# Lecture 17: Conservativity

Goal : Thm [FR]

$$\mathrm{DMod}(\mathrm{Bun}_G)^{temp} \xrightarrow{\mathrm{coeff}^{enh}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}})$$
$$\mathrm{Shv}_{Nilp}(\mathrm{Bun}_G)^{temp} \xrightarrow{\mathrm{coeff}^{enh}} \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{restr})$$

are conservative.

where $\mathrm{Shv}_{Nilp} \subset \mathrm{DMod}$ full subcat of ind (reg holonomic D-mods).

## (I) What's $\mathrm{coeff}^{enh}$ ?

Motivation "Autom side" = "Galois side"

$$\begin{tikzcd} \mathrm{DMod}(\mathrm{Bun}_G) \arrow[r, "\mathrm{Conj}", "\sim"'] \arrow[d, shift left] & \mathrm{Ind}\,\mathrm{Coh}_{Nilp}(\mathrm{LS}_{\check{G}}) \arrow[d, shift left, "\text{(adj pair)}"] \\ \mathrm{DMod}(\mathrm{Bun}_G)^{temp} \arrow[u, hook, shift left] \arrow[r, "\mathrm{Conj}", "\sim"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[u, hook, shift left] \end{tikzcd}$$

& similarly for $\mathrm{Shv}_{Nilp}$ v.s. $\mathrm{LS}_{\check{G}}^{restr}$.

Consider spectral action $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{DMod}(\mathrm{Bun}_G) \longrightarrow \mathrm{DMod}(\mathrm{Bun}_G)$

from action of $\mathrm{Rep}(\check{G})$ (+ Kac-Moody localization)

By duality
$$\begin{tikzcd} \mathrm{DMod}(\mathrm{Bun}_G) \arrow[r] \arrow[dr, "\mathrm{coeff}^{enh}"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \otimes \mathrm{DMod}(\mathrm{Bun}_G) \arrow[d, "\mathrm{Id}\otimes \mathrm{coeff}"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \end{tikzcd}$$

where $\mathrm{coeff} = \Gamma \circ \mathrm{coeff}^{enh} : \mathrm{DMod}(\mathrm{Bun}_G) \to \mathrm{Vect}$.

$$\begin{tikzcd} & \mathrm{Bun}_N \arrow[dl, "p"'] \arrow[d] \arrow[dr] & \\ \mathrm{Bun}_G & \mathbb{G}_m & * \end{tikzcd}$$

"Poincaré" $\to p$

Then $\mathrm{coeff}(-) := \Gamma_*(p^!(-) \otimes^! \exp_{\mathrm{Bun}_N})$ [shift].

**Lem**

$$\begin{tikzcd} \mathrm{DMod}(\mathrm{Bun}_G) \arrow[rr, "\mathrm{coeff}"] \arrow[dr] & & \mathrm{Vect} \\ & \mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

**Lem** $\mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \curvearrowright \mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}}$ $\hookleftarrow$ stable under Satake actions.

$\Rightarrow$ $\mathrm{coeff}^{\mathrm{enh}}$ also factors through $\mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}}$.

**Rmk** $\mathcal{C} \to \mathcal{A}$ $\mathcal{A}$-linear $\Leftrightarrow$ $\mathcal{C} \to \mathrm{Vect}$ (when $\mathcal{A}$ is rigid)

b/c $\mathcal{A}$ rigid $\Rightarrow$ $\mathrm{Vect} \rightleftarrows \mathcal{A}$

(II) Why this functor $\mathrm{coeff} : \mathrm{DMod}(\mathrm{Bun}_G) \longrightarrow \mathrm{Vect}$ ?

$$\begin{tikzcd} (S^\circ \xrightarrow{W^\circ} \mathrm{Gr}_G) \in \mathrm{Whit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, "\text{Casselman-Shalika}"] \arrow[d, shift right] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \arrow[d, shift right] & \mathrm{triv}_{\mathrm{Ran}} \arrow[d, mapsto] \\ (\mathrm{Bun}_G^{N\text{-gen}} \hookleftarrow \mathrm{Bun}_N) \in \mathrm{Whit}(\mathrm{Bun}_G^{N\text{-gen}}) \arrow[u, shift right] \arrow[d, shift right, "(!\text{-push})"'] & & \\ \mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}} \arrow[u, shift right, "\mathrm{conj}"'] \arrow[r, "\sim"] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[uu, shift right, "\text{Sym monoidal}"'] & \mathcal{O}_{\mathrm{LS}_{\check{G}}} \end{tikzcd}$$

$S^\circ = N(K)\backslash G(K)/G(\mathcal{O})$

$\Rightarrow$ $\mathcal{O}_{\mathrm{LS}_{\check{G}}}$ should go to $p_!(\exp_{\mathrm{Ran}})$ (from $\mathrm{Bun}_G^{N\text{-gen}}$).

$\rightsquigarrow$ passing to duality & right adjoint:

$$\mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}} \longrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \xrightarrow{\ \Gamma\ } \mathrm{Vect}$$

should by given by coeff.

(III) What is $\mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}}$ ?

Local geom Langlands:

$\mathrm{DMod}(\mathcal{L}G)\text{-mod}$ (+ strong $G(K)$-action) $\xleftrightarrow[\sim]{\mathrm{conj}}$ (certain modification of $\mathrm{ShvCat}(\mathrm{LS}_{\check{G}}(\mathring{D}))$).

punctured disc

__Upshot__ $\mathrm{DMod}(LG)\text{-mod}^{\mathrm{temp}} \overset{\text{Conj}}{\underset{\sim}{\longleftrightarrow}} \mathrm{ShvCat}(\mathrm{LS}_{\check{G}}(\mathring{D}))$.

$$\mathcal{C} \longmapsto \mathrm{Whit}(\mathcal{C}).$$

__Def__ $\mathcal{C}$ is __tempered__ if $\mathrm{Whit}(\mathcal{C}) \subset \mathcal{C}$ generates $\mathcal{C}$ as a $\mathrm{DMod}(LG)$-mod.
(namely, nondegenerate Whit.)

Conj Define $\mathrm{BiWhit} := \mathrm{Whit}\backslash D(LG)/\mathrm{Whit}$.
Then (1) $\mathrm{Whit}(LG) \otimes_{\mathrm{BiWhit}} \mathrm{Whit}(\mathcal{C}) \longrightarrow \mathcal{C}$ is fully faithful.
(2) $\mathrm{BiWhit} = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathring{D}))$.

Much easier in unramified case:

__Claim__ "Derived Satake":

$$\begin{tikzcd} \mathrm{Sph}_G\text{-mod} \arrow[d, shift right, hook] & \mathcal{D} \arrow[d, mapsto] & \mathcal{C}^{L^+G} \\ \mathrm{DMod}(LG)\text{-mod} \arrow[u, shift right] & \mathrm{DMod}(\mathrm{Gr}_G)\otimes_{\mathrm{Sph}_G}\mathcal{D} & \mathcal{C}. \arrow[u, mapsto] \end{tikzcd}$$

adjoint (& fully faithful).

__Fact__ If $\mathcal{C}$ is unramified,
$\mathcal{C}$ is tempered iff $\mathcal{C}^{L^+G} \xrightarrow{\mathrm{Av}_!} \mathrm{Whit}\,\mathcal{C}$ is conservative.

__Fact__ If $\mathcal{C}$ is unramified,

$$\begin{tikzcd} \mathcal{C}^{\mathrm{temp}} \arrow[r, shift left, hook] & \mathcal{C} \arrow[l, shift left, two heads] \arrow[r, shift left, two heads] & \mathcal{C}^{\mathrm{antitemp}} \arrow[l, shift left, hook] \\ \mathrm{Sph}_G^{\mathrm{temp}} \arrow[r, shift left, hook] \arrow[u, "(-)\otimes_{\mathrm{Sph}_G}\mathcal{C}"] \arrow[d, equal] & \mathrm{Sph}_G \arrow[l, shift left, two heads] \arrow[r, shift left, two heads] & \mathrm{Sph}_G^{\mathrm{antitemp}} \arrow[l, shift left, hook] \arrow[d, phantom, "\wr"] \\ \mathrm{Sph}_G^{\mathrm{left\text{-}cpt}} & & \mathrm{Sph}_G^{\leq -\infty} \end{tikzcd}$$

std t-str (not left-complete)

(IV) __What is $\mathrm{DMod}(\mathrm{Bun}_G)^{\mathrm{temp}}$?__

For any $x \in X$, $\mathrm{Sph}_{G,x} \circlearrowright \mathrm{DMod}(\mathrm{Bun}_G)$
$\rightsquigarrow \mathrm{DMod}(\mathrm{Bun}_G)^{*\text{-temp}}$ (by $\mathrm{Sph}_G\text{-mod} \rightleftarrows \mathrm{DMod}(LG)\text{-mod}$).

__Thm__ [FR] This def'n does not depend on choice of $x$.

**Local picture** $\quad D(\mathcal{L}G)\text{-mod}^{temp} \simeq \mathrm{ShvCat}(\mathrm{LS}_{\check{G}}(\mathring{D})) = \mathrm{QCoh}(\mathrm{LS}_{\check{G}}(\mathring{D}))\text{-mod}$

**Conj** $\quad D(LG)^{left\text{-}temp} \simeq D(LG)^{right\text{-}temp}$

**Global shadow** $\qquad$ Use this to describe temp condition.

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Bun}_G)^{temp} \arrow[r] \arrow[d, shift right] & G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) \supseteq G(\mathbb{O})\backslash G(K_x)/G(\mathcal{O}_x). \arrow[d, shift right] \\
\mathrm{Whit}(\mathrm{Bun}_G^{N\text{-}gen}) \arrow[r] \arrow[u, shift right, two heads] & (N(\mathbb{A}),\chi)\backslash G(\mathbb{A})/G(\mathbb{O}) \arrow[u, shift right, two heads]
\end{tikzcd}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \updownarrow \qquad\quad \updownarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Whit} \qquad \mathrm{Bun}_G^{N\text{-}gen}$

**Proof of conservativity** (main thm)

**Step 1** [AGKRRV]

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Bun}_G)^{temp} \arrow[r] \arrow[d, shift right, "\text{Verdier quot}"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, shift right, "\text{formal cplt}"'] \\
\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{temp} \arrow[r] \arrow[u, shift right, bend right] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}^{restr}) \arrow[u, shift right, "!\text{-push}"']
\end{tikzcd}$$

$\downarrow$ Higgs bundle

**Step 2** $\quad \mathrm{Nilp}_{irreg} \hookrightarrow \mathrm{Nilp} \subseteq T^*\mathrm{Bun}_G = \{(\mathcal{F}_G, e) \mid e \in \Gamma(X, \mathfrak{g}^*_{\mathcal{F}_G} \otimes \Omega_X)\}.$

**Thm** $\quad \mathrm{Shv}_{\mathrm{Nilp},irreg}(\mathrm{Bun}_G) \subset \mathrm{DMod}(\mathrm{Bun}_G)^{antitemp}.$

**Conj** $\quad \mathrm{DMod}_{irreg}(\mathrm{Bun}_G) \simeq \mathrm{DMod}(\mathrm{Bun}_G)^{antitemp}.$

(Intuitively: "$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})^{irred} \longleftrightarrow \mathrm{QCoh}(\mathrm{LS}_{\check{G}})^{non\text{-}sm}$".)

**Step 3** $\quad \mathcal{F} \in \mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)^{constr'ble}$

Define char cycle $\quad CC(\mathcal{F}) := \sum_{\alpha \in \mathrm{Irr}(\mathrm{Nilp})} C_{\alpha,\mathcal{F}}[\alpha], \qquad C_{\alpha,\mathcal{F}} \in \mathbb{Z}.$

**Thm** $\quad \chi(\mathrm{coeff}(\mathcal{F})) = (-1)^{\dim \mathrm{Bun}_G} C_{\mathrm{Kos},\mathcal{F}}.$

$\qquad T^*\mathrm{Bun}_G = \underline{\mathrm{Maps}}(X, \mathfrak{g}/(G\times\mathbb{G}_m)) \times_{\underline{\mathrm{Map}}(X, \mathbb{B}\mathbb{G}_m)} \{\Omega_X\}.$

$\mathfrak{g}/\!/G \xrightarrow{s'} \mathfrak{g}^{reg}/G \subset \mathfrak{g}/G$.

$(\mathfrak{g}/G \to \mathfrak{g}/\!/G)$.

$$\begin{tikzcd}
\rightsquigarrow & T^*\mathrm{Bun}_G \arrow[rr, "\text{Hitchin}"] & & \Gamma(X, \mathfrak{g}/\!/G \overset{\mathbb{G}_m}{\times} \Omega_X) \arrow[ll, bend left] \\
& \mathrm{Kos}^{glob} \arrow[u, hook] \arrow[urr, leftrightarrow, "="'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Nilp}^{reg} \arrow[r] & 0 \\
f^{glob} \arrow[u, hook] &
\end{tikzcd}$$

(+ Smoothness as $\mathfrak{g}^{reg}/G$ sm).

&

Let $\mathrm{Nilp}^{Kos}$ be the unique irr comp containing $f^{glob}$.

$$\begin{tikzcd}
& f^{glob} \arrow[dl] \arrow[rr] & & \mathrm{Kos}^{glob} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Diamond"] & & \\
\mathrm{Nilp} \arrow[dr, hook] & & T^*\mathrm{Bun}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_N \arrow[dl, "(!)"'] \arrow[dr] & & \mathrm{Bun}_N \arrow[dl, "df"'] \arrow[dr] & \\
& T^*\mathrm{Bun}_G & & T^*\mathrm{Bun}_N & & \mathrm{pt}.
\end{tikzcd}$$

$\mathrm{Kos}^{glob}$ = fibre product.

<u>Step 4</u> By Step 3,

If $\mathrm{Nilp}^{Kos} \subseteq SS(\mathcal{F})$,

then $C_{Kos,\mathcal{F}} \neq 0 \Rightarrow \chi(\mathrm{coeff}(\mathcal{F})) \neq 0$

$\Rightarrow \mathrm{coeff}(\mathcal{F}) \neq 0$.

For any $\check{\Lambda}^+$-valued divisor $D$ on $X$,

$\rightsquigarrow V^D \in \mathrm{Rep}(\check{G})_X$.

If the sing supp $SS(\mathcal{F} \star V^D) \supset \mathrm{Nilp}^{Kos}$

then $\mathrm{coeff}(\mathcal{F} \star V^D) \neq 0$

$\Rightarrow \mathrm{coeff}_{enh}(\mathcal{F}) \neq 0$.

Thm If $SS(\mathcal{F}) \not\subset \mathrm{Nilp}_{irreg}$,
then $\exists D$ s.t. $\mathrm{Nilp}^{Kos} \subseteq SS(\mathcal{F} \star V^D)$.

Now if $\mathcal{F}$ is s.t. $\mathrm{Coeff}^{enh}(\mathcal{F}) = 0$,

$\Rightarrow \forall D,\ \mathrm{Nilp}^{Kos} \not\subset SS(\mathcal{F} \star V^D)$

$\Rightarrow SS(\mathcal{F}) \subset \mathrm{Nilp}_{irreg}$

$\Rightarrow \mathcal{F}$ is anti-tempered. $\square$