

Laumon III

Last time, we were showing the Vanishing Theorem implies the Clean Theorem. There was one unfinished claim:

Lemma: For fixed $M_k \in \text{Bun}_k$, $\mathcal{J} \in \text{Tor}^l$, $\hat{M} \in \text{Bun}_k, M_k \rightarrow \mathcal{J}$

such that $\hat{M} \oplus \mathcal{J} \in \mathcal{C}_k$, consider the stacks

$$\left\{ \begin{array}{c} \begin{array}{c} M_k \\ \downarrow \\ 0 \rightarrow \hat{M} \rightarrow M \rightarrow \mathcal{J} \rightarrow 0 \end{array} \\ \downarrow \\ M/M_k \end{array} \mid M \in \mathcal{C}_k \right\} =: H$$

$$\downarrow$$

$$\in \text{Tor}.$$

Then:

$$\Gamma_!(\mathcal{L}_E|_H) = 0.$$

In this lecture we will deduce this lemma from a result about Laumon's sheaf: $\mathcal{L}_E \in \text{Perf}(\text{Tor})$.

Remark: First, there is a technical point we did not discuss last time: we will put one more assumption on the good open \mathcal{C}_k , such that

$$\hat{M} \oplus \mathcal{J} \in \mathcal{C}_k \Rightarrow M \in \mathcal{C}_k.$$

This can be guaranteed by the following:

TODO: Find a line bundle \mathcal{L}^{est} , and define

$$\mathcal{C}_k = \{ M \in \text{Coh}_k \mid \text{Hom}(M, \mathcal{L}^{\text{est}}) = 0 \}.$$

$$(\hat{M} \oplus \mathcal{J} \in \mathcal{C}_k \Leftrightarrow \hat{M} \in \mathcal{C}_k \Rightarrow M \in \mathcal{C}_k).$$

This allows us to ignore \mathcal{L}_k in the proof of the lemma.

Proof of the Lemma: Last time we explained how to deduce the case $M_k \rightarrow J$ from the

vanishing theorem. Namely, in this case:

$$\begin{array}{ccccccc} 0 & \rightarrow & M_k^\circ & \rightarrow & M_k & \rightarrow & J \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \tilde{M} & \rightarrow & M & \rightarrow & J \rightarrow 0 \end{array}$$

$$\Leftrightarrow M_k^\circ \xrightarrow{\quad} \tilde{M}$$

It follows:

$$H = \left(\prod_{B_{M_k}} \mathcal{E}_{M_k} \right) \times \text{Mod}_k \times \left(\prod_{B_{\tilde{M}}} \mathcal{E}_{\tilde{M}} \right)$$

$$\left(B_{M_k} \leftarrow \text{Mod}_k \rightarrow B_{\tilde{M}} \right)$$

$$\Gamma(\mathcal{L}_{\tilde{M}}) = \text{AV}_{\mathbb{Z}} \left(\mathcal{S}_{M_k} \right) \Big|_{\tilde{M}}^* \otimes \text{line}$$

is (Thm V).
0

Here we can apply the vanishing theorem because

$$\deg(\tilde{M}) - \deg(M_k) > nk(z-2) + l.$$

In general, write

$$\begin{array}{ccc} M_k & \longrightarrow & J \\ & \searrow & \nearrow \\ & J' & \end{array}$$

Then:

$$H = \left\{ \begin{array}{ccccccc} 0 & \rightarrow & M_k^0 & \rightarrow & M_k & \rightarrow & J' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & M^0 & \rightarrow & M & \rightarrow & J' \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & M & \rightarrow & M & \rightarrow & J \rightarrow 0 \end{array} \right\}$$

$$I := \left\{ \begin{array}{ccccccc} 0 & \rightarrow & M_k^0 & \rightarrow & M_k & \rightarrow & J' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & M & \rightarrow & M' & \rightarrow & J' \rightarrow 0 \end{array} \right\}$$

$\text{Tor}^{\text{ext}} := \{ \text{short exact seq. in Tor} \}$

$$\text{Tor} \xleftarrow{h} \text{Tor}^{\text{ext}} \xrightarrow{i} \text{Tor}$$

$$\downarrow e$$

$$\text{Tor}$$

$$\begin{array}{ccc} H & \longrightarrow & H' \\ \downarrow & & \downarrow \\ \text{Tor} \xleftarrow{e} \text{Tor}^{\text{ext}} & \xrightarrow{(h,i)} & \text{Tor} \times \text{Tor} \end{array}$$

where

$$H \rightarrow \text{Tor}^{\text{ext}} \quad \begin{array}{c} M/M_k^0 \\ \parallel \\ M/M_k \end{array} \quad \begin{array}{c} J/J' \\ \parallel \\ J \end{array}$$

$$\dots \rightarrow 0 \rightarrow M'/M_k \rightarrow M/M_k \rightarrow M/M' \rightarrow 0.$$

The above square is not Cartesian, but we still have base-change isomorphism for it. This is because

$$H \longrightarrow \text{Tor}^{\text{ext}} \times_{(\text{Tor} * \text{Tor})} H'$$

is an affine bundle, with fiber at

$$0 \rightarrow J_k \rightarrow J_e \rightarrow J_2 \rightarrow 0, \quad 0 \rightarrow M_k \hookrightarrow M \rightarrow J_k \rightarrow 0$$

given by the affine spaces

$$0 \rightarrow M_k \xrightarrow{r} M' \rightarrow J_k \rightarrow 0$$

$$\text{Ext}^1(J_e, M_k) \times_{\text{Ext}^1(J_k, M_k)} \text{pt}$$

Note that the dimension of the above affine space is constant on each connected component of the base.

$$\begin{array}{ccc} \bullet & \xrightarrow{! \text{-push}} & \\ \uparrow * \text{-pull} & = & \uparrow * \text{-pull} \\ \bullet & & \xrightarrow{! \text{-push}} \end{array} \quad [-2 \text{dim}]$$

It follows that we only need to show

$$\underbrace{(k, \xi), e^* \mathcal{L}_E}_{\cap} \Big|_{H'} \xrightarrow{P} 0$$

$D(\text{Tor} * \text{Tor})$

By the surjective case treated before, we only need to prove the following result, which is the main goal of this lecture.

□

Prop: If E is geometrically irreducible, then:

← Laumon started with this assumption, I don't think it is necessary.

$$(h, \xi)_! \circ e^* \mathcal{L}_E = \mathcal{L}_E \boxtimes \mathcal{L}_E$$

$$\text{Tor} \xleftarrow{e} \text{Tor}^{\text{ext}} \xrightarrow{(h, \xi)} \text{Tor} \times \text{Tor}$$

Proof of the Prop: Recall the regular semi-simple locus:

$$\rightarrow \mathcal{O}_{x_1 + \dots + x_d | x_i + x_j} = \text{Tor}^{\text{rss}, (d)} \subset \text{Tor}^d$$

$$\downarrow \quad \downarrow$$

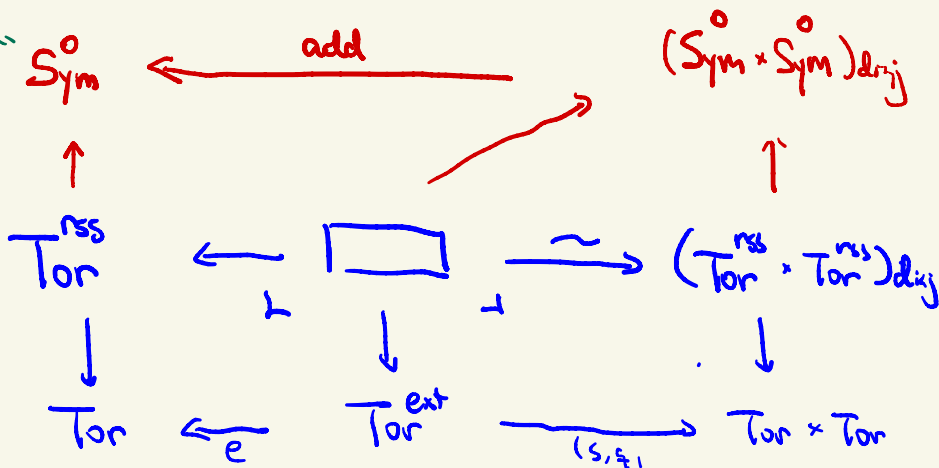
$$X^{(d)}\text{-ding} \subset X^{(d)}$$

← local system

$$\mathcal{L}_E^d = \text{IC} \left(\mathcal{E}^{(d)} \Big|_{\text{Tor}^{\text{rss}, (d)} [d]} \right)$$

$$(\mathcal{E}^{(d)} = r_! (\mathcal{E}^d)^{\text{St}}, \quad r: X^d \rightarrow X^{(d)})$$

$\frac{1}{d} (X^{(d)}\text{-ding})$



⇒

$$\text{LHS} \Big|_{r_{ss}, d_{ij}} \cong \underbrace{(\text{add}^* \text{Sym}(E))}_{\text{is}} \Big|_{r_{ss}, d_{ij}}$$

$$\text{RHS} \Big|_{r_{ss}, d_{ij}} \cong \underbrace{(\text{Sym}(E) \otimes \text{Sym}(E))}_{\text{is}} \Big|_{r_{ss}, d_{ij}}$$

Note that

$$\text{RHS} \cong \text{IC}(\text{RHS} \Big|_{r_{ss}, d_{ij}}).$$

Hence we only need

$$\text{LHS} = \text{IC}(\text{LHS} \Big|_{r_{ss}, d_{ij}}).$$

Recall

$$\mathcal{L}_E^d = (\text{Spr}_E^d) \text{Sd} \oplus \dots$$

is
 Spr_E^d

Hence we only need to show

$$(h, g): e^*(\text{Spr}_E^d) = \text{IC}(\dots \Big|_{r_{ss}, d_{ij}}).$$

The question is local for the étale topology and we can reduce to the case $X = \mathbb{A}^1$,

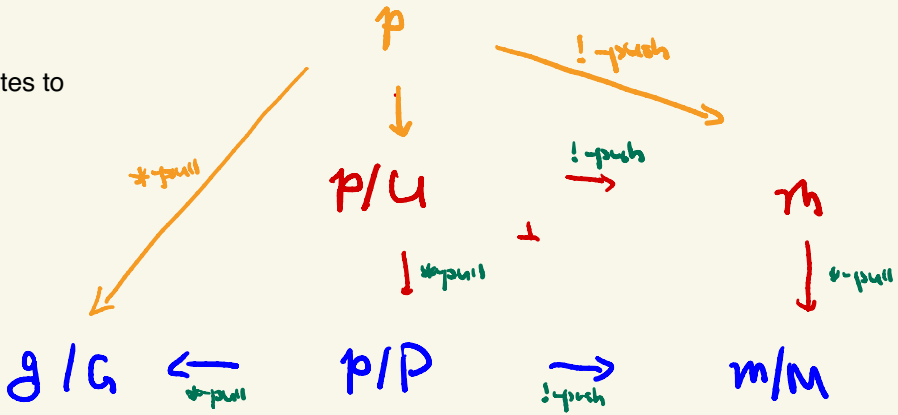
$$E = (\overline{\mathbb{Q}}_l)^{\otimes n} \quad \text{and further to the case} \quad E = \overline{\mathbb{Q}}_l.$$

Now we restrict to the connected component

$$(d = d_1 + d_2)$$

$$\text{Tor}^d \longleftarrow \text{Tor}^{\text{ext}, d_1, d_2} \longrightarrow \text{Tor}^{d_1} * \text{Tor}^{d_2}.$$

It translates to



$$P_{d_1, d_2} = \begin{pmatrix} \boxed{GL_{d_1}} & \boxed{U_{d_1, d_2}} \\ 0 & \boxed{GL_{d_2}} \end{pmatrix}$$

$$G := GL_n$$

$$P := P_{d_1, d_2} = U \rtimes M$$

$$M := GL_{d_1} \times GL_{d_2}$$

By base-change, we need to show

$$\text{Per}(g/G) \xrightarrow{* \text{-pull}} D(p) \xrightarrow{! \text{-push}} D(m)$$

$$\begin{array}{ccc} \Downarrow & & \Downarrow \\ \text{Spr} & \xrightarrow{\quad \quad \quad} & \text{IC} \left(\begin{array}{c} | \\ \text{rss, disj} \end{array} \right) \text{ [shift]} \\ & & \text{(eig. val. disj.)} \end{array}$$

Now

$$\begin{array}{ccccccc} \tilde{g}/G & \leftarrow & \tilde{g} & \leftarrow & \tilde{p} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ g/G & \leftarrow & g & \leftarrow & p & \rightarrow & m \end{array}$$

Need $! \text{-push}(G_B)$ to be a shifted IC-extension.

$\text{Ad}_g(b)$

$$\tilde{p} = \tilde{g} \{ (x, g) \mid x \in p \cap b^g, g \in G/B \}$$

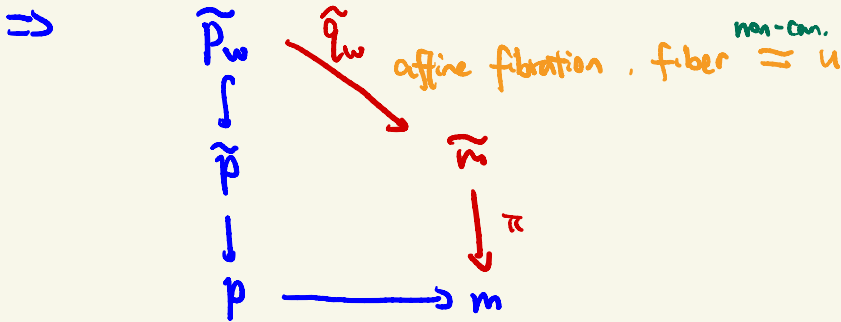
$$G = \bigcup_{w \in W_m \setminus W_G} P_w B \quad (W_m \setminus W_G = (S_{d_1} \times S_{d_2}) \setminus S_{d_1 d_2})$$

$$\Rightarrow \tilde{\mathcal{P}} = U \tilde{\mathcal{P}}_w .$$

$$\begin{aligned} \tilde{\mathcal{P}}_w &= \{ (x, g) \mid x \in P \cap B^d, g \in P \cap B/B \} \\ &= \{ (x, g) \mid x^{h^i} \in P \cap B^w, h \in P / P \cap B^w \} \end{aligned} \quad (g = hw)$$

We can choose each w such that it has minimal possible length. This implies:

$$P \cap B^w = (M \cap B) \cdot (U \cap B^w)$$



Now the desired complex has a filtration whose graded pieces are

$$\pi_! \tilde{\mathcal{G}}_{w,1}(\bar{\mathcal{O}}_u) \cong \pi_!(\bar{\mathcal{O}}_u)[-2 \dim u](-\dim u)$$

Each piece is a pure perverse sheaf with weight independent of w . Hence so is the desired complex. Now we win by

Thm (Deligne): Any pure perverse sheaf is (geometrically) semi-simple.

Namely the theorem implies the above filtration splits.

□