$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$

- Last time:

  <u>Conj</u>: (de-Rham setting)

  $$\mathrm{DMod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_N(\mathrm{LS}_{\check{G}})$$

- How about other sheaf theories?

- Betti setting:

  Classify sheaves on Bun_G, viewed as a stack in complex geometry.

  - For f.t. affine scheme S over $\mathbb{C}$.

    $$\mathrm{Shv}^{\mathrm{all}}(S) := \mathrm{Shv}^{\mathrm{all}}(S(\mathbb{C})).$$

    (put no constructible conditions!)

  - $\mathcal{Y}$ algebraic stack

    $$\mathrm{Shv}^{\mathrm{all}}(\mathcal{Y}) := \lim_{\substack{S \to \mathcal{Y} \\ \text{smooth}}} \mathrm{Shv}^{\mathrm{all}}(S) \quad \text{(connected by pullbacks)}$$

    $$\simeq \operatorname{colim}_{\substack{S \to \mathcal{Y} \\ \text{smooth}}} \mathrm{Shv}^{\mathrm{all}}(S) \quad \text{(connected by !-push)}$$

> In general, $i \longmapsto C_i$ presentable
>
> $L_{ij} : C_i \rightleftarrows C_j : R_{ij}$ adjoint.
>
> then $\underset{\text{connected by } L_{ij}}{\operatorname{colim} C_i} \simeq \underset{\text{connected by } R_{ij}}{\lim C_i}$

- Want to understand $\mathrm{Shv}^{\mathrm{all}}(\mathrm{Bun}_G)$

- In de-Rham setting, classified by de-Rham local system,
  (A.G.)

$$LS^{dR}_{\check{G}} := \mathrm{Conn}_{\check{G}} = \left\{ (\mathcal{F}_{\check{G}}, \nabla) \,\middle|\, \begin{array}{l} \mathcal{F}_{\check{G}} \ \check{G}\text{-torsor} \\ \nabla \ \text{principle conn.} \end{array} \right\}$$

- In Betti setting, classified by Betti local systems.
  (topology)

$$LS^{Betti}_{\check{G}} := \{ \text{"}\pi_1(X(\mathbb{C})) \to \check{G}\text{"} \}$$

Need to be careful about base-point.
Also, it is a derived stack.

- $LS^{dR}_{\check{G}} \not\simeq LS^{Betti}_{\check{G}}$!

  - dR depends on X, but Betti only on $X(\mathbb{C})$.

  - Example: $g=1$, $X=E$, $e \in E$ base-point.

$$LS^{dR,\,rigide}_{GL_1} \longrightarrow \mathrm{Pic}^{\circ}_{E} \simeq E \quad \text{is an } \mathbb{A}^1\text{-bundle}$$

Picard scheme

trivialized at e.
(kill the derived & stacky part)

$$LS^{Betti,\,rigide}_{GL_1} \simeq \mathbb{G}_m \times \mathbb{G}_m$$

(Their $\mathbb{C}$-points should be bijective by R.H.
But the alg. geo. structure are different because
R.H. uses exp)

- Conj (Betti setting)

$$Shv^{all}(Bun_G) \overset{\text{false!}}{\simeq} \mathrm{IndCoh}_{N}(LS^{Betti}_{\check{G}})$$

This is not true even for GL_1. $X = E$.

$$\mathrm{Bun}_{GL_1} \simeq \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d_E \times \mathbb{B}\mathbb{G}_m \simeq E \times \mathbb{Z} \times \mathbb{B}\mathbb{G}_m$$

$$LS^{Betti}_{GL_1} \simeq (\mathbb{G}_m \times \mathbb{G}_m) \times \mathbb{B}\mathbb{G}_m \times pt \times_{\mathbb{A}^1} pt$$

$Shv(\mathbb{Z}) = QCoh(\mathbb{B}\mathbb{G}_m)$

$Shv(\mathbb{B}\mathbb{G}_m) \simeq QCoh(pt \times_{\mathbb{A}^1} pt)$

$$Shv(E) \not\simeq QCoh(\mathbb{G}_m \times \mathbb{G}_m)$$

$$\cup \qquad \qquad //$$

$QLisse(E) = \left\{ \begin{array}{l} \text{complex whose coh. in usual t-structure} \\ \text{are ind. of f.d. local systems} \end{array} \right\}$

$QLisse \simeq Shv_0 = \{$ complexes with $SS = 0 \}$.

- For S smooth scheme, $\mathcal{N} \subset T^*S$ conical Lagrangian

$$Shv^{all}_{\mathcal{N}}(S) \subset Shv^{all}(S)$$

(Last time $M \in Coh(S)$, $SS(\mathcal{F}) \subset T^*[-1]S$).

"Def": $\mathcal{F} \in Shv^{all}_{\mathcal{N}}(S)$ iff $\forall\, v_s \in T_s S$, $v_s \perp \mathcal{N}_s$ $\Rightarrow$ $\mathcal{F}$ is "lisse" along the direction of v.

(locally acyclic)

(Rmk: For regular hol. D-module M, $Sol(M)$ perverse sheaf

$$SS(M) = SS(Sol(M)).$$)

"$SS(\mathcal{F})$ measures how far $\mathcal{F}$ is away from being lisse".

$$T^*\mathrm{Bun}_G \simeq \mathrm{Higgs}_G = \{ (\mathcal{F}_G, s) \mid s \in H^0(X, \mathfrak{g}^*_{\mathcal{F}_G} \otimes \Omega_X) \}$$

$$\cup\!\!\!\mid$$

$$\mathrm{Nilp}.$$

<u>Conj</u>: (Betti Setting)

$$\underbrace{\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}}_{}(\mathrm{Bun}_G) \simeq \underbrace{\mathrm{IndCoh}_{\mathrm{Nilp}}}_{}(\mathrm{LS}_{\check{G}})$$

important & non-technical   technical.

<u>Why believe this</u>?

- True for torus.
- Not too small: conjecturally all Hecke-eigensheaves has $SS \subset \mathrm{Nilp}$.
- Not too big: $\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}$ is compactly generated but $\mathrm{Shv}^{\mathrm{all}}$ is not.

In general $\mathrm{Shv}^{\mathrm{all}}_{N}(S)$ is cpt. gen.

$\exists$ stratification of S (depending on N)

s.t. any $\mathcal{F} \in \mathrm{Shv}^{\mathrm{all}}_{N}(S)$ is loc. constant on each stratum.

- More motivations to be given.

- First, the global Nilpotent cone

$$\mathrm{Nilp} \subseteq T^*\mathrm{Bun}_G \qquad \text{is very import.}$$

(just like $\mathcal{N} \subset \mathfrak{g}$).

Hitchin fibration:

$$\begin{tikzcd} T^*\mathrm{Bun}_G \arrow[r] & \mathcal{A}_G \arrow[d, equal] \\ (\ \Gamma(X, \mathfrak{g}^*_{\mathcal{G}} \otimes \Omega_X) \arrow[r] & \Gamma(X, \mathfrak{g}^* /\!/ G \otimes \Omega_X)\ ) \end{tikzcd}$$