Laumon III

Last time, we were showing the Vanishing Theorem implies the Clean Theorem. There was one unfinished claim:

Lemma: For fixed $M_k \in \mathrm{Bun}_k$, $J \in \mathrm{Tor}^d$, $\mathring{M} \in \mathrm{Bun}_k$, $M_k \to J$

such that $\mathring{M} \oplus J \in C_k$, consider the stacks

$$\begin{tikzcd} & & M_k \arrow[d, hook] \arrow[dr] & & \\ 0 \arrow[r] & \mathring{M} \arrow[r] & M \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

$$\left\{ 0 \to \mathring{M} \to M \to J \to 0 \;\middle|\; M \in C_k \right\} =: H$$

$$\mapsto M/M_k \qquad\qquad H \to \; \in \mathrm{Tor}.$$

Then:

$$\Gamma_!\left(\mathcal{L}_E|_H\right) \simeq 0.$$

In this lecture we will deduce this lemma from a result about Laumon's sheaf: $\mathcal{L}_E \in \mathrm{Perv}(\mathrm{Tor})$.

Remark: First, there is a technical point we did not discuss last time: we will put one more assumption on the good open $\mathcal{C}_k$, such that

$$\mathring{M} \oplus J \in \mathcal{C}_k \Rightarrow M \in \mathcal{C}_k.$$

This can be guaranteed by the following:

TODO: Find a line bundle $\mathcal{L}^{est}$, and define

$$\mathcal{C}_k = \{ M \in \mathrm{Coh}_k \mid \mathrm{Hom}(M, \mathcal{L}^{est}) = 0 \}.$$

$$\left(\mathring{M} \oplus J \in \mathcal{C}_k \Leftrightarrow \mathring{M} \in \mathcal{C}_k \Rightarrow M \in \mathcal{C}_k \right).$$

This allows us to ignore ℓ_k in the proof of the lemma.

Proof of the Lemma: Last time we explained how to deduce the case $M_k \twoheadrightarrow J$ from the vanishing theorem. Namely, in this case:

$$\begin{tikzcd} 0 \arrow[r] & \mathring{M}_k \arrow[r] \arrow[d, hook] & M_k \arrow[r] \arrow[d, hook] & J \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathring{M} \arrow[r] & M \arrow[r] & J \arrow[r] & 0 \end{tikzcd}$$

$$\Longleftrightarrow \qquad \mathring{M}_k \hookrightarrow \mathring{M}$$

It follows:

$$H = \{\mathring{M}_k\} \underset{\mathrm{Bun}_k}{\times} \mathrm{Mod}_k \underset{\mathrm{Bun}_k}{\times} \{\mathring{M}\}$$

$$(\ \mathrm{Bun}_k \longleftarrow \mathrm{Mod}_k \longrightarrow \mathrm{Bun}_k\ )$$

$$\Gamma_!(\mathcal{L}_E|_H) = \mathrm{Av}_E(\delta_{\mathring{M}_k})\Big|^*_{\mathring{M}} \otimes \text{line}$$
$$\wr\!\!\wr \quad (\text{Thm V}).$$
$$0$$

Here we can apply the vanishing theorem because $\deg(\mathring{M}) - \deg(\mathring{M}_k) > nk(2g-2)+\ell$.

In general, write

$$\begin{tikzcd} M_k \arrow[rr] \arrow[dr, two heads] & & J \\ & J' \arrow[ur, hook] & \end{tikzcd}$$

Then:

$$
H = \left\{
\begin{tikzcd}
0 \arrow[r] & \mathring{M}_k \arrow[r] \arrow[d, hook] & M_k \arrow[r] \arrow[d, hook] & J' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathring{M} \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d, hook] & J' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathring{M} \arrow[r] & M \arrow[r] & J \arrow[r] & 0
\end{tikzcd}
\right\}
$$

$$
H' := \left\{
\begin{tikzcd}
0 \arrow[r] & \mathring{M}_k \arrow[r] \arrow[d, hook] & M_k \arrow[r] \arrow[d, hook] & J' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathring{M} \arrow[r] & M' \arrow[r] & J' \arrow[r] & 0
\end{tikzcd}
\right\}
$$

$$
\mathrm{Tor}^{ext} := \{ \text{short exact seq. in } \mathrm{Tor} \}
$$

$$
\begin{tikzcd}
\mathrm{Tor} & \mathrm{Tor}^{ext} \arrow[l, "k"'] \arrow[r, "q"] \arrow[d, "e"] & \mathrm{Tor} \\
& \mathrm{Tor} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& H \arrow[r] \arrow[d] \arrow[dl] & H' \arrow[d] \\
\mathrm{Tor} & \mathrm{Tor}^{ext} \arrow[l, "e"'] \arrow[r, "{(k,q)}"] & \mathrm{Tor} \times \mathrm{Tor}
\end{tikzcd}
$$

where

$$
\begin{tikzcd}
H \arrow[r] & \mathrm{Tor}^{ext} & & \mathring{M}/\mathring{M}_k \arrow[d, phantom, "\cong" rotate=90] & & J/J' \arrow[d, phantom, "\cong" rotate=90] & \\
\cdots \arrow[r, mapsto] & 0 \arrow[r] & M'/M_k \arrow[rr, phantom] & & M/M_k \arrow[r] & M/M' \arrow[r] & 0 .
\end{tikzcd}
$$

$$
\cdots \longmapsto 0 \to M'/M_k \to M/M_k \to M/M' \to 0 , \quad M'/M_k \cong \mathring{M}/\mathring{M}_k, \quad M/M' \cong J/J'
$$

The above square is not Cartesian, but we still have base-change isomorphism for it. This is because

$$H \longrightarrow \mathrm{Tor}^{ext} \times_{(\mathrm{Tor} \times \mathrm{Tor})} H'$$

is a affine bundle, with fiber at

$$0 \to J_k \to J_e \to J_q \to 0, \quad 0 \to \mathring{M}_k \hookrightarrow \mathring{M} \to J_k \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & \mathring{M}_k \arrow[r, hook] \arrow[d] & \mathring{M} \arrow[r] \arrow[d] & J_k \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M_k \arrow[r, hook] & M' \arrow[r] & J_k \arrow[r] & 0 \end{tikzcd}$$

given by the affine spaces

$$\begin{tikzcd} \mathrm{Ext}^1(J_e, M_k) \arrow[dr] & \times & pt \arrow[dl] \\ & \mathrm{Ext}^1(J_k, M_k) & \end{tikzcd}.$$

Note that the dimension of the above affine space is constant on each connected component of the base.

$$\begin{tikzcd} \bullet \arrow[r, "! \text{-push}"] & \\ \bullet \arrow[u, "* \text{-pull}"] & \end{tikzcd} = \begin{tikzcd} & \\ \arrow[r, "! \text{-push}"'] & \arrow[u, "* \text{-pull}"'] \end{tikzcd} [-2\dim]$$

It follows that we only need to show

$$\underbrace{(k, q)_! \, e^* \mathcal{L}_E}_{\in D(\mathrm{Tor} \times \mathrm{Tor})} \Big|_{H'} \overset{?}{\longmapsto} 0 .$$

By the surjective case treated before, we only need to prove the following result, which is the main goal of this lecture.

$\square$

Laumon started with this assumption. I don't think it is necessary.

Prop: If E is geometrically irreucible, then:

$$(k,q)_! \circ e^* \mathcal{L}_E = \mathcal{L}_E \boxtimes \mathcal{L}_E$$

$$\mathrm{Tor} \xleftarrow{e} \mathrm{Tor}^{ext} \xrightarrow{(k,q)} \mathrm{Tor} \times \mathrm{Tor}$$

Proof of the Prop: Recall the regular semi-simple locus:

$$\begin{tikzcd}
\{\mathcal{O}_{x_1+\cdots+x_d} \mid x_i \neq x_j\} = \mathrm{Tor}^{rss,(d)} \arrow[r, phantom, "\subset"] \arrow[d] & \mathrm{Tor}^d \arrow[d] \\
X^{(d)}_diag \arrow[r, phantom, "\subset"] & X^{(d)}
\end{tikzcd}$$

local system

$$\mathcal{L}_E^d = IC\left(E^{(d)}|_{\mathrm{Tor}^{rss,(d)}} [d] \right)$$

$$\left(E^{(d)} = r_!(E^d)^{S_d}, \quad r: X^d \to X^{(d)} \right)$$

$$\begin{tikzcd}
\coprod_d (X^{(d)}_diag) = \overset{\circ}{\mathrm{Sym}} & & (\overset{\circ}{\mathrm{Sym}} \times \overset{\circ}{\mathrm{Sym}})_{disj} \arrow[ll, "add"'] \\
\mathrm{Tor}^{rss} \arrow[u] \arrow[d] & \square \arrow[l] \arrow[r, "\sim"] \arrow[d] \arrow[ur] & (\mathrm{Tor}^{rss} \times \mathrm{Tor}^{rss})_{disj} \arrow[u] \arrow[d] \\
\mathrm{Tor} & \mathrm{Tor}^{ext} \arrow[l, "e"] \arrow[r, "(s,q)"'] & \mathrm{Tor} \times \mathrm{Tor}
\end{tikzcd}$$

$\Rightarrow$

$$\mathrm{LHS}|_{rss,disj} \simeq (\mathrm{add}^* \, \mathrm{Sym}(E))|_{rss,disj}$$

$$\wr$$

$$\mathrm{RHS}|_{rss,disj} \simeq (\mathrm{Sym}(E) \boxtimes \mathrm{Sym}(E))|_{rss,disj}$$

Note that

$$\mathrm{RHS} \simeq \mathrm{IC}(\mathrm{RHS}|_{rss,disj}).$$

Hence we only need

$$\mathrm{LHS} \simeq \mathrm{IC}(\mathrm{LHS}|_{rss,disj}).$$

Recall

$$\mathcal{L}_E^d \simeq (\mathrm{Spr}_E^d)^{S_d} \oplus \cdots$$

$$\wr$$

$$\mathrm{Spr}^d$$

Hence we only need to show

$$(h, q)_! \, e^*(\mathrm{Spr}_E) \simeq \mathrm{IC}(\cdots|_{rss,disj}).$$

The question is local for the étale topology and we can reduce to the case $X = \mathbb{A}^1$,

$E = (\overline{\mathbb{Q}}_\ell)^{\oplus n}$ and further to the case $E = \overline{\mathbb{Q}}_\ell$.

Now we restrict to the connected component $(d = d_1 + d_2)$

$$\mathrm{Tor}^d \longleftarrow \mathrm{Tor}^{ext, d_1, d_2} \longrightarrow \mathrm{Tor}^{d_1} \times \mathrm{Tor}^{d_2}.$$

It translates to

$$\begin{tikzcd}
 & \mathfrak{p} \arrow[d] \arrow[dl, "*\text{-pull}"'] \arrow[dr, "!\text{-push}"] & \\
 & \mathfrak{p}/U \arrow[r, "!\text{-push}"] \arrow[d, "*\text{-pull}"] \arrow[dr, phantom, "\perp"] & \mathfrak{m} \arrow[d, "*\text{-pull}"] \\
\mathfrak{g}/G & \mathfrak{p}/P \arrow[l, "*\text{-pull}"'] \arrow[r, "!\text{-push}"'] & \mathfrak{m}/M
\end{tikzcd}$$

$$P_{d_1,d_2} = \begin{pmatrix} \boxed{GL_{d_1}} & \boxed{U_{d_1,d_2}} \\ 0 & \boxed{GL_{d_2}} \end{pmatrix}$$

$G := GL_d$

$P := P_{d_1,d_2} \simeq U \rtimes M$

$M := GL_{d_1} \times GL_{d_2}$

By base-change, we need to show

$$\begin{array}{ccc}
\mathrm{Perv}(\mathfrak{g}/G) \xrightarrow{*\text{-pull}} D(\mathfrak{p}) & \xrightarrow{!\text{-push}} & D(\mathfrak{m}) \\
\cup & & \cup \\
\mathrm{Spr} & \longmapsto & IC\left(\Big|_{\substack{rss,disj}} \right) [\text{shift}] \ (\text{eig. val. disj.})
\end{array}$$

Now

$$\begin{tikzcd}
\tilde{\mathfrak{g}}/G \arrow[d] & \tilde{\mathfrak{g}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & \tilde{\mathfrak{p}} \arrow[l] \arrow[d] \arrow[drr] \arrow[dl, phantom, "\lrcorner"] & & \\
\mathfrak{g}/G & \mathfrak{g} \arrow[l] & \mathfrak{p} \arrow[l] \arrow[rr] & & \mathfrak{m}
\end{tikzcd}$$

Need $!\text{-push}(\overline{\mathbb{Q}}_\ell)$ to be a shifted IC-extension.

$$\tilde{\mathfrak{p}} = \{ (x, g) \mid x \in \mathfrak{p} \cap \underbrace{\mathfrak{b}^g}_{Ad_g(\mathfrak{b})},\ g \in G/B \}$$

$$G = \bigcup_{w \in W_M \backslash W_G} PwB \quad (W_M \backslash W_G = (S_{d_1} \times S_{d_2}) \backslash S_{d_1+d_2})$$

$$\Rightarrow \quad \tilde{\mathfrak{p}} = \bigcup \tilde{\mathfrak{p}}_w .$$

$$\tilde{\mathfrak{p}}_w = \{ (x, g) \mid x \in \mathfrak{p} \cap \mathfrak{b}^g ,\ g \in PwB/B \} \qquad (g = hw)$$

$$= \{ (x, g) \mid x^{h^{-1}} \in \mathfrak{p} \cap \mathfrak{b}^w ,\ h \in P / P \cap B^w \}$$

We can choose each w such that it has minimal possible length. This implies:

$$P \cap B^w = (M \cap B) \cdot (U \cap B^w)$$

$$\Rightarrow \quad \begin{tikzcd}
\tilde{\mathfrak{p}}_w \arrow[d, hook] \arrow[dr, "\tilde{q}_w"] & \\
\tilde{\mathfrak{p}} \arrow[d] & \tilde{\mathfrak{m}} \arrow[d, "\pi"] \\
\mathfrak{p} \arrow[r] & \mathfrak{m}
\end{tikzcd}$$

affine fibration, fiber $\overset{\text{non-can.}}{\simeq} \mathfrak{u}$

Now the desired complex has a filtration whose graded pieces are

$$\pi_! \, \tilde{q}_{w,!} (\overline{\mathbb{Q}}_\ell) \simeq \pi_! (\overline{\mathbb{Q}}_\ell) [-2 \dim \mathfrak{u}] (-\dim \mathfrak{u})$$

Each piece is a pure perverse sheaf with weight independent of w. Hence so is the desired complex. Now we win by

Thm (Deligne): Any pure perverse sheaf is (geometrically) semi-simple.

Namely the theorem implies the above filtration splits.

$\square$