Beilinson-Drinfeld

Last time:

For any D_X-scheme $\mathcal{Y} \xrightarrow{\pi} X$, introduced

$$\mathrm{Op}_{\check{G}}(\mathcal{Y}) = \left\{ \begin{array}{l} \check{B}\text{-torsor } \mathcal{F}_{\check{B}} \text{ on } \mathcal{Y}; \\ \text{connection } \nabla \text{ on } \mathcal{F}_{\check{G}} := \check{G} \overset{\check{B}}{\times} \mathcal{F}_B \text{ relative to } \pi; \\ c(\nabla) \in \Gamma(\mathcal{Y}, (\check{\mathfrak{g}}^{-1}/\check{\mathfrak{b}})_{\mathcal{F}_{\check{B}}} \otimes \pi^* \Omega_X) \ \& \text{ is generic} \end{array} \right\}$$

$\rightsquigarrow$ a D_X-scheme, denoted by $\mathrm{Op}^{D}_{\check{G}} \longrightarrow X$.

Assume $\check{G}$ is of adjoint type.

<u>Prop</u>: $\mathrm{Op}_{\check{G}}(X)$ is an $\mathrm{Hitch}(X)$-torsor.

<u>Cor</u>:

$$\underset{\displaystyle A}{\underset{\shortparallel}{\Gamma(\mathrm{Op}_{\check{G}}(X), \mathcal{O})}} \overset{gr}{\rightsquigarrow} \underset{\displaystyle A^{cl}}{\underset{\shortparallel}{\Gamma(\mathrm{Hitch}(X), \mathcal{O})}}$$

<u>Prop</u> follows from the following results.

<u>Lem 1</u>: $\mathrm{Op}_{\check{G}}(X)$ is non-empty.

<u>Lem 2</u>: $\mathrm{Op}^{D}_{\check{G}}$ is a $W(\Omega_X \otimes \mathfrak{c})$-torsor.

Here $\Omega_X \otimes \mathfrak{c}$ is viewed as a vector bundle on X.

$$\mathrm{Sch}_{/X_{dR}} \longrightarrow \mathrm{Sch}_{/X}, \quad \mathcal{Y}^D \longmapsto \mathcal{Y}^D \underset{X_{dR}}{\times} X$$

(forgetting the "D"-structure)

has a right adjoint, a.k.a. Weil restriction, a.k.a. jets construction

$$\mathrm{Sch}_{/X} \xrightarrow{W} \mathrm{Sch}_{/X_{dR}}$$

$$(W(\Omega_X \otimes \mathfrak{c}))(D_x) = \mathrm{Jets}_x(\Omega_X \otimes \mathfrak{c})$$

Lem 3: Choose principle $sl_2 \to \check{\mathfrak{g}}$ (e, f, h).

There exists an $\check{B}$-torsor on X (depending on e) such that the underlying $\check{B}$-torsor of any $\check{G}$-oper is canonically isomorphic to its pullback along $Y \to X$.

Notation: Denote this $\check{B}$-torsor by $\mathcal{F}_{\check{B}}^{op}$.

(The corresponding map $pt \to Bun_{\check{B}}$ depends on e; But its image in $\pi_0(Bun_{\check{B}}(k))$ does not).

Cor 2

$$\begin{tikzcd} Op_{\check{G}}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & LS_{\check{G}}(X) \arrow[d] \\ * \arrow[r, "\mathcal{F}_{\check{G}}^{op}"'] & Bun_{\check{G}}(X) \end{tikzcd}$$

Idea of proof:

Step 1: First suppose $\mathcal{F}_{\check{B}}$ is trivial and X admits a coordinate (e.g. this can be achieved by replacing X & Y by Zariski opens)

Claim: There is a unique trivialization of $\mathcal{F}_{\check{B}}$ s.t. ∇ is of the form

$$\nabla = d + (f + p)dt, \quad p \in \mathfrak{g}^e.$$

Note that by the claim

Lem 1 $\Rightarrow$ Lem 3.

Claim $\Leftarrow$ Kostant's Lem.

$$\begin{array}{c} Ad(B) \overset{\text{free}}{\curvearrowright} f + b \\ \uparrow \text{ section} \\ f + \mathfrak{g}^e \end{array}$$

Step 2:

- Case of PGL_2, oper-condition gives a trivialization of

$$(\check{\mathfrak{g}}/\check{\mathfrak{b}})_{\mathcal{F}_{\check{B}}} \otimes \pi^*\Omega_X \rightsquigarrow \check{\mathfrak{n}}_{\mathcal{F}_{\check{B}}} \simeq \pi^*\Omega_X.$$

$$\check{\mathfrak{n}}_{\mathcal{F}_{\check{B}}} \otimes \Omega_X \simeq \pi^*\Omega_X^{\otimes 2}$$

The claim implies $Op_{\check{G}}(Y)$ is a possibly empty torsor of $\pi^*\Omega_X^{\otimes 2}$.

It is indeed nonempty because

$$H^1(X, \Omega_X^{\otimes 2}) = 0 \quad (\text{if } g>1)$$

& direct construction when $g=0,1$.

$\Rightarrow$ Lem 1, 2, 3 for PGL_2 $(\Omega_X \otimes G \simeq \Omega_X^{\otimes 2})$.

Cor 2 $\Leftarrow$ $R\Gamma(X, \check{\mathfrak{n}}_{\mathcal{F}_{\check{B}}^{op}} \otimes \Omega_X) \simeq R\Gamma(X, \check{\mathfrak{g}}_{\mathcal{F}_{\check{B}}^{op}} \otimes \Omega_X)$

$$\left(\begin{array}{ccccccc} 0 \subset & \check{\mathfrak{n}}_{\mathcal{F}_{\check{B}}^{op}} & \subset & \check{\mathfrak{b}}_{\mathcal{F}_{\check{B}}^{op}} & \subset & \check{\mathfrak{g}}_{\mathcal{F}_{\check{B}}^{op}} & \text{iterated} \\ & \downarrow & & \downarrow & & \downarrow & \text{extension} \\ & \Omega_X & & \mathcal{O}_X & & \Omega_X^{-1} & \end{array} \right)$$

Step 3:

$(\mathcal{F}_{\check{B}_0}^{op}, \nabla_0)$ PGL_2-oper.

induce along $PGL_2 \longrightarrow \check{G}$ (adjoint type)

gives a $\check{G}$-oper. $\boxed{\text{Lem 1}}$

$\check{B}_0$ (= Borel of PGL_2) $\curvearrowright$ $\check{\mathfrak{g}}^e$.

$$(\check{\mathfrak{g}}^e)_{\mathcal{F}_{\check{B}_0}^{op}} \otimes \Omega_X \simeq G \otimes \Omega_X \qquad \boxed{\text{Lem 2}}$$

Cor 2 is similar to PGL_2-case. $\square$

$(\hat{\mathfrak{a}}_{crit} / \mathfrak{L}^+\mathfrak{g}\, \hat{\mathfrak{a}}_{crit})^{\mathfrak{L}^+G}$

<u>Notation</u>:

$(\mathfrak{Z}_{:})$ $\qquad$ is

$$\mathfrak{Z}_{crit,x} := Z(\hat{\mathfrak{a}}_{crit}) \qquad \mathfrak{z}_{crit}(D_x) = \mathrm{End}(\mathbb{V}_{crit,x})$$

$\mathfrak{z}_{crit} \in \mathrm{CAlg}(\mathrm{DMod}(X))$, $\quad \mathrm{Spec}\, \mathfrak{z}_{crit}$, D-schem

$\mathfrak{z}^{cl}_{crit}(D) = \mathrm{Sym}(\mathfrak{L}\mathfrak{g}/\mathfrak{L}^+\mathfrak{g})^{\mathfrak{L}^+G}$

$\mathrm{Spec}(\mathfrak{z}_{crit}) \simeq \mathrm{Op}_{\check{G}}$ torsor for

$\mathrm{Spec}(\mathfrak{z}^{cl}_{crit}) \simeq \mathbb{V}(\Omega_x \otimes \check{\mathfrak{c}})$

<u>Thm</u> [FF]:

- $\mathrm{Spf}\, \mathfrak{Z}_{crit,x} \simeq \mathrm{Op}_{\check{G}}(\overset{\circ}{D}_x)$
- $\mathrm{Spec}\, \mathfrak{z}_{crit}(D_x) \simeq \mathrm{Op}_{\check{G}}(D_x)$
- $\underline{\mathrm{Spec}\, \mathfrak{z}_{crit}} \simeq \mathrm{Op}_{\check{G}}$ as D_x-scheme.

- Idea of proof:

<u>Step 1</u>: $\mathbb{V}_{crit,x}$, x moves $\rightsquigarrow$ chiral algebra $\mathbb{V}_{crit}$.

$$\hat{\mathfrak{g}}_{crit}\text{-mod} \simeq \mathbb{V}_{crit}\text{-mod}^{ch}.$$

(chiral "=" factorization "=" VOA)

It follows formally $\quad \mathfrak{z}_{crit} = Z(\mathbb{V}_{crit})$

$\Downarrow$

$\hat{\mathfrak{g}}_{crit}$-mod lives over $\mathrm{Spf}\, \mathfrak{z}_{crit}(\overset{\circ}{D}_x)$

Hence

$$\mathrm{Spf}\, \mathfrak{Z}_{crit,x} \longrightarrow \mathrm{Spf}\, \mathfrak{z}_{crit}(\overset{\circ}{D}_x)$$

<u>Step 2</u>: Fact: This is iso.

Hence only needs to show $\mathrm{Spec}\, \mathfrak{z}_{crit} \simeq \mathrm{Op}_{\check{G}}$.

Only needs to show $\mathrm{Spec}\, \mathfrak{z}_{crit}(D) \simeq \mathrm{Op}_{\check{G}}(D)$ compatible with $\mathrm{Aut}(D)$-actions.

Step 3 : Construct $\mathrm{Spec}\, \mathfrak{z}_{crit}(D) \longrightarrow \mathrm{Op}_{\check{G}}(D)$. as follows.

Need: a $\check{G}$-torsor on $\mathrm{Spec}\, \mathfrak{z}_{crit}(D) \times D$.

w/ a connection rel. D, w/ a $\check{B}$-reduction

satisfying oper conditions, equivariant for $\mathrm{Aut}(D)$ actions.

$\Updownarrow$ $\left(\mathrm{Aut}^0(D) \subset \mathrm{Aut}(D)\right.$ fixing centers of disk.$\left.\right)$

An $\mathrm{Aut}(D)$-equ. $\check{G}$-torsor $\mathcal{P}_{\check{G}}$ on $\mathrm{Spec}\, \mathfrak{z}_{crit}(D)$

w/ $\mathrm{Aut}^0(D)$-equ $\check{B}$-reduction $\mathcal{P}_{\check{B}}$

w/ $\mathcal{P}_{\check{T}} \simeq \mathrm{triv} \otimes \underbrace{\rho(\omega_D / t\omega_D)}_{\text{a line}}$ $\mathrm{Aut}^0(D)$ equ.

$$\begin{tikzcd}
\mathrm{Rep}(\check{G}) \arrow[r, "F"] \arrow[d, "\mathrm{Sat}"] & \mathfrak{z}_{crit}(D)\text{-mod} \\
\mathrm{DMod}(\mathrm{Gr}_G)^{L^+G}_{crit} \arrow[r, "\Gamma"] & \hat{\mathfrak{g}}_{crit}\text{-mod}^{L^+G} \arrow[u, "\underline{\mathrm{Hom}}(\mathbb{V}_{crit}, -)"]
\end{tikzcd}$$
$(\Gamma(\delta_{1,crit}) = \mathbb{V}_{crit})$

$$\boxed{\text{Fact: } \begin{array}{c} \Gamma \circ \mathrm{Sat}(-) \\ \text{is} \\ F(-) \otimes_{\mathfrak{z}_{crit}(D)} \mathbb{V}_{crit} \end{array}}$$

F has a canonical $\mathrm{Aut}(D)$-equ. sym. mon. structure

$\rightsquigarrow \mathcal{P}_{\check{G}}$.

Now need $\rho(\omega_D / t\omega_D) \longrightarrow F(V^{\check{\lambda}})$ satisfying Plucker.

$(L_0 = -t\partial_t \in \mathrm{Lie}(\mathrm{Aut}^0))$

$$F(V^{\check\lambda}) = \Gamma(\mathrm{Gr}_G, IC^{\check\lambda}_{crit})^{\mathfrak{L}^+G}$$

lowest L_0-eigenvalue of $\Gamma(\quad)$ is $-\langle\check\lambda,\rho\rangle$

eigenspace dim. $=1$, it is $\mathfrak{L}^+G$-inv.

$\rightsquigarrow$ the desired line.

<u>Step 4</u>: The map in Step 3 is an isomorphism. $\square$

<u>Construction</u>:

Construct

$$\begin{tikzcd} \mathfrak{Z}_{crit}(X) \arrow[r] & \Gamma(\mathrm{Bun}_G, D_{crit}) \arrow[d, squiggly, "gr"] \\ \mathfrak{Z}^{cl}_{crit}(X) \arrow[r] & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}) \end{tikzcd}$$

as follows.

Local picture:

$$\mathfrak{Z}_{crit}(D_x) \longrightarrow \Gamma(\mathrm{Bun}_G, D_{crit})$$

$$\parallel$$

$$(\hat{U}_{crit} / \mathfrak{L}^+\mathfrak{g}\,\hat{U}_{crit})^{\mathfrak{L}^+G}$$

In general, if $(\mathfrak{h}, K) \curvearrowright \tilde{Y}$, $Y := \tilde{Y}/K$, then

$$(U(\mathfrak{h}) / \mathfrak{k}\, U(\mathfrak{h}))^K \longrightarrow \Gamma(Y, D_Y).$$

x moves $\rightsquigarrow$ $\mathfrak{Z}_{crit} \longrightarrow \Gamma(\mathrm{Bun}_G, D_{crit}) \otimes \mathcal{O}_X$

compatible with connections.

taking H_{∇}.

<u>But</u>: This needs $\mathfrak{z}_{crit}$ & $\Gamma(\mathrm{Bun}_G, D_{crit})$ to be commutative.

<u>Fact</u>: Both are true.

<u>Now</u>:

$$\begin{tikzcd}
A = \Gamma(\mathrm{Op}_{\check{G}}(X), \mathcal{O}) \simeq \mathfrak{z}_{crit}(X) \arrow[r] & \Gamma(\mathrm{Bun}_G, D_{crit}) \arrow[d, squiggly, "gr"] \\
A^{cl} \simeq \Gamma(\mathrm{Hitch}(X), \mathcal{O}) \simeq \mathfrak{z}^{cl}_{crit}(X) \arrow[r] & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O})
\end{tikzcd}$$

Given $\sigma \in \mathrm{Op}_{\check{G}}(X)$, $\underline{\sigma} \in \mathrm{LS}_{\check{G}}(X)$,

$$\mathrm{Aut}_\sigma := D_{crit} \otimes_{\mathfrak{z}_{crit}(X)} k_\sigma \in \mathrm{DMod}(\mathrm{Bun}_G).$$

<u>Thm</u>: Aut_σ is a Hecke eigensheaf for $\underline{\sigma}$.

<u>Localization</u>:

$$\hat{\mathfrak{g}}_{crit}\text{-mod}^{\mathfrak{L}^+G} \xrightarrow{\mathrm{Loc}_x} \mathrm{DMod}_{crit}(\mathrm{Bun}_G)$$

<u>Fact</u>: $\mathrm{Loc}_x(\mathbb{V}_{crit}) \simeq D_{crit}$.

Proof:

$$\mathrm{Sat}_x(V) \star \mathbb{D}_{crit} \simeq \mathrm{Loc}_x(\mathrm{Sat}_x(V) \star \mathbb{V}_{crit,x})$$
$$\simeq \mathrm{Loc}_x(\Gamma(\mathrm{Gr}_{G,x}, \mathrm{Sat}_x(V)))$$

By definition

$$\Gamma(\mathrm{Gr}_G, \mathrm{Sat}(V)) = F(V) \underset{\mathfrak{Z}_{crit}(D)}{\otimes} \mathbb{V}_{crit}$$

and $F(V)$ is the bundle on $\mathrm{Spec}(\mathfrak{Z}_{crit}(D))$ $\overset{\text{is}}{\ldots}$ $\mathrm{Op}_{\check{G}}(D)$

associated to the universal $\check{G}$-torsor.

Let $\mathcal{F}$ be the universal $\check{G}$-torsor on $\mathrm{Op}_{\check{G}}(X)$, so that

$$F(V) \simeq V_{\mathcal{F}} \underset{\mathfrak{Z}_{crit}(X)}{\otimes} \mathfrak{Z}_{crit}(D_x)$$

Then

$$\mathrm{Sat}_x(V) \star \mathbb{D}_{crit} \simeq \mathrm{Loc}_x(V_{\mathcal{F}} \underset{\mathfrak{Z}_{crit}(X)}{\otimes} \mathbb{V}_{crit,x})$$
$$\simeq V_{\mathcal{F}} \underset{\mathfrak{Z}_{crit}(X)}{\otimes} \mathbb{D}_{crit}.$$

tensoring with $- \underset{\mathfrak{Z}_{crit}(X)}{\otimes} k_\sigma$.

$$\mathrm{Sat}_x(V) \star \mathrm{Aut}_\sigma \simeq V_{\mathcal{F}}|_\sigma \otimes \mathrm{Aut}_\sigma.$$

Let x moves. $\square$.

In above, the key idea is

$$\begin{tikzcd} \mathrm{KL}^{\mathrm{Vac\text{-}gen}}_{\mathrm{crit},x} \arrow[r, "\mathrm{Loc}_x"] & \mathrm{DMod}_{\mathrm{crit}}(\mathrm{Bun}_G) \\ \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(D_x)) \arrow[u, phantom, "\circlearrowleft"] \arrow[r] & \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(X)) \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

Here Vac-gen means the part generated by the vacuum.

Next time:

$$\mathrm{KL}_{\mathrm{crit},x} \simeq \mathrm{QCoh}\big(\overbrace{\mathrm{Op}_{\check{G}}(\mathring{D}_x) \underset{\mathrm{LS}_{\check{G}}(\mathring{D}_x)}{\times} \mathrm{LS}_{\check{G}}(D_x)}^{\mathrm{Op}^{\mathrm{unr}}_{\check{G}}(\mathring{D}_x)}\big)$$

and Loc_x can be enhanced to

$$\mathrm{KL}_{\mathrm{crit},x} \underset{\mathrm{Op}^{\mathrm{unr}}_{\check{G}}(\mathring{D}_x)}{\otimes} \mathrm{Op}^{\mathrm{unr}}_{\check{G}}(X-x) \longrightarrow \mathrm{DMod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$