrm{Bun}_k) \longrightarrow D(\mathrm{Bun}_k)$

$$\parallel$$

$$\overrightarrow{h}_!\,(\mathcal{L}_E^{(d)} \tilde{\boxtimes} -)$$

(sends $D(\mathrm{Bun}_k^a)$ to $D(\mathrm{Bun}_k^{a+d})$)

Need

$$\mathrm{Av}_E(\mathcal{W}_!)\big|_{C_k^{\mathrm{tor}=0}} = 0.$$

Note that

$$C_k^{\mathrm{tor}=0} \subset \bigcup_{d > nk(2g-1)} \mathrm{Bun}_k^d \quad \text{(by ②)}.$$

$\Leftarrow$

Thm (V): For $d > nk(2g-2)$,

$$\mathrm{Av}_E^d = 0.$$

$\boxed{l-1 \Rightarrow l}$ By ind. hyp., only need closedness along

$$\underset{Y}{\overline{C}_k^{1,\mathrm{tor}\le l}} \longleftarrow \underset{U}{C_k^{1,\mathrm{tor}\le l} \cup \overline{C}_k^{1,\mathrm{tor}<l}}$$

The complement is

$$Z := \{ \Omega_x^{k-1} \longrightarrow M_k \mid \text{not injective}, \mathrm{Tor}(M_k) = l \}$$

$$Z \hookrightarrow Y \supset U$$

Unlike before Y is not a bundle over Z: no way to produce a $(\mathrm{Tor} = l)$ coherent sheaf from a $(\mathrm{Tor} \le l)$ one in a canonical way.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ॥
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ functorial

But we can make this possible by doing a <u>smooth</u> base-change.

<u>Motivation</u>: Any $(\mathrm{Tor} \le l)$ M_k can be written as

$$0 \to \mathring{M}_k \hookrightarrow M_k \longrightarrow J \to 0$$

with $\mathring{M}_k \in \mathrm{Bun}_k$, $J \in \mathrm{Tor}^l$.

And vice versa.

$\mathring{M}_k \oplus J$ is $\mathrm{Tor} = l$

$$\tilde{\mathcal{Y}} = \left\{ \begin{array}{l} \Omega^{k-1} \to M_k \\ 0 \to \mathring{M}_k \hookrightarrow M_k \to T \to 0 \end{array} \right\}.$$

Lem: $\tilde{\mathcal{Y}} \longrightarrow \mathcal{Y}$ is smooth.

Pmf: True even without the section $\Omega^{k-1} \to M_k$.

$$\widetilde{\mathrm{Coh}}_k^{\mathrm{Torsl}} \longrightarrow \mathrm{Coh}_k^{\mathrm{Torsl}} \quad \text{is smooth.}$$

Both stacks are smooth. Only need smoothness for fibers.

Relative tangent space $\mathrm{Hom}(\mathring{M}_k, T)$, no dimension jump. $\square$.

$$\begin{tikzcd} \tilde{\mathcal{Z}} \arrow[r, hook] \arrow[d] & \tilde{\mathcal{Y}} \arrow[r, phantom, "\supset"] \arrow[d] & \tilde{\mathcal{U}} \arrow[d] \\ \mathcal{Z} \arrow[r, hook] & \mathcal{Y} \arrow[r, phantom, "\supset"] & \mathcal{U} \end{tikzcd}$$

$$\tilde{\mathcal{Z}} = \left\{ \begin{array}{l} \Omega^{k-1} \to M_k \mid \text{not inj} \\ 0 \to \mathring{M}_k \to M_k \to T \to 0 \end{array} \;\middle|\; \underbrace{\mathrm{Tor}(M_k) = \ell}_{} \right\}$$

$$\Downarrow$$

$$M_k = \mathring{M}_k \oplus T.$$

$$\Downarrow$$

$$\Omega^{k-1} \dashrightarrow T \subset M_k$$

$$\|$$

$$\left\{ \Omega^{k-1} \to T,\ \mathring{M}_k \;\middle|\; \mathring{M}_k \oplus T \in C_k \right\}$$

We almost get the projection $\tilde{\mathcal{Y}} \to \tilde{Z}$:

$$\left(\begin{array}{l} \Omega^{k-1} \to M_k \\ 0 \to \mathring{M}_k \to M_k \to T \to 0 \end{array} \right)$$

$$\downarrow$$

$$\left(\Omega^{k-1} \to M_k \to T \, , \, \mathring{M}_k \right)$$

But need $\mathring{M}_k \oplus T \in \mathcal{C}_k$

$$\tilde{\mathcal{Y}} = \left\{ \left(\quad \right) \middle| \ \underline{M_k} \, , \, \underline{\mathring{M}_k \oplus T} \in \mathcal{C}_k \right\}$$

Rmk: $\mathbb{G}_m$ acts on $\tilde{\mathcal{Y}}$ as scalars on $\mathrm{Ext}^1(T, \mathring{M}_k)$. This extends to $\mathbb{A}^1$-action, $\tilde{Z}$ is the fixed locus.

Rmk: $\tilde{Z} \neq Z$: in $\tilde{Z}$, $M_k = \mathring{M}_k \oplus T$ is a structure

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{i}"] & \tilde{Y} \arrow[d, "\hat{q}"] & \tilde{U} \arrow[l, "\tilde{j}"'] \end{tikzcd} \quad (\hookrightarrow U \subset C_k')$$

$$\tilde{Z}$$

Contraction principle : only need

$$\hat{q}_! \circ \tilde{j}_! \ \mathcal{F}_k|_{\tilde{U}} = 0$$

Recall $\mathcal{F}_k$ can be calculated as

$$\begin{tikzcd} \underbrace{\mathrm{Bun}_{N_k}^{c,coh}}\arrow[r, "\pi"] \arrow[d] \arrow[dr] & \mathrm{Coh}_k' \supset C_k' \\ \mathrm{Tor} & \mathrm{Gau} \end{tikzcd}$$

$$\pi_! (\mathcal{L}_E|_\cdot \otimes \exp_\psi|_\cdot) \Big|_{C_k'} .$$

By base change, consider

$$\mathrm{Bun}_{N_k}^{c,coh} \times_{\mathrm{Coh}_k'} \hat{U} \xrightarrow{\tilde{\pi}} \tilde{U} \xrightarrow{\hat{q}\circ\tilde{j}} \tilde{Z}$$

$$\left\{ \begin{array}{l} \overset{\text{fix}}{(0 \subset M_1 \subset \cdots \subset M_k)} \in \mathrm{Bun}_{N_k} \\ M_k \hookrightarrow M \\ 0 \to \overset{\circ}{M} \to M \to J \to 0 \end{array} \;\middle|\; \begin{array}{l} M \\ \overset{\circ}{M} \oplus q \end{array} \in C_k, \quad |J| = \ell \right\}$$

$$\longrightarrow \Big\{ \underbrace{\Omega_x^{k-1} \to J, \ \overset{\circ}{M}}_{\text{fix}} \ \Big| \ \cdots \Big\} \quad \text{even fix } M_k \to J$$

where $\Omega_x^{k-1} = M_1 \hookrightarrow M_k \hookrightarrow M \to J$.

Need vanishing of !-push for certain sheaf.

Fiber over the fixed thing

$$\left\{ \begin{tikzcd} 0 \arrow[r] & \overset{\circ}{M}_k \arrow[r] \arrow[d] & M_k \arrow[r] \arrow[d] & J \arrow[d] & \\ 0 \arrow[r] & \overset{\circ}{M} \arrow[r] & M \arrow[r] & J \arrow[r] & 0 \end{tikzcd} \right\} \cong : \text{1-ty}$$

$$\Big\downarrow \simeq \quad \boxed{\text{If } M_k \twoheadrightarrow J}$$

$$\Big\{ \overset{\circ}{M}_k \xrightarrow[\text{modification of degree } d.]{} \overset{\circ}{M} \Big\} \longrightarrow \mathrm{Tor}$$

Hence if $M_k \twoheadrightarrow J$ is surjective, then only need to show

$$\mathcal{L}_E\Big|_{\{\mathring{M}_k \to \mathring{M}\}} \xmapsto{\text{! push}} 0.$$

$$\left\{\begin{array}{l} \deg M_k = 0 \Rightarrow \deg \mathring{M}_k = -\deg J = -l \\ \deg(\mathring{M}) > nk(2g-2) \quad \text{by } (2) \\ d > nk(2g-2) + l \end{array}\right. \Longrightarrow \text{Thm U}$$

When $M_k \to J$ is not surjective.

$$\begin{tikzcd} 0 \arrow[r] & \mathring{M}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\bullet"] & M_k \arrow[r, two heads] \arrow[d] & J' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathring{M} \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d] \arrow[dr, phantom, "\bullet"] & J' \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & \mathring{M} \arrow[r] & M \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

$$0 \to M'/M_k \to M/M_k \to J/J' \to 0$$

$$\begin{tikzcd}
& H_J \arrow[r] \arrow[d] & H_{J'} \arrow[d] \\
\underset{(M/M_k)}{\mathrm{Tor}} & \underset{J_b}{\mathrm{Tor}^{\subset}} \arrow[l, "\mathrm{big}"'] \arrow[r, "(\mathrm{sml},\, \mathrm{quot})"] & \underset{T_s(M'/M_k)\quad (J/J')}{\mathrm{Tor} \times \mathrm{Tor}}
\end{tikzcd}$$

$$H_J \longrightarrow H_{J'} \times_{(\mathrm{Tor} \times \mathrm{Tor})} \mathrm{Tor}^{\subset}$$

is a fibration fibered in affine space

$$\mathrm{Ext}^1(J_b, M_k) \times_{\mathrm{Ext}^1(T_s, M_k)} \{0 \to M_k \to M' \to T_s \to 0\}$$

$\Rightarrow$ base-change up to [dim]

$$\Rightarrow \quad \Gamma_c\left((s,q)_! \circ b^* \mathcal{L}_E \Big|_{H_{J'}} \right) = 0$$

Prop: $(s,q)_! \circ b^* \mathcal{L}_E \simeq \mathcal{L}_E \otimes \mathcal{L}_E$

(Next time)

$\rightsquigarrow$ reduce to the surjective case

$\square$