Lecture 12: From Aut'_E to Aut_E

<u>Previous lectures</u>

Have
$$\begin{array}{ccc} & Gr_n^{reg} \tilde{\times} Bun_N \xrightarrow{\ \pi\ } Bun'_n \subset Coh'_n. & \\ \swarrow & & \searrow^{ev} \\ \mathrm{Tor} & & \mathbb{G}_a \end{array}$$

Constructed $\mathrm{Aut}'_E \in D(Coh'_n)$.

<u>Thm</u> On $C'_n \subset Bun'_n$ a "good open",
$\mathrm{Aut}'_E|_{C'_n}$ is perverse & irred.

<u>Thm</u> (Descent) If $C_n \cap Bun_n^d \neq \emptyset$, then
$\mathrm{Aut}'_E|_{C'_n \cap (Bun_n^d)'}$ descends to $\mathrm{Aut}_E|_{C_n \cap Bun_n^d}$.
(Can consider $\rho : C'_n \cap Bun'_n \to C_n \cap Bun_n$ smooth.)

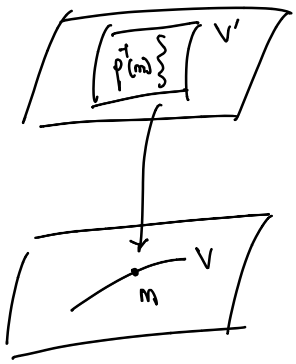
<u>Note</u> $\mathrm{Aut}_E|_{C_n \cap Bun_n^d} \neq 0$ is automatically perv & irred.

<u>Thm</u> (Hecke) $\mathrm{Aut}_E|_{C_n \cap Bun_n}$ has a unique ext'n to a Hecke eigensheaf Aut_E.

<u>Thm</u> (Cuspidal) Aut_E is cuspidal.
$\uparrow$ vanishing thm.

<u>Thm</u> [FGKV] If there is a "cuspidal Hecke-eigenform" on $Bun_n(\mathbb{F}_q)$,
then its restriction on $Bun'_n(\mathbb{F}_q)$ is equal to $\mathrm{tr}(\mathrm{Frob}; \mathrm{Aut}'_E)$.

Descent picture:



$Cn \cap Bun_n'$

$\downarrow p$

$Cn \cap Bun_n$

- $Aut'_E|_{Cn \cap (Bun_n^d)'}$ $= IC(\text{loc sys on } V')$, V' smooth $\rightsquigarrow$ V smooth (up to shrink)
- Aim to descend from proj bundle.

Want (i) $V' \cap p^{-1}(m) \hookrightarrow p^{-1}(m)$ open & dense.

($\rightsquigarrow$ insert this as an assumption.)

Can always descend loc sys along $p^{-1}(V)$.

(ii) $Aut'_E|_{p^{-1}(V)}$ is a loc sys (as oppose to perv sheaf).

Granting (i) (ii), get Thm D by $\pi_1(\mathbb{P}^m) = pt$.

<u>Motivated lemma</u>: If Y sm stack & $K \in Perv(Y)$ irred then

K is a loc sys $\Leftrightarrow$ Euler char of K at any geom pt on $Y(\mathbb{F}_q)$ is a const func.

Prop $\chi(Aut'_E)$ = const on the fibers of $Bun_n' \to Bun_n$ & is nonzero over $Bun_n \cap Cn$.

Prop + Lem $\Rightarrow$ χ const on $p^{-1}(m)$, $Aut'_E|_{V'}$ loc sys $\Rightarrow$ χ same const on V'.

$\Rightarrow$ χ const on $p^{-1}(V)$

$\overset{\text{Lem}}{\Rightarrow}$ (ii) (with (i) supposed.)

Thm (Deligne, applied as a blackbox)

If $f: Y_1 \to Y_2$ proper + $K_1, K_2 \in D(Y_1)$ étale locally iso,
then $\chi(f_!(K_1)) = \chi(f_!(K_2))$.

Application in proving Prop

For E_1, E_2 (even reducible) $\mathcal{L}_{E_1} \overset{\text{locally}}{\simeq} \mathcal{L}_{E_2}$

$\Rightarrow$ If π was proper then $\chi(\mathrm{Aut}'_{E_1}) = \chi(\mathrm{Aut}'_{E_2})$.

(π can be compactified using Drinfeld compactification.)

In order to prove Prop for E,
it's enough to show it for some E'.

$\rightsquigarrow$ Can assume $\exists$ cuspidal Hecke eigenfunctor for E

$\overset{\text{[FGKV]}}{\rightsquigarrow}$ $\mathrm{tr}(\mathrm{Frob}, \mathrm{Aut}'_E)$ is constructed along fibers of ρ.

$\rightsquigarrow$ const χ.

Def Hecke-Laumon stack:

$$\mathrm{Coh}_k \xleftarrow{\;\overleftarrow{h}\;} \mathcal{HL}_k^d \xrightarrow{\;\overrightarrow{h}\;} \mathrm{Tor}^d \times \mathrm{Coh}_k$$

$$M \longleftarrow\!\shortmid (0 \to M' \to M \to T \to 0) \longmapsto (T, M').$$

Hecke-Laumon functor:

$$\mathrm{HL}_k^d : D(\mathrm{Coh}_k) \longrightarrow D(\mathrm{Tor}^d \times \mathrm{Coh}_k)$$

by $\mathrm{HL}_k^d := \overrightarrow{h}_! \overleftarrow{h}^* (-)[\text{shift}](\text{twist})$.

Say $K \in \mathrm{Perv}(\mathrm{Coh}_k)$ is a Hecke-Laumon eigensheaf
if (1) $\mathrm{HL}_k^d(K) \simeq \mathcal{L}_E^d \boxtimes K$.

$$
\begin{array}{ccc}
(\mathrm{Id}\otimes HL_k^{d_2})\circ HL_k^{d_1}(K) & \xrightarrow{\ \alpha\ } & (\mathrm{Id}\otimes HL_k^{d_2})\circ(\mathcal{L}_E^{d_1}\boxtimes K) \\
 & & \downarrow \simeq \\
\downarrow \simeq & \circlearrowright & \mathcal{L}_E^{d_1}\boxtimes\mathcal{L}_E^{d_2}\boxtimes K \\
 & & \uparrow \simeq\ \alpha\ (\text{Lau III}) \\
(HL_0^{d_1}\otimes \mathrm{Id})\circ HL_k^{d}(K) & \xrightarrow{\ \alpha\ } & (HL_0^{d_1}\otimes \mathrm{Id})(\mathcal{L}_E^{d}\boxtimes K)
\end{array}
$$

(2)

Explaination of (2):

(a)

$$
\mathrm{Tor}^{ext,d_1,d_2} \longleftarrow \bullet \quad \left\{\begin{array}{l} 0\to M'\to M\to T'\to 0 \\ 0\to M''\to M'\to T''\to 0\end{array}\right\}
$$

$$
\begin{array}{ccccccc}
 & & \swarrow & \bullet & \searrow & & \\
 & \mathcal{H}\mathcal{L}_k^{d_1} & & & & \mathrm{Tor}^{d_1}\times \mathcal{H}\mathcal{L}_k^{d_2} & \\
\overleftarrow{h}\swarrow & & \searrow\overrightarrow{h} & & \overleftarrow{h}\swarrow & & \searrow\overrightarrow{h} \\
\mathrm{Coh}_k & & \mathrm{Tor}^{d_1}\times\mathrm{Coh}_k & & & & \mathrm{Tor}^{d_1}\times\mathrm{Tor}^{d_2}\times\mathrm{Coh}_k. \\
\Downarrow & & \Downarrow & & & & \Downarrow \\
M & & (T,M) & & & & (T,T',M'')
\end{array}
$$

$$\Rightarrow (\mathrm{Id}\otimes HL_k^{d_2})\circ HL_k^{d_1} \simeq ((k,q)_!\, e^*\otimes \mathrm{Id})\, HL_k^{!}$$

$$\simeq (HL_0^{d_1}\times \mathrm{Id})\, HL_k^{d} \qquad (d=d_1+d_2)$$

$$\left(\begin{array}{c} \underline{\text{Recall}}\quad \mathrm{Tor}\xleftarrow{\ e\ }\mathrm{Tor}^{ext}\xrightarrow{(k,q)}\mathrm{Tor}\times\mathrm{Tor} \\ \Rightarrow (k,q)_!\, e^*\simeq HL_0^{d_1}. \end{array}\right)$$

(b) __Prop__ (Lau III) $\mathcal{L}_E\in \mathrm{Perv}(\mathrm{Coh}_0)$ is an HL-eigensheaf.

__Rmk__ Can also define HL-eigensheaf for Coh_k', $\mathcal{C}_k'$, $\bar{\mathcal{C}}_k'$, etc.

__Claim__ Using induction, can show $\mathcal{F}_k\in\mathrm{Perv}(\mathcal{C}_k')$ is an HL-eigensheaf.

$\Downarrow$ $\mathrm{Aut}_E|_{\mathcal{C}_n'\cap \mathrm{Bun}_n}$ satisfies HL-eigen property.

__Question__ HL-eigen $\overset{?}{\rightsquigarrow}$ Hecke-eigen.

Prop If $S \in \mathrm{Perv}(\mathcal{C}_n \cap \mathrm{Bun}_n)$ satisfies HL for E

DS Verdier dual satisfies HL for E^*

then S satisfies Hecke-eigen property for E.

Prop $\Rightarrow$ $\mathrm{Aut}_E|_{\mathcal{C}_n \cap \mathrm{Bun}_n}$ satisfies "Hecke^n", i.e.

$$\mathcal{C}_n \cap \mathrm{Bun}_n \xrightarrow{h} \mathcal{C}_n \cap \mathrm{Bun}_n$$

$$M \longmapsto M(-x)$$

with $h^*(\mathrm{Aut}_E|_{\mathcal{C}_n \cap \mathrm{Bun}_n}) \simeq \mathrm{Aut}_E|_{\mathcal{C}_n \cap \mathrm{Bun}_n} \otimes \Lambda^n E_x$.

Step 1 (1) $!*$-extend $\mathrm{Aut}_E|_{\mathcal{C}_n \cap \mathrm{Bun}_n^d}$ to $\mathrm{Aut}_E|_{\mathrm{Bun}_n^d}$.

(2) Uniquely extend to Aut_E using Hecke^n.

(3) Show Aut_E satisfies Hecke^1 by using compatibility of Hecke^1, Hecke^n on $\mathcal{C}_n$.

Proof of Prop By lecture 3, only need to show S satisfies Hecke^1.

$$X \longrightarrow \mathrm{Tor}^1, \qquad \mathrm{Coh}_n \longleftarrow \mathcal{HL}_n^1 \longrightarrow \mathrm{Coh}_n$$

$$\mathcal{HL}_n^1 \longrightarrow \mathrm{Tor}^1.$$

Consider $\mathring{\mathcal{HL}}_n^1 \subset \mathcal{HL}_n^1$ by removing split SES.

$$\mathcal{HL}_n^1 = \{0 \to M' \to M \to T \to 0 : \deg T = 1\}.$$

$$\mathring{\mathcal{HL}}_1 \times_{\mathrm{Tor}^1} X = \{(x, 0 \to M' \to M) : x \in X,\ \mathrm{supp}\, M/M' = x\}.$$

$$\mathcal{H}_n^1 = \{M' \subset M \subset M'(x) : \deg M/M' = 1\}$$

$$\rightsquigarrow \mathring{\mathcal{HL}}_1 / \mathbb{G}_m \times_{\mathrm{Tor}^1} X = \mathcal{H}_n^1.$$

Only need to show $\mathcal{H}_n^1(\mathrm{Aut}_E|_{\mathcal{C}_n \cap \mathrm{Bun}_n})$ is perverse.

Need a lemma:

Lem $\mathcal{E} \xrightarrow{\pi} B$ is a vec bun, $\mathbb{P}_{\mathcal{E}} \xrightarrow{\bar{\pi}} B$.

If $K \in Perv(\mathcal{E})$ $\mathbb{G}_m$-equivariant, and $\pi_! K, \pi_* K \in Perv$

then $\bar{\pi}(\bar{K})$ is perverse.