

Lecture 17: Conservativity

Goal: Thm [FR]

$$\begin{aligned} \mathcal{D}\text{Mod}(\text{Bun}_G)^{\text{temp}} &\xrightarrow{\text{coeff}^{\text{enh}}} \text{QCoh}(\text{LS}_G) \\ \text{Shv}_{\text{nilp}}(\text{Bun}_G)^{\text{temp}} &\xrightarrow{\text{coeff}^{\text{enh}}} \text{QCoh}(\text{LS}_G^{\text{restr}}) \end{aligned}$$

are conservative.

where $\text{Shv}_{\text{nilp}} \subset \mathcal{D}\text{Mod}$ full subcat of ind (reg holonomic D-mods).

(I) What's $\text{coeff}^{\text{enh}}$?

Motivation "Autom side" = "Galois side"

$$\begin{array}{ccc} \mathcal{D}\text{Mod}(\text{Bun}_G) & \xrightarrow[\sim]{\text{conj}} & \text{IndCoh}_{\text{nilp}}(\text{LS}_G) \\ \uparrow \downarrow & & \uparrow \downarrow \text{(adj pair)} \\ \mathcal{D}\text{Mod}(\text{Bun}_G)^{\text{temp}} & \xrightarrow[\sim]{\text{conj}} & \text{QCoh}(\text{LS}_G) \end{array}$$

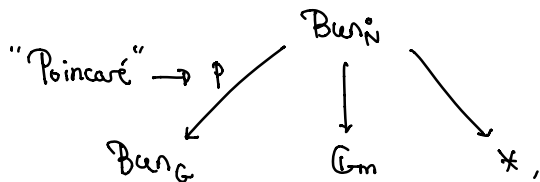
& similarly for Shv_{nilp} v.s. $\text{LS}_G^{\text{restr}}$.

Consider spectral action $\text{QCoh}(\text{LS}_G) \otimes \mathcal{D}\text{Mod}(\text{Bun}_G) \longrightarrow \mathcal{D}\text{Mod}(\text{Bun}_G)$
from action of $\text{Rep}(\check{G})$ (+ Kac-Moody localization)

By duality $\mathcal{D}\text{Mod}(\text{Bun}_G) \longrightarrow \text{QCoh}(\text{LS}_G) \otimes \mathcal{D}\text{Mod}(\text{Bun}_G)$

$$\begin{array}{ccc} & \circlearrowleft & \downarrow \text{Id} \otimes \text{coeff} \\ \text{coeff}^{\text{enh}} & \searrow & \text{QCoh}(\text{LS}_G) \end{array}$$

where $\text{coeff} = \Gamma \circ \text{coeff}^{\text{enh}} : \mathcal{D}\text{Mod}(\text{Bun}_G) \rightarrow \text{Vect}$.



Then $\text{coeff}(-) := \Gamma_{\text{pt}}(\rho^!(-) \otimes^! \exp_{\text{Bun}_G})$ [shift].

Lem $\text{DMod}(\text{Bun}_G) \xrightarrow{\text{coeff}} \text{Vect}$

\searrow

$\text{DMod}(\text{Bun}_G)^{\text{temp}} \xrightarrow{\exists} \text{Vect}$

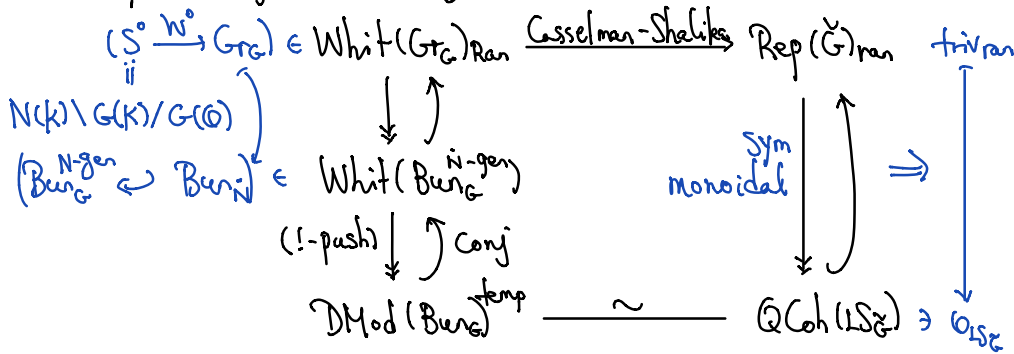
Lem $\text{QCoh}(\text{LS}_G) \subset \text{DMod}(\text{Bun}_G)^{\text{temp}}$ \int stable under Satake actions.

$\Rightarrow \text{coeff}^{\text{enh}}$ also factors through $\text{DMod}(\text{Bun}_G)^{\text{temp}}$.

Prnk $\mathcal{C} \rightarrow \mathcal{A}$ \mathcal{A} -linear $\Leftrightarrow \mathcal{C} \rightarrow \text{Vect}$ (when \mathcal{A} is rigid)

b/c \mathcal{A} rigid $\Rightarrow \text{Vect} \leftarrow \mathcal{A}$

(II) Why this functor $\text{coeff} : \text{DMod}(\text{Bun}_G) \rightarrow \text{Vect}$?



$\Rightarrow Q_{LS_G}$ should go to $P!(\text{exp}_{\text{ran}})$ (from $\text{Bun}_G^{N\text{-gen}}$).

\hookrightarrow passing to duality & right adjoint:

$\text{DMod}(\text{Bun}_G)^{\text{temp}} \rightarrow \text{QCoh}(\text{LS}_G) \xrightarrow{\Gamma} \text{Vect}$

should be given by coeff .

(III) What is $\text{DMod}(\text{Bun}_G)^{\text{temp}}$?

Local geom Langlands: $\text{DMod}(\mathbb{A}_G)\text{-mod} \xleftarrow{\text{conj}} \text{(certain modification of } \text{ShvCat}(\text{LS}_G(\mathbb{D}))\text{)}$

punctured disc \downarrow

(+ strong $G(k)$ -action)

Upshot $\text{DMod}(LG)\text{-mod} \xrightarrow[\sim]{\text{Conj}} \text{ShuCat}(LS_{\mathbb{Z}}(\mathbb{D})).$

$$\mathcal{C} \longmapsto \text{Whit}(\mathcal{C}).$$

Def \mathcal{C} is tempered if $\text{Whit}(\mathcal{C}) \subset \mathcal{C}$ generates \mathcal{C} as a $\text{DMod}(LG)\text{-mod}$.
(namely, nondegenerate Whit .)

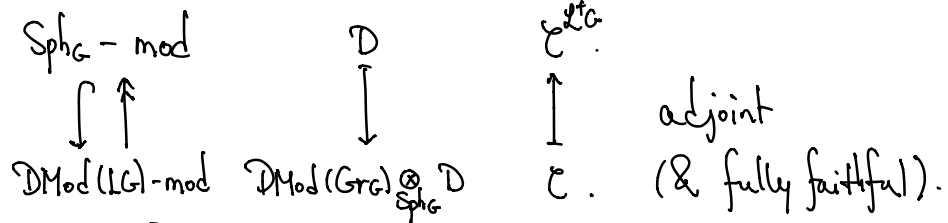
Conj Define $\text{BiWhit} := \text{Whit} \backslash \text{D}(LG) / \text{Whit}$.

Then (i) $\text{Whit}(LG) \otimes_{\text{BiWhit}} \text{Whit}(\mathcal{C}) \rightarrow \mathcal{C}$ is fully faithful.

(ii) $\text{BiWhit} = \text{QCoh}(LS_{\mathbb{Z}}(\mathbb{D}))$.

Much easier in unramified case:

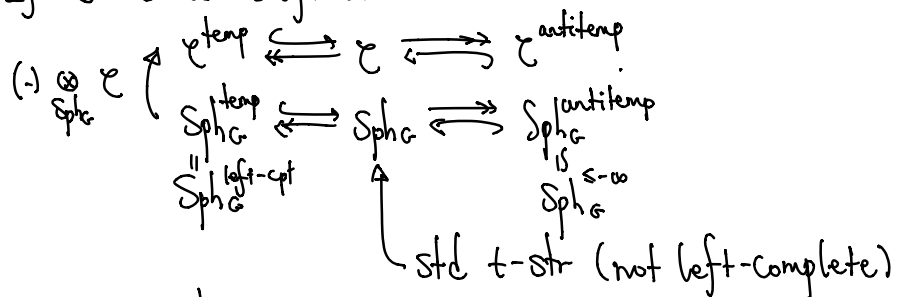
Claim "Derived Satake":



Fact If \mathcal{C} is unramified,

\mathcal{C} is tempered iff $\mathcal{C}^{LG} \xrightarrow{Av!} \text{Whit } \mathcal{C}$ is conservative.

Fact If \mathcal{C} is unramified,



(iv) What is $\text{DMod}(Bun_G)^{\text{temp}}$?

For any $x \in X$, $\text{Sph}_{G,x} \subset \text{DMod}(Bun_G)$

$\hookrightarrow \text{DMod}(Bun_G)^{\text{temp}}$ (by $\text{Sph}_G\text{-mod} \xleftrightarrow{\quad} \text{DMod}(LG)\text{-mod}$).

Thm [FR] This defin does not depend on choice of x .

Local picture $D(LG)\text{-mod}^{\text{temp}} \approx \text{ShvCat}(LS\mathbb{Z}(\bar{D}))$
 $\approx \text{QCoh}(LS\mathbb{Z}(\bar{D}))\text{-mod}$

Conj $D(LG)^{\text{left-temp}} \approx D(LG)^{\text{right-temp}}$

Global shadow

Use this to describe temp condition.

$$\begin{array}{ccc} D\text{Mod}(B_{\text{un}})^{\text{temp}} & \longrightarrow & G(F) \backslash G(A) / G(\mathcal{O}) \overset{\uparrow}{\cong} G(\mathbb{Q}) \backslash G(\mathbb{R}) / G(\mathbb{Z}). \\ \uparrow \uparrow & & \downarrow \uparrow \\ \text{Whit}(B_{\text{un}})^{\text{N-gen}} & \longrightarrow & (N(A, X) \backslash G(A) / G(\mathcal{O})) \\ & & \downarrow \uparrow \\ & & \text{Whit } B_{\text{un}}^{\text{N-gen}} \end{array}$$

Proof of conservativity (main thm)

Step 1 [ACKRRV]

$$\begin{array}{ccc} D\text{Mod}(B_{\text{un}})^{\text{temp}} & \longrightarrow & \text{QCoh}(LS\mathbb{Z}) \\ \text{Verdier} \downarrow \uparrow & & \text{formal} \downarrow \uparrow \\ \text{Quot} & & \text{cpt} \downarrow \uparrow \\ \text{Shv}_{\text{nilp}}(B_{\text{un}})^{\text{temp}} & \longrightarrow & \text{QCoh}(LS\mathbb{Z}^{\text{rest}}) \\ & & \downarrow \uparrow \\ & & \text{Higgs bundle} \end{array}$$

Step 2 $\text{Nilp}_{\text{imag}} \hookrightarrow \text{Nilp} = T^*B_{\text{un}} = \{(F, e) \mid e \in T(x, \mathcal{O}_{F, G}^* \otimes \Omega_x)\}$.

Thm $\text{Shv}_{\text{nilp, imag}}(B_{\text{un}}) \subset D\text{Mod}(B_{\text{un}})^{\text{antitemp}}$

Conj $D\text{Mod}_{\text{imag}}(B_{\text{un}}) \approx D\text{Mod}(B_{\text{un}})^{\text{antitemp}}$

(Intuitively: "IndCoh_{nilp}(LS \mathbb{Z})^{imag} \leftrightarrow QCoh(LS \mathbb{Z})^{non-sm}".)

Step 3 $\mathcal{F} \in \text{Shv}_{\text{nilp}}(B_{\text{un}})^{\text{constribe}}$

Define char cycle $cc(\mathcal{F}) := \sum_{\alpha \in \text{Irr}(\text{Nilp})} C_{\alpha, \mathcal{F}} [\alpha]$, $C_{\alpha, \mathcal{F}} \in \mathbb{Z}$.

Thm $\chi(\text{coeff}(\mathcal{F})) = (-1)^{\dim B_{\text{un}}} C_{\text{Kos}, \mathcal{F}}$.

$T^*B_{\text{un}} = \text{Maps}(x, \mathcal{O}_x / (G \times G_m)) \times_{\text{Map}(x, \mathbb{P}G_m)} \{\Omega_x\}$.

$$\mathcal{O}/G \xrightarrow{s^1} \mathcal{O}^{reg}/G \subset \mathcal{O}/G.$$

$$(\mathcal{O}/G \rightarrow \mathcal{O}/G).$$

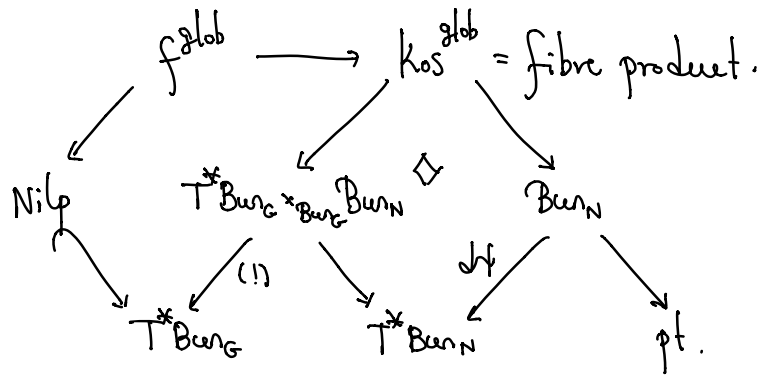
$$\hookrightarrow T^*B_{unG} \xrightarrow{\text{Hitchin}} \Gamma(X, \mathcal{O}/G \otimes \Omega_X^{G_m})$$

\uparrow \downarrow
 $k_{S^{glob}}$ \leftarrow \rightarrow $=$

$$\text{Nilp}^{reg} \longrightarrow 0 \quad (+ \text{Smoothness as } \mathcal{O}^{reg}/G \text{ sm}).$$

\uparrow
 f^{glob}

Let $\text{Nilp}^{k_{S^{glob}}}$ be the unique irr comp containing f^{glob} .



Step 4 By Step 3.

$$\text{If } \text{Nilp}^{k_{S^{glob}}} \subseteq \text{SS}(\mathcal{F}),$$

$$\text{then } C_{k_{S^{glob}}, \mathcal{F}} \neq 0 \Rightarrow \chi(\text{coeff}(\mathcal{F})) \neq 0$$

$$\Rightarrow \text{coeff}(\mathcal{F}) \neq 0.$$

For any λ^+ -valued divisor D on X .

$$\hookrightarrow X^D \in \text{Rep}(G)_X.$$

$$\text{If the sing supp } \text{SS}(\mathcal{F} * V^D) \supset \text{Nilp}^{k_{S^{glob}}}$$

$$\text{then } \text{coeff}(\mathcal{F} * V^D) \neq 0$$

$$\Rightarrow \text{coeff}(\mathcal{F}) \neq 0.$$

Thm If $SS(\mathcal{F}) \not\subseteq \text{Nilp}_{\text{reg}}$,
then $\exists D$ s.t. $\text{Nilp}^{\text{kas}} \subseteq SS(\mathcal{F} \star V^D)$.

Now if \mathcal{F} is s.t. $\text{coeff}^{\text{enh}}(\mathcal{F}) = 0$,

$\Rightarrow \forall D, \text{Nilp}^{\text{kas}} \not\subseteq SS(\mathcal{F} \star V^D)$

$\Rightarrow SS(\mathcal{F}) \subset \text{Nilp}_{\text{reg}}$

$\Rightarrow \mathcal{F}$ is anti-tempered. □