Vanishing Conjectures

Thm: E irreducible, $\operatorname{rank}(E) = n > m$

$D(-)$ can be any sheaf theory.

Vanishing Conjecture

$$Av_E^d : D(\mathrm{Bun}_m) \longrightarrow D(\mathrm{Bun}_m)$$

$d = d_2 - d_1$

"upper modification by $\mathcal{L}_E^d$"

$$\mathrm{Bun}_m^{d_1} \leftarrow \mathrm{Mod}_m^{d_1,d_2} \rightarrow \mathrm{Bun}_m^{d_2}$$
$$\downarrow$$
$$\mathrm{Tor}^d$$

$$Av_E^d = 0 \quad \text{for} \quad d > nm(2g-2).$$

$g > 1$ for simplicity.

Goal: Explain why we expect this in geometric Langlands.

For simplicity, look at the de-Rham setting: $k = \mathbb{C}$, $D(-) = \mathrm{DMod}(-)$

Conj: $\mathrm{DMod}_{coh}(\mathrm{Bun}_G) \simeq \mathrm{Coh}(\mathrm{LS}_{\check{G}})$

Conj: $\mathrm{DMod}(\mathrm{Bun}_G) \simeq$ "$\mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})$"

$\cup$ $\mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ $\quad$ $\cap$ $\mathrm{IndCoh}(\mathrm{LS}_{\check{G}})$

Expect: $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \curvearrowright \mathrm{DMod}(\mathrm{Bun}_G)$

This expected action is not arbitrary: for $E \in \mathrm{LS}_{\check{G}}$,

$$pt \xrightarrow{E} \mathrm{LS}_{\check{G}} \quad \overset{\text{fiber functor}}{\rightsquigarrow} \quad \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{(\cdot)_E} \mathrm{Vect}$$

Require

$$\mathrm{DMod}(\mathrm{Bun}_G) \underset{\mathrm{QCoh}(\mathrm{LS}_{\check{G}})}{\otimes} \mathrm{Vect} \simeq \mathrm{DMod}(\mathrm{Bun}_G)_{E\text{-Hecke}}.$$

$(\cdot)_E$

Def: For $x \in X$

$$\mathrm{Rep}(\check{G}) \otimes D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G)$$

"modification by $\mathrm{Rep}(\check{G}) \xrightarrow{\text{s.t.}} D(G(O_x)\backslash G(K_x)/G(O_x))$".

$$\mathrm{Rep}(\check{G}) \otimes D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G \times X).$$

Say $\mathcal{K} \in D(\mathrm{Bun}_G)$ is E-Hecke-eigen if

$$V \otimes K \longmapsto K \boxtimes V_E$$

& higher compatibilities:

$$\mathrm{Rep}(\check{G})^I \otimes D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G \times X^I) \ldots\ldots$$

It turns out the expected action

$$\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \curvearrowright D(\mathrm{Bun}_G)$$

is completely determined by the above requirement.

Duality:

$$\mathrm{Rep}(\check{G}) \otimes D(X) \xrightarrow{H \ \overset{\text{temperity}}{\circlearrowleft}} \mathrm{End}(D(\mathrm{Bun}_G))$$

For DMod, !-pull *-push more natural

It is a functor, but not monoidal!

$$\underset{H_x(V_1)}{H(V_1, \delta_x)} \circ H(V_2, \delta_y) \neq \underset{\substack{\| \\ 0}}{H(V_1 \otimes V_2, \delta_x \otimes \delta_y)}$$

In fact commute.

$$\mathrm{Rep}(\check{G})^{\otimes 2} \otimes D(X^2) \longrightarrow \mathrm{End}(D(\mathrm{Bun}_G))$$

$$H_{x,y}(V_1, V_2) = \begin{cases} H_x(V_1) \circ H_y(V_2) & x \neq y \\ \underline{H_x(V_1 \otimes V_2)} & x = y \end{cases}$$

(b/c Sat is monoidal).

The pushout

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Rep}(\check{G})^{\otimes 2} \otimes D(X) \arrow[r, hook] \arrow[d] & \mathrm{Rep}(\check{G}) \otimes D(X^2) \arrow[d] \\ \mathrm{Rep}(\check{G}) \otimes D(X) \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

$\mathrm{Rep}(\check{G})_{X^2}$

is monoidal, and acts on $D(\mathrm{Bun}_G)$

To describe what happens for X^I, $|I| > 2$,

$$\mathrm{Ran} = \{ I \subset X(\mathbb{C}) \text{ finite} \}$$

$$= \operatorname{colim} X^I \qquad (I \twoheadrightarrow J,\ X^J \hookrightarrow X^I)$$

Define $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$ to be a clever assemblation of

$\mathrm{Rep}(\check{G})^{\otimes I} \otimes D(X^I)$'s. It is a category "parametrised" by Ran.

$$\mathrm{Rep}(\check{G})_{\underline{x}} \simeq \bigotimes_{x \in \underline{x}} \mathrm{Rep}(\check{G})$$

$\underline{x} \in \mathrm{Ran}$

"factorization category".

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \xrightarrow{\text{monoidal}} \mathrm{End}(D(\mathrm{Bun}_G))$$

$$\left(\begin{array}{l} V_{\underline{x}} \in \mathrm{Rep}(\check{G})_{\underline{x}} \\ W_{\underline{y}} \in \mathrm{Rep}(\check{G})_{\underline{y}} \\ V_{\underline{x}} \otimes W_{\underline{y}} \in \mathrm{Rep}(\check{G})_{\underline{x} \cup \underline{y}} \end{array} \right)$$

$$\begin{tikzcd} \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[r, "\text{monoidal}"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \\ \mathrm{Rep}(\check{G})_{\underline{x}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$\prod_{x \in \underline{x}} \mathbb{B}\check{G}_x \longleftarrow \mathrm{LS}_{\check{G}}$$

$$(E_x) \longleftarrow\!\shortmid E$$

Thm (Gaitsgory-Lurie): (A local-to-global result)

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$

is a Verdier quotient (right adjoint is fully faithful)

The proof of this theorem is formal:

$$LS_{\check{G}} = \Big\{ \text{"horizontal" section for } \underbrace{B\check{G} \times X}_{\text{trivial connection}} \longrightarrow X \Big\}$$

$B\check{G} \times X$ can be replaced by any Z equipped with connections relative to X

The expected $QCoh(LS_{\check{G}}) \curvearrowright D(Bun_G)$ has no choice because the action of $Rep(\check{G})_{Ran}$ is known.

<u>Thm</u> (Generalized Vanishing Conjecture).

$$Rep(\check{G})_{Ran} \longrightarrow End(DMod(Bun_G))$$

"spectral action"

(uniquely) factors through $QCoh(LS_{\check{G}})$.

This theorem is proved using Beilinson-Drinfeld's theory of localization of Kac-Moody rep'n's.

Thm GenVan $\Rightarrow$ Thm Van (de-Rham setting).

What's $Av_E^d \in End(D(Bun_m))$?

Claim : (Up to normalization) $(\check{G} = GL_m)$

$$\begin{tikzcd}
Rep(\check{G})_{Ran} \arrow[r] \arrow[d] & End(DMod(Bun_m)) \\
QCoh(LS_{\check{G}}) \arrow[ur, "\text{Thm (Gen-Van)}"] &
\end{tikzcd}$$

$Sym^d(St_m \otimes E) \longmapsto Av_E^d$

("supported" on $X^{(d)} \to Ran$)

tautological

fiber at $E' \in LS_{\check{G}}$ is

$H^\bullet(X^{(d)}, Sym^d((E')^* \otimes E)) \longmapsto 0$ (for $d \gg 0$) (Thm Van)

dual due to convention

Recall

$$\mathcal{L}_E^d|_{Gr^{reg}_{m,x}} \simeq \bigoplus_{|\lambda| = d} (IC_\lambda \otimes \chi_\lambda \otimes E_x^d)^{S_d}$$

Satake

$$\simeq (St_m^d \otimes E_x^d)^{S_d}$$

$$\simeq Sym^d(St_m \otimes E_x)$$

Only need to show:

Lem: $d > mn(2g-2) = (2g-2)\cdot \mathrm{rank}(E'\otimes E)$

$$H^\bullet(X^{(d)}, \mathrm{Sym}^d(E'\otimes E)) = 0.$$

Proof: E irre. & $\mathrm{rank}(E) > \mathrm{rank}(E')$

$$\Rightarrow H^0(X, E'\otimes E) = 0$$
$$H^2(X, E'\otimes E) = 0 \quad \text{(duality)}$$

$$\Rightarrow \dim H^1(X, E'\otimes E) = (2g-2)\cdot \mathrm{rank}(E'\otimes E)$$

$$\Rightarrow H^\bullet(X^{(d)}, \mathrm{Sym}^d(E'\otimes E))$$
$$\simeq H^\bullet(X, E'\otimes E)^{(d)} \quad \text{(Kunneth)}$$
$$\simeq \Lambda^d H^1(X, E'\otimes E) \simeq 0. \qquad \square.$$

Rmk:

- For other sheaf theory:

  $LS_{\check{G}}$ replaced by $LS_{\check{G}}^{res}$ ([AGKRRV])

- $LS_{\check{G}}^{res}(S) = \{ \mathrm{Rep}(\check{G}) \xrightarrow{\text{monoidal}} \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \}$

  quasi-lisse vs lisse
  similar to
  quasi-coh vs coh.

$$LS^{u}_{\check{G}}(S) = \{ \operatorname{Rep}(\check{G}) \xrightarrow{\text{monoidal}} QCoh(S) \otimes DMod(X) \}$$

(de-Rham)

- $$\text{“} \operatorname{Rep}(\check{G})^{QLisse}_{Ran} \text{”} \twoheadrightarrow QCoh(LS^{res}_{\check{G}})$$

$$\begin{tikzcd} \text{“}\operatorname{Rep}(\check{G})^{QLisse}_{Ran}\text{”} \arrow[r, two heads] \arrow[d] & QCoh(LS^{res}_{\check{G}}) \arrow[dl, "\underline{\text{Thm}}"] \\ End(D_{Nilp}(Bun_G)) & \end{tikzcd}$$

Nilp $\rightsquigarrow$ Singular support

The theorems in [AGKRRV] does not imply Thm Van because $D_{Nilp} \neq D$. (Need learn more to be sure on this).

- For G other than GL_n.

__Conj__ (Vanishing conjecture for general reductive)

If E is irreducible, $V \in \operatorname{Rep}(\check{G})$, $\operatorname{rank}(E) > \dim V$, then

$$\begin{array}{ccc} Sym^{(d)}(V \otimes E) & \longmapsto & 0 \\ \cap & & \cap \\ \operatorname{Rep}(\check{G})_{Ran} & \longrightarrow & End(D(Bun_G)) \end{array}$$

for $d > \dim(V \otimes E) \cdot (2g-2)$.

(In de-Rham, this follows from Thm Gen-Van
For l-adic, ..., this is a conjecture.)

Gaitsgory's origin proof of Thm Van.

Step 1: $Av_{E^*}^{-d}$ both left and right adjoint to Av_E^d.

(define using "lower modification") $\quad (\mathbb{D} Av_{E^*}^{-d} = Av_E^d \mathbb{D})$

Need $Av_E^d \circ Av_{E^*}^{-d} = 0$

· **Lem**: $Av_E^d(\bullet)$ is cuspidal.

$Av_{E^*}^{-d}(\ )$

$\Rightarrow$ Only need $Av_E^d(\text{cuspidal}) = 0$

Step 2:

Prop: Av_E^d is t-exact (for perverse t-structure).

Step 3:

By Prop, it is enough to show $\chi(Av_E^d(\mathcal{F})) = 0$

Euler char.

Standard argument, reduce to the case $E = \text{triv}$

(But Prop. only holds for irreducible E).

Step 4: (Braverman).

$$Av^d_{1\oplus\cdots\oplus 1} = \bigoplus_{d_1+\cdots+d_n=d} Av_1^{d_1} \circ \cdots \circ Av_1^{d_n}$$

$\Rightarrow$ Only need to show if $d > m(2g-2)$,

$Av_1^d(\text{cuspidal}) = 0$

direct calculation.

"L-function of cusp. auto. rep. is a polynomial deg $\le m(2g-2)$"

by Jacquet-Godement [I don't understand].

Braverman's calculation is a geometrization.

Why $\mathrm{Av}_E^d(-)$ is cuspidal ?

$m = m_1 + m_2$

$CT : D(\mathrm{Bun}_m) \to D(\mathrm{Bun}_{m_1} \times \mathrm{Bun}_{m_2})$ $\begin{pmatrix} \boxed{GL_{m_1}} & \boxed{\,|\,|\,|\,} \\ & \boxed{GL_{m_2}} \end{pmatrix}$

$$CT \circ \mathrm{Av}_E^d \overset{\text{almost}}{=} \bigoplus \left(\mathrm{Av}_E^{d_1} \otimes \mathrm{Av}_E^{d_2}\right) \circ CT$$

filtration   pure geometric.   graded $\| \; 0$ By induction.

Digression: Last time

Prop: $\mathcal{F} \in D(\mathrm{Bun}_n)$ E-Hecke eigensheaf

then $\mathcal{F}$ is cuspidal

Proof:
$$\mathrm{Av}_E^d(\mathcal{F}) = \mathcal{F} \otimes H^\bullet(X^{(d)}, \mathrm{Sym}^d(E^* \otimes E))$$
(predicted by spectral action)

$$CT \circ \mathrm{Av}_E^d(\mathcal{F}) = CT(\mathcal{F}) \otimes H^\bullet(\underbrace{\cdots}_{\neq 0})$$
$$\|$$

$$\bigoplus (\mathrm{Av}_E^{d_1} \circledast \mathrm{Av}_E^{d_2})\, CT(\mathcal{F})$$

Thm. Van. $\quad \| \quad \Rightarrow \quad CT(\mathcal{F}) = 0.$

$\qquad\qquad\qquad 0$

Why Av_E^d is t-exact ?

$\Uparrow$ (Springer like we did before)

Av_E^1 is t-exact

But this is false !

Nevertheless:

<u>Lem</u> (Super technical)

There exists a Verdier quotient

$$\begin{tikzcd} D(\mathrm{Bun}_G) \arrow[r, two heads] \arrow[d, "\mathrm{Av}_E^1"] & \tilde{D}(\mathrm{Bun}_G) \arrow[d, dashed, "\widetilde{\mathrm{Av}}_E^1"] \\ D(\mathrm{Bun}_G) \arrow[r, two heads] & \tilde{D}(\mathrm{Bun}_G) \end{tikzcd}$$

s.t.

$\widetilde{\mathrm{Av}}_E^1$ is t-exact.

Moreover: $D(\mathrm{Bun}_G)^{\mathrm{cusp}} \to D(\mathrm{Bun}_G) \to \tilde{D}(\mathrm{Bun}_G)$

is conservative

Gaitsgory's construction of $\check{D}$ is very complicated.

<u>But</u>: there is a modern choice:

$$\check{D}(\mathrm{Bun}_G) := D(\mathrm{Bun}_G)^{\mathrm{temp}}.$$

<u>Ramanujan</u>: tempered & Eis (tempered) gives everything.