FLE

Last time:

We explained

- $\operatorname{Spec} \mathfrak{Z}_{crit} = \mathrm{Op}^{\mathcal{D}}_{\check{G}}$ as D-scheme
- quantization of Hitchin system

$$\begin{tikzcd}
A_x := \Gamma(\mathrm{Op}_{\check{G}}(X), \mathcal{O}) = \mathfrak{Z}_{crit}(X) \arrow[r] & \Gamma(\mathrm{Bun}_G, \mathcal{D}_{crit}) & (\mathfrak{L}_x\mathfrak{g}, \mathfrak{L}^+_x G) \curvearrowright \mathrm{Bun}_G^{\mathrm{level}_x} \arrow[l, squiggly] \\
\mathfrak{Z}_{crit}(\mathcal{D}_x) \simeq \left(\hat{U}_{crit} / \mathfrak{L}^+_x\mathfrak{g}\,\hat{U}_{crit}\right)^{\mathfrak{L}^+_x G} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

- $\sigma \in \mathrm{Op}_{\check{G}}(X)$. $\mathrm{Aut}_\sigma := \mathcal{D}_{crit} \otimes_A k_\sigma$ is Hecke-eigen for σ

The proof of the Hecke-eigenproperty uses two key inputs:

①
$$\begin{tikzcd}
\hat{\mathfrak{g}}_{crit}\text{-mod}_x^{\mathbb{W}\text{-gen}} \arrow[r, "\mathrm{Loc}_x"] & \mathrm{DMod}_{crit}(\mathrm{Bun}_G) & \text{(quant. Hitch.)} \\
\mathrm{QCoh}(\mathrm{Op}_{\check{G}}(\mathcal{D}_x)) \arrow[u, phantom, "\circlearrowright"] \arrow[r, "\text{pullback}"] & \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(X)) \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

② $$\mathrm{Sat}(V) \star \mathbb{W}_{crit} = \Gamma(\mathrm{Gr}_G, \mathrm{Sat}(V)) \simeq \mathcal{F}(V) \otimes_{\mathfrak{Z}_{crit}(\mathcal{D})} \mathbb{W}_{crit}$$

Here $\mathcal{F}(V)$ is the vector bundle on

$$\mathrm{Op}_{\check{G}}(\mathcal{D}) \simeq \operatorname{Spec}(\mathfrak{Z}_{crit}(\mathcal{D}))$$

provided by $V \in \mathrm{Rep}(\check{G})$ and the universal $\check{G}$-torsor.

Today, generalize both of them: which is called Fundamental Local Equivalence (FLE)

Thm [F-G, ...]

t-exact

- $\hat{\mathfrak{g}}_{crit}\text{-mod}^{\mathfrak{L}^+G} \simeq \mathrm{IndCoh}(\mathrm{Op}^{unr}_{\check{G}})$

compatible with Satake.

Def:

$$\mathrm{Op}^{unr}_{\check{G},x} := \mathrm{Op}_{\check{G}}(\overset{\circ}{D}_x) \underset{\mathrm{LS}_{\check{G}}(\overset{\circ}{D}_x)}{\times} \mathrm{LS}_{\check{G}}(D_x).$$

$$\mathrm{Op}^{unr}_{\check{G}} \neq \mathrm{Op}_{\check{G}}(D) =: \mathrm{Op}^{reg}_{\check{G}}.$$

$\mathbb{C}$-point: $\nabla = d + \omega$ $\quad \omega \in \mathfrak{g}^{-1}((t))dt$

$\omega \bmod \mathfrak{b}((t))$, generic

such that under certain $G((t))$-gauge, $\nabla \mapsto \nabla'$

$$\nabla' - d \in \mathfrak{g}[[t]]\,dt.$$

However, it is not true that $\exists\, \nabla' \sim \nabla$ s.t.

$\nabla' - d \in \mathfrak{g}^{-1}[[t]]\,dt$ & $(\nabla' - d) \bmod \mathfrak{b}[[t]]$ is generic (const. terms $\neq 0$)

In fact, for any dominant weight $\check{\lambda}$,

$$\mathrm{Op}^{reg,\check{\lambda}}_{\check{G}} \hookrightarrow \mathrm{Op}^{unr}_{\check{G}}$$

$\nabla' - d \in \mathfrak{g}^{-1}[[t]]\,dt$

& $(\nabla' - d) \bmod \mathfrak{b}[[t]]\,dt$ is of the form

$$\sum_i t^{\langle \check{\lambda}, \alpha_i \rangle} \cdot \varphi_i \cdot (-\alpha_i)\,dt, \quad \varphi_i \in k[[t]],\ \varphi_i(0) \neq 0.$$

Fact: $\mathrm{Op}^{\mathrm{unr}}_{\check{G}}(k) \simeq \bigcup_{\check{\lambda}} \mathrm{Op}^{\mathrm{reg},\check{\lambda}}_{\check{G}}(k)$.

$$\mathrm{Op}^{\mathrm{unr}}_{\check{G}} \simeq \coprod_{\check{\lambda}} \mathrm{Op}^{\mathrm{unr},\check{\lambda}}_{\check{G}}$$

$$\mathrm{Op}^{\mathrm{reg},\check{\lambda}}_{\check{G}} \hookrightarrow \left(\mathrm{Op}^{\mathrm{unr},\check{\lambda}}_{\check{G}}\right)_{\mathrm{red}}$$

$$\boxed{\begin{gathered} \mathrm{FLE}_{\mathrm{crit}}(\mathbb{W}^{\check{\lambda}}_{\mathrm{crit}}) = \mathfrak{z}^{\mathrm{reg},\check{\lambda}} = \mathrm{End}(\mathbb{W}^{\check{\lambda}}_{\mathrm{crit}}). \\ \mathrm{Spec}\, \mathfrak{z}^{\mathrm{reg},\check{\lambda}} \simeq \mathrm{Op}^{\mathrm{reg},\check{\lambda}}_{\check{G}}. \end{gathered}}$$

$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$ lives over $\mathrm{Spf}\, \mathfrak{Z}_{\mathrm{crit}} \simeq \mathrm{Op}_{\check{G}}(\overset{\circ}{D})$

The theorem implies

$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathfrak{L}G}$ is supported on $\mathrm{Op}^{\mathrm{unr}}_{\check{G}}$.

unramified on geometric side $\Leftrightarrow$ unramified in spectral side.

This is in fact proven by [FG] before the theorem.

The functor $\mathrm{FLE}_{\mathrm{crit}}$ is Drinfeld–Sokolov reduction:

$\Psi(M) := C^{\frac{\infty}{2}}(n((t)), M \otimes \chi)$ is the underlying vector space.

$\chi : n((t)) \to \mathbb{G}_a$

<u>Thm</u>

$$\hat{\mathfrak{g}}_{crit}\text{-mod}_{Whit} \xrightarrow[\sim]{\Psi} \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathring{D})).$$

Satake action:

$$\mathrm{Dmod}(\mathfrak{L}^+G\backslash \mathfrak{L}G/\mathfrak{L}^+G)_{crit} = \mathrm{IndCoh}\left(\mathrm{LS}_{\check{G}}(D) \times_{\mathrm{LS}_{\check{G}}(\mathring{D})} \mathrm{LS}_{\check{G}}(D)\right)$$

$$\hat{\mathfrak{g}}_{crit}\text{-mod}^{\mathfrak{L}^+G} \simeq \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{unr})$$

In FG paper, they used abelian categories and abelian Satake. But this is also true.

Why FLE_{crit}?

$$\mathrm{crit}_G + c\,\mathrm{Kil}_G \xleftrightarrow{\text{dual}} \mathrm{crit}_{\check{G}} + \frac{1}{c}\mathrm{Kil}_{\check{G}}.$$

$$\kappa \qquad\qquad\qquad \check{\kappa}$$

<u>Thm</u> (......) $\quad c \neq 0$.

$$\hat{\mathfrak{g}}_\kappa\text{-mod}^{\mathfrak{L}^+G} \overset{\mathrm{FLE}_\kappa}{\simeq} \mathrm{Whit}(D(\mathrm{Gr}_{\check{G}})_{\check{\kappa}})$$

[KL] $\qquad$ // (original conjecture by Lurie)

$$\mathrm{Rep}_q(G)$$

If $c \to \infty$, $\hat{\mathfrak{g}}_\kappa\text{-mod}^{\mathfrak{L}^+G} \rightsquigarrow \mathrm{Rep}(G)$.

$q \to 1$

$$\mathrm{Whit}_{crit}(D(Gr_{\check{G}})) = \mathrm{Rep}(G)$$

(C-S).

If $c \to 0$ $\quad \mathrm{Whit}(D(Gr_{\check{G}})_{\check{\kappa}}) \rightsquigarrow \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}^{unr})$.

Why FLE_κ ? quantum local geometric Langlands.

$$\mathfrak{L}G\text{-mod}_\kappa \overset{\text{Conj}}{\simeq} \mathfrak{L}\check{G}\text{-mod}_{\check{\kappa}}$$

$$\hat{\mathfrak{g}}_\kappa\text{-mod} \longmapsto \mathrm{Whit}(D(\mathfrak{L}\check{G})_{\check{\kappa}})$$

$$\mathrm{Whit}(D(\mathfrak{L}G)_\kappa) \longmapsto \hat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}$$

$$D_\kappa(Gr_G) \longmapsto D_{\check{\kappa}}(Gr_{\check{G}})$$

$$D_\kappa(Fl_G) \longmapsto D_{\check{\kappa}}(Fl_{\check{G}})$$

$$D_\kappa(Bun_G^{local})\longmapsto D_{\check{\kappa}}(Bun_{\check{G}}^{local})$$

$\kappa \to crit \qquad \check{\kappa} \to \infty$

$$\mathfrak{L}\check{G}\text{-mod}_\infty \rightsquigarrow \mathrm{ShvCat}(LS_{\check{G}}(\mathring{D}))$$

$$\mathrm{Whit}(D(\mathfrak{L}\check{G})_\infty) \rightsquigarrow \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathring{D}))$$

$$\hat{\mathfrak{g}}_\infty\text{-mod} \rightsquigarrow \mathrm{IndCoh}(LS_{\check{G}}(\mathring{D}))$$

$$D_\infty(Gr_G) \rightsquigarrow \mathrm{IndCoh}(LS_{\check{G}}(D))$$

$$\mathcal{L}G\text{-mod}_{\mathrm{crit}} \overset{\mathrm{Conj}}{\simeq} \mathrm{ShvCat}(\mathrm{LS}_{\check{G}}(\mathring{D}))$$

$$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod} \longmapsto \mathrm{IndCoh}(\mathrm{Op}_{\check{G}}(\mathring{D}))$$

In other words, if $\mathcal{C}$ is the universal $\mathcal{L}G$-module cat's on $\mathcal{L}S_{\check{G}}(\mathring{D})$, then

$$\mathcal{C}|_{\mathrm{Op}_{\check{G}}(\mathring{D})} \text{ is } \hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}.$$

Note indeed $\hat{\mathfrak{g}}_{\mathrm{crit}}$-mod lives on $\mathrm{Op}_{\check{G}}(\mathring{D})$
$$= \mathrm{Spec}\, \mathfrak{Z}_{\mathrm{crit}}.$$

Factorization FLE

$$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}} \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{unr}}_{\check{G}})_{\mathrm{Ran}}$$

<u>Thm:</u>

$$\begin{array}{ccc}
\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}} & \xrightarrow[\sim]{\mathrm{FLE}_{\mathrm{crit}}} & \mathrm{IndCoh}(\mathrm{Op}^{\mathrm{unr}}_{\check{G}})_{\mathrm{Ran}} \\
\downarrow{\scriptstyle \mathrm{Loc}} & & \downarrow{\scriptstyle \mathrm{Poinc}^{\mathrm{spec}}} \\
\mathrm{Dmod}(\mathrm{Bun}_G)_{\mathrm{crit}} & \xrightarrow{\mathbb{L}} & \mathrm{IndCoh}(\mathrm{LS}_{\check{G}})
\end{array}$$

(Recall $\mathbb{L}$ is constructed using spectral. decomp. & duality)

$$\begin{array}{ccc} \downarrow \scriptstyle \mathrm{coeff} & & \downarrow \scriptstyle \mathrm{P}^{\mathrm{Spec}} \\ \mathrm{Whit}_{\mathrm{crit}}(\mathrm{Gr}_G)_{\mathrm{Ran}} & \xrightarrow[\sim]{\mathrm{FLE}_\infty} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \end{array}$$

Here $\mathrm{Poinc}^{\mathrm{Spec}}$ is the functor

$$\mathrm{Op}_{\check{G}}(\overset{\circ}{X})^{\mathrm{unr}} = \underbrace{\mathrm{Op}_{\check{G}}(\overset{\circ}{X}) \times_{\mathrm{LS}_{\check{G}}(\overset{\circ}{X})} \mathrm{LS}_{\check{G}}(X)}_{\downarrow \, \mathrm{push}} \xrightarrow{\mathrm{pull}} \mathrm{Op}_{\check{G}}(\overset{\circ}{D}) \times_{\mathrm{LS}_{\check{G}}(\overset{\circ}{D})} \mathrm{LS}_{\check{G}}(D)$$

$$\mathrm{LS}_{\check{G}}(X)$$

Later we will see essentially IL being equivalence is proven using this diagram and many other things.

Haven't proven vanishing conjecture yet!

$$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathfrak{L}^+G}_{x_1,\dots,x_n} = \bigotimes_{x_i} \hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathfrak{L}^+G}_{x_i} \qquad \underline{x} = (x_1,\dots,x_n)$$

$$\begin{array}{ccc} \mathrm{Loc}_{\underline{x}} : \hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathfrak{L}^+G}_{\underline{x}} & \longrightarrow & \mathrm{Dmod}(\mathrm{Bun}_G) \\ \circlearrowleft & & \circlearrowleft \\ \mathrm{QCoh}(\mathrm{Op}^{\mathrm{unr}}_{\check{G},\underline{x}}) & & \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(X)) \end{array}$$

Fact: $\mathrm{Loc}_{\underline{x}}$ can be enhanced to

$$\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{x}^{\mathcal{L}^+G} \underset{\mathrm{QCoh}(\mathrm{Op}^{\mathrm{unr}}_{\check{G},\underline{x}})}{\otimes} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}(X-\underline{x})^{\mathrm{unr}})$$

$$\longrightarrow \mathrm{DMod}(\mathrm{Bun}_G)$$

$\rightsquigarrow$ $\mathrm{Rep}(\check{G})_{\underline{x}}$-action on $\mathrm{Im}(\mathrm{Loc}_{\underline{x}})$

factors through $\mathrm{LS}_{\check{G}}(X)$.

Claim: When $\underline{x}$ longer and longer, generate all cuspidal.

In fact: for any q.cpt $U \subset \mathrm{Bun}_G$.

$$\begin{tikzcd}
\hat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}^{\mathcal{L}^+G}_{\mathrm{Ran}} \arrow[r] \arrow[dr, "\text{Vanishes quotient}"'] & D(\mathrm{Bun}_G)_{\mathrm{cusp}} \arrow[d] \\
& D(U)_{\mathrm{cusp}}
\end{tikzcd}$$

Reduces to deal with Eis. By induction & compatibility of FLG with Eis/CT.