Spectral Decomposition & $LS^{rest}_{\check{G}}$

· Last time, we introduced

<u>Conj</u> (Betti GLC)

$$Shv^{all}_{Nilp}(Bun_G) \simeq IndCoh_{Nilp}(LS^{Betti}_{\check{G}})$$

$$\circlearrowleft$$
$$QCoh(LS^{Betti}_{\check{G}})$$

How to obtain $QCoh(LS^{Betti}_{\check{G}})$-action via local-to-global method ?

· Recall in de Rham setting:

$$\begin{array}{ccc} Rep(\check{G})_{Ran} & \twoheadrightarrow & QCoh(LS^{dR}_{\check{G}}) \\ \text{Satake} \downarrow & & \\ Dmod(\mathcal{L}^+G \backslash \mathcal{L}G/\mathcal{L}^+G)_{Ran} & & \end{array}$$

Kernel is very large & mysterious (vanishing conjectures ...)

But in Betti setting, the similar story is much clean.

<u>Def</u>: For any homotopy type $\mathcal{Y} \in Spc$ (topological space up to homotopy)

$$LS(\mathcal{Y}) := \{ \text{local systems on } \mathcal{Y} \}$$

$$= \underset{\mathcal{Y}}{\operatorname{colim}}\, Vect \simeq \underset{\mathcal{Y}}{\lim}\, Vect.$$

self-dual

$Shv^{all}_0(X)$
$\|$
$LS(X(\mathbb{C}))$

(View $\mathcal{Y}$ as an ∞-groupoid. $\mathcal{Y} \longrightarrow DGCat$ constant diagram with value Vect).

(Warn: $LS(\mathcal{Y})$ is not determined by $\pi_1(\mathcal{Y})$ even if $\mathcal{Y}$ connected.)

$f: \mathcal{Y}_1 \to \mathcal{Y}_2$, $f^\dagger : LS(\mathcal{Y}_2) \longrightarrow LS(\mathcal{Y}_1)$ $\quad f_\dagger : LS(\mathcal{Y}_1) \longrightarrow LS(\mathcal{Y}_2)$

a non-obvious functor.

$pt \hookrightarrow S^1$ produce ∞ - Jordan block.

Consider the $\underset{\sim}{\text{algebraic}}$ stack $\underline{\mathrm{Hom}}(Y, \mathbb{B}\check{G})$

$$\left(\mathrm{Hom}(S, \underline{\mathrm{Hom}}(Y, \mathbb{B}\check{G})) = \text{“}\mathrm{Hom}(S \times \underline{Y}, \mathbb{B}\check{G})\text{”} = \left\{ \begin{array}{l} \text{right t-exact } \otimes\text{-functor} \\ \mathrm{Rep}(\check{G}) \to \mathrm{QCoh}(S) \otimes \underset{\sim}{\mathrm{LS}(Y)} \end{array} \right\} \right)$$

I made a mistake last time

$$\left(\underline{\mathrm{Hom}}(X(\mathbb{C}), \mathbb{B}\check{G}) = \mathrm{LS}^{\mathrm{Betti}}_{\check{G}} \right)$$

$\underline{\text{Thm}}$: Knowing $\mathrm{QCoh}(\underline{\mathrm{Hom}}(Y, \mathbb{B}\check{G}))$-action on $\mathcal{C}$

$$\Updownarrow$$

knowing

$$\mathrm{Rep}(\check{G})^{\otimes I} \xrightarrow{\text{monoidal}} \mathrm{End}(\mathcal{C}) \otimes \mathrm{LS}(Y^I)$$

+ compatibilities: $I \to J$, $Y^J \xrightarrow{\Delta} Y^I$

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})^{\otimes I} \arrow[r] \arrow[d] & \mathrm{End}(\mathcal{C}) \otimes \mathrm{LS}(Y^I) \arrow[d, "\Delta^!"] \\
\mathrm{Rep}(\check{G})^{\otimes J} \arrow[r] & \mathrm{End}(\mathcal{C}) \otimes \mathrm{LS}(Y^J)
\end{tikzcd}$$

Recall in the de Rham setting (and $Y = X(\mathbb{C})$).

$\mathrm{LS}(Y^I)$ replaced by $\mathrm{DMod}(X)$.

Via duality,

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})^I \otimes \mathrm{DMod}(X^J) \arrow[r, "\Delta_*"] \arrow[d] & \mathrm{Rep}(\check{G})^I \otimes \mathrm{DMod}(X^I) \arrow[d] \\
\mathrm{Rep}(\check{G})^J \otimes \mathrm{DMod}(X^J) \arrow[r] & \mathrm{End}(\mathcal{C})
\end{tikzcd}$$

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} := \operatorname*{colim}_{\substack{\mathrm{TwArr} \\ \rotatebox{90}{$\in$} \\ (I \to J)}} \mathrm{Rep}(\check{G})^I \otimes \mathrm{DMod}(X^J)$$

$\boxed{\text{monoidal}}$

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \longrightarrow \mathrm{End}(\mathcal{C}).$$

But in the topological setting

$$LS(Y^{\mathcal{J}}) \xrightarrow{\Delta_!} LS(Y^{I}) \quad \text{is not monoidal!}$$

- Hence it is not obvious these actions can be controlled by a <u>single</u> monoidal category (e.g. via taking colimits naively)

But the theorem says $QCoh(\underline{\mathrm{Hom}}(Y, \mathbb{B}\check{G}))$ does this.

- As a comparison, in de-Rham setting

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}} \twoheadrightarrow QCoh(LS^{dR}_{\check{G}})$$

has a big kernel.

(Heuristically, the kernel is caused by the difference between

$D\text{Mod}(X)$ vs $LS(X(\mathbb{C}))$

But no one knows how to give a convenient desc. such that knowing when $\mathrm{Rep}(\check{G})_{\mathrm{Ran}}$-action factors through $QCoh(LS^{dR}_{\check{G}})$).

<u>Thm</u> ([NY], [AGKRRV])

<u>Def</u>: $\mathcal{F} \in Shv^{all}(Bun_G)$ is <u>Hecke-lisse</u> iff

$$\mathrm{Rep}(\check{G}) \otimes Shv^{all}(Bun_G) \xrightarrow{H} Shv^{all}(Bun_G \times X)$$

$$\cup$$

$$Shv^{all}(Bun_G) \otimes LS(X(\mathbb{C}))$$

$$H(V, \mathcal{F}) \in$$

$$\mathrm{Shv}^{all}_{Nilp}(\mathrm{Bun}_G) \simeq \mathrm{Shv}^{all}(\mathrm{Bun}_G)^{\text{Hecke-lisse}}.$$

Cor:

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{Betti}}_{\check{G}}) \curvearrowright \mathrm{Shv}^{all}_{Nilp}(\mathrm{Bun}_G).$$

The above theorem actually works in any sheaf theory context. $\mathrm{Shv}(-)$

- "Full" Betti setting, $\Bbbk = \mathbb{C}$, $e = \mathbb{C}$, Shv^{all}
- constructible Betti setting, $\Bbbk = \mathbb{C}$, $e = \mathbb{C}$, $\mathrm{Ind}(\mathrm{Shv}_{cons})$
- ⋯⋯ étale setting, $e = \mathbb{Z}/\ell$, $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}}_\ell$ …
- "Full" de Rham setting $\Bbbk = e = \bar{k}$, $\mathrm{char}(\Bbbk) = 0$ DMod
- holonomic de-Rham setting $\mathrm{Ind}(\mathrm{DMod}_{hol})$
  $\mathrm{Ind}(\mathrm{DMod}_{hol,reg})$

Replace $\mathrm{LS}(X(\mathbb{C}))$ by $\mathrm{Shv}_0(X) \subset \mathrm{Shv}(X)$

$$\mathrm{Shv}_0(X) = \{\text{zero singular support}\}.$$

(Ex: in cons. Betti

$\mathrm{Shv}_0(X)$ is $\mathrm{QLisse}(X(\mathbb{C})) \subsetneq \mathrm{LS}(X(\mathbb{C}))$

∞-Jordan block is not QLisse.)

In any constructible setting, write $\mathrm{QLisse}(X) := \mathrm{Shv}_0(X)$.

$$LS^{rest}_{\check{G}}(S) := \left\{ \begin{array}{l} \text{right t-exact sym. monoidal functors} \\ \mathrm{Rep}(\check{G}) \longrightarrow \mathrm{QCoh}(S) \otimes \mathrm{QLisse}(X) \end{array} \right\}.$$

This makes sense in any constructible context.

<u>Cor</u>: $\mathrm{QCoh}(LS^{rest}_{\check{G}}) \curvearrowright \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$

in constructible settings.

$k = \mathbb{C}$, $e = \mathbb{C}$.

$$\begin{tikzcd} & LS^{rest}_{\check{G}} \arrow[dl] \arrow[dr] & \\ LS^{dR}_{\check{G}} & & LS^{Betti}_{\check{G}} \end{tikzcd}$$

roughly speaking:

- $\pi : LS^{Betti}_{\check{G}} \longrightarrow LS^{coarse}_{\check{G}}$ coarse moduli problem

$$\coprod_{\bar{\sigma}} (\pi^{-1}(\bar{\sigma}))^{\wedge} = LS^{rest}_{\check{G}}.$$

<u>Fact</u>: $\pi(\sigma_1) = \pi(\sigma_2)$ iff σ_1, and σ_2 have the same semi-simplification.

Similarly for $LS^{rest}_{\check{G}} \longrightarrow LS^{dR}_{\check{G}}$.

<u>Conj</u>: (might be false; constructible setting)

$$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{Nilp}(LS_{\check{G}}^{rest})$$

(Dennis told me they can prove LHS is a direct summand of RHS corresponding to some connected components)

Now $k = \mathbb{F}_q$. ℓ-adic setting.

$$\mathrm{Fr}: \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G) \longrightarrow \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G).$$

<u>Conj</u>:

$$\mathrm{Tr}(\mathrm{Fr}; \mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)) \simeq C^\infty(\mathrm{Bun}_G(\mathbb{F}_q)).$$

$\Downarrow$ V. Laf. work (unramified)

What's LHS? Categorical trace.

- For vector space V. dualizable = fin. dim.

  $f: V \to V$.

  $$k \xrightarrow{\text{unit}} V \otimes V^* \xrightarrow{f \otimes \mathrm{Id}} V \otimes V^* \xrightarrow{\langle , \rangle} k$$

  is given by a number in k.

  Claim: this number is $\mathrm{Tr}(f; V)$.

- For DG category $\mathcal{C}$. if $\mathcal{C}$ is dualizable. ($\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)$ is!)

  $F: \mathcal{C} \to \mathcal{C}$

$$\mathrm{Vect} \to \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{F \otimes \mathrm{Id}} \mathcal{C} \otimes \mathcal{C}^\vee \longrightarrow \mathrm{Vect}$$

is given by an object in Vect.

<u>Def</u>: $\mathrm{Tr}(F; \mathcal{C}) :=$ this object.

More generally: $\quad V \in \mathrm{Rep}(\check{G})^{\otimes I}$.

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \xrightarrow{\mathrm{Fr}} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \xrightarrow{H(V,-)} \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \otimes \mathrm{QLisse}(X^I)$$

$\mathrm{Tr}(\text{this functor}) \in \mathrm{QLisse}(X^I)$

<u>Conj</u>: This is the shtuka cohomology

<u>Conj</u> $\Rightarrow$ V. Laf.

$$\mathrm{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \curvearrowright \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

formally $\rightsquigarrow$ $\quad \mathrm{Tr}(\mathrm{Fr}, \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G))$ can be upgraded to a quasi-coherent sheaf on $(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{\mathrm{Frob}=\mathrm{Id}}$

(such that global section is $\mathrm{Tr}(\mathrm{Fr}; -)$).

$(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})^{\mathrm{Frob}=\mathrm{Id}} = \{\text{Weil } \check{G}\text{-local systems}\}$.