Categorical Conjectures.

Conj: $\mathbb{k} = \mathbb{C}$, $e = \mathbb{C}$

$$\mathrm{DMod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}}) \quad (\mathrm{dRLS}_{\check{G}})$$

What's IndCoh_{Nilp}? Why?

Fact: $\mathrm{LS}_{\check{G}}$ is singular.

- **Ex:** $X = \mathbb{P}^1$, $\mathrm{LS}_{\check{G}}(\mathbb{P}^1) = \mathbb{B}\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \mathbb{B}\check{G}$ — derived stack

Why do we care about the derived structure?

$$\mathrm{DMod}(\mathrm{Bun}_G(\mathbb{P}^1)) = \mathrm{DMod}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$$
$$\not\simeq \mathrm{Rep}(\check{G})$$

as ∞-categories (even triangulated categories)

For singular schemes / stacks, ($\mathcal{O}_Y$ bounded)

$$\mathrm{Perf}(Y) \subsetneqq \mathrm{Coh}(Y)$$

- **Ex:** $\mathrm{Spec}\, k[\varepsilon]$, $\deg \varepsilon = -1$. then $k \in \mathrm{Coh}$, $k \notin \mathrm{Perf}$

$$(\cdots A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \to k)$$

$$\mathrm{QCoh}(Y) \subsetneqq \mathrm{IndCoh}(Y)$$

Y nice enough. is $\mathrm{IndPerf}(Y)$

There are categories between Perf and Coh

$$\mathrm{Sing}(Y) = \mathrm{Spec}_{/Y}(H^1(T(Y))) \quad T^*[-1]Y$$

$\emptyset$ if Y smooth.

For any $\mathcal{F} \in \mathrm{Coh}(Y)$, can define conical $\mathrm{supp}(\mathcal{F}) \subset \mathrm{Sing}(Y)$

$$\mathrm{Coh}_N(Y) := \{\mathcal{F} \in \mathrm{Coh}(Y) \mid \mathrm{Supp}(\mathcal{F}) \subset N\}$$

$$\mathrm{Coh}_0(Y) \simeq \mathrm{Perf}(Y)$$

Ex: $Y = pt \times_V pt$, $H^\bullet(T(Y)) = V$ $\quad \mathrm{Sing}(Y) = V^*$

$$\mathrm{IndCoh}(pt \times_V pt) \overset{KD}{\simeq} (\mathrm{Sym}\, V[-2])\text{-mod}$$

$$\mathcal{F} \in (\mathrm{Sym}\, V[-2])\text{-mod}$$

$H^\bullet(\mathcal{F}) \in \mathrm{Sym}\, V\text{-mod}^\heartsuit = \mathrm{QCoh}(V^*)^\heartsuit$.

$\mathrm{Supp}(\mathcal{F}) := \mathrm{supp}(H^\bullet(\mathcal{F}))$ (conical b/c graded)

For a scheme Y and $y \in Y$, local behavior is controlled by

$y \times_Y y = * \times_{T_y(Y)} *$ $\rightsquigarrow$ produce $\mathrm{Supp}(\mathcal{F})$

For stacks, using descent.

What's $T\, LS_{\check{G}}$?

Given $(\mathcal{F}, \nabla) \in LS_{\check{G}}$, $\check{\mathfrak{g}}_{\mathcal{F}}$ is a vector bundle w/ connection,

$$T_{(\mathcal{F},\nabla)} LS_{\check{G}} = H_{dR}(X, \check{\mathfrak{g}}_{\mathcal{F}})[1]$$

$$= H^\bullet(X, \underset{-1}{\check{\mathfrak{g}}_{\mathcal{F}}} \xrightarrow{ad(\nabla)} \underset{0}{\check{\mathfrak{g}}_{\mathcal{F}} \otimes \Omega_X})[1]$$

$$T^*_{(\mathcal{F},\nabla)} LS_{\check{G}} = H_{dR}(X, \check{\mathfrak{g}}^*_{\mathcal{F}})[-1]$$

$$\Rightarrow \mathrm{Sing}\, LS_{\check{G}} = \{ (\mathcal{F}, \nabla, A) \mid A \in H^0_{dR}(X, \check{\mathfrak{g}}^*_{\mathcal{F}}) \}$$

horizontal section.

$$\mathrm{Nilp} \subset \mathrm{Sing}\, LS_{\check{G}} : \{ (\mathcal{F}, \nabla, A) \mid A \text{ is nilpotent} \}$$

(identify $\check{\mathfrak{g}}^*$ with $\check{\mathfrak{g}}$)

Conj: $\Bbbk = \mathbb{C}$, $e = \mathbb{C}$

$$\mathrm{DMod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).$$

Why expect this?

- Reason 1: derived Satake:

Thm: $\mathrm{DMod}(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}((pt \times_{\check{\mathfrak{g}}} pt)/\check{G})$

$\mathrm{Sing}(pt \times_{\check{\mathfrak{g}}} pt) \simeq (\check{\mathfrak{g}})^* \simeq \check{\mathfrak{g}} \supset \mathrm{Nilp}.$

- Reason 2: Eisenstein series:

• $$G \longleftarrow P \longrightarrow M$$

$$\mathrm{Bun}_G \xleftarrow{\ p\ } \mathrm{Bun}_P \xrightarrow{\ q\ } \mathrm{Bun}_M$$

$\mathrm{Eis}_! := p_! \circ q^* : \mathrm{DMod}(\mathrm{Bun}_M) \longrightarrow \mathrm{DMod}(\mathrm{Bun}_G)$

right adjoint $\mathrm{CT}_* = q_* \circ p^!$ continuous.

• But

$$\mathrm{LS}_{\check{G}} \xleftarrow{\ \check{p}\ } \mathrm{LS}_{\check{P}} \xrightarrow{\ \check{q}\ } \mathrm{LS}_{\check{M}}$$

$\check{p}_* \circ \check{q}^* : \mathrm{QCoh}(\mathrm{LS}_{\check{M}}) \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$ does not have continuous right adjoint

| $\check{p}$ is proper: $\check{p}_*$ sends Coh to Coh but Perf to ~~not Perf~~ $\mathrm{Coh}_{\mathrm{Nilp}}$.

But why Nilp ?

$$\check{p}_* \circ \check{q}^* (\mathrm{Perf}(LS_{\check{M}})) \subset \mathrm{Coh}_{Nilp}(LS_{\check{G}}).$$

• <u>Fact</u> : $\mathrm{IndCoh}_{Nilp}(LS_{\check{G}})$ is generated by

$$\mathrm{Eis}^{spec}_{\check{P}} (\mathrm{Perf}(LS_{\check{M}})) \quad \text{for all } \check{M} \text{ (including } \check{G}).$$

<u>Conj</u> :

$$\mathrm{DMod}(\mathrm{Bun}_G)_{temp} \simeq \mathrm{QCoh}(LS_{\check{G}})$$

TBD.

• <u>Fact</u> : $\mathrm{DMod}(\mathrm{Bun}_G)$ is generated by

$$\mathrm{Eis}_P (\mathrm{DMod}(\mathrm{Bun}_M)_{temp, cpt}) \text{ for all } M.$$

(Conj tempered $\Leftrightarrow$ Conj all)

Ideologically :

$\mathrm{DMod}(\mathrm{Bun}_G)_{temp}$ is the part that can be detected by Whittaker.

<u>Recall</u> :

$$\mathrm{Whit}(\mathrm{Bun}_G^{N\text{-gen}}) \xrightarrow{!\text{-push}} \mathrm{DMod}(\mathrm{Bun}_G)$$

$$\text{"}\textstyle\int_{ch} \mathrm{Whit}(G_{\mathcal{K}}) = \int_{ch} \mathrm{Rep}(\check{G})\text{"}$$

know generates but not know Verdier quotient yet. $\searrow$ $\text{DMod}(\text{Bun}_G)_{temp}$ $\nearrow$ by def

$\cup$ known

$\text{DMod}(\text{Bun}_G)_{cusp}$.

In reality: define $\text{DMod}(\text{Bun}_G)_{temp}$ using other method.

$$\text{Sph}_{G,x} \curvearrowright \text{DMod}(\text{DMod}(\text{Bun}_G))$$

$$\text{Sph}_{G,x} \simeq \text{DMod}(G(O_x)\backslash G(K_x)/G(O_x))$$

$$\simeq \text{ (t-exact)}$$

$$\text{IndCoh}_{Nilp}(pt \times_{\check{g}} pt/\check{G})$$

$$\uparrow\downarrow$$

$$\text{QCoh}(pt \times_{\check{g}} pt/\check{G})$$

(Also the left completion)

<u>Def</u> $\mathcal{F} \in \text{DMod}(\text{Bun}_G)$ is tempered iff $\text{Sph}_{G,x} \curvearrowright \mathcal{F}$ factors through its left completion.

<u>Fact</u>: This does not depend on x.

What about other sheaf-theories?

Betti setting.

Betti-version. Rep of π_1

Conj: $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$

What's $\mathrm{Shv}_{\mathrm{Nilp}}$?

- Singular support for sheaves:

$$\mathcal{F} \in \mathrm{Shv}(Y), \quad \mathrm{Supp}(\mathcal{F}) \subseteq T^*Y$$

defined using micro-local analysis (nearby cycles).

($\mathcal{F}$ is local system $\Leftrightarrow$ $\mathrm{Supp}(\mathcal{F}) = 0$)

$$T_{\mathcal{F}} \mathrm{Bun}_G \simeq H^{\cdot}(X, \mathfrak{g}_{\mathcal{F}})[1]$$

$$T^*_{\mathcal{F}} \mathrm{Bun}_G \simeq H^{\cdot}(X, \mathfrak{g}^*_{\mathcal{F}} \otimes \Omega_X) \quad (\simeq \mathfrak{g}_{\mathcal{F}})$$

Higgs_G

$$\mathrm{Nilp} \subset \mathrm{Higgs}_G.$$