Main Thm:

$$\mathbb{L}_G : \mathrm{DMod}_{crit}(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})$$

compatible with derived Satake, Eisenstein series, duality etc.

Recall:

Generalized vanishing conjecture ⇒

$$\begin{tikzcd}
\mathrm{DMod}_{crit}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G^0"] \arrow[d, "\mathrm{coeff}_G"'] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}}) \arrow[d, "\Gamma^{spec}_{\check{G}}"] \\
\mathrm{Whit}_{crit}(\mathrm{Gr}_G)_{\mathrm{Ran}} \arrow[r, phantom, "\simeq"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}}
\end{tikzcd}$$

($\mathbb{L}_G^0$ will be $\Psi \circ \mathbb{L}_G$, $\Psi : \mathrm{IndCoh}_{Nilp} \twoheadrightarrow \mathrm{QCoh}$)

-
$$\begin{tikzcd}
\mathrm{DMod}_{crit}(\mathrm{Bun}_G)^c \arrow[r, "\mathbb{L}_G^0"] \arrow[ddr] & \mathrm{QCoh}(\mathrm{LS}_{\check{G}})^{>-\infty} \arrow[d, phantom, "\wr"] \\
 & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})^{>-\infty} \arrow[d, hook] \\
 & \mathrm{IndCoh}_{Nilp}(\mathrm{LS}_{\check{G}})
\end{tikzcd}$$
(by [FR])

$\mathbb{L}_G$ is obtained by ind-completion.

Shorthand:

$$\mathrm{D}(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_G} \mathrm{QC}^!(\mathrm{LS}_{\check{G}})$$

By construction, compatible with derived Satake

Thm: (Eis, CT)

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] \arrow[d, shift left, "\mathrm{CT}_*^-"] & \mathrm{QC}^!(\mathrm{LS}_{\check{G}}) \arrow[d, shift left, "\mathrm{CT}^-_{spec}"] \\
\mathrm{D}(\mathrm{Bun}_M) \arrow[r, "\mathbb{L}_M"] \arrow[u, shift left, "\mathrm{Eis}_!^-"] & \mathrm{QC}^!(\mathrm{LS}_{\check{M}}) \arrow[u, shift left, "\mathrm{Eis}^-_{spec}"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{Bun}_{P^-} \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_M
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathrm{LS}_{\check{P}^-} \arrow[dl] \arrow[dr] & \\
\mathrm{LS}_{\check{G}} & & \mathrm{LS}_{\check{M}}
\end{tikzcd}$$

Proof of Eis :

$$\begin{tikzcd}
 & \mathrm{Whit}(G) \arrow[rr, "\sim"] & & \mathrm{Rep}(\check{G}) \\
\mathrm{Whit}(M) \arrow[ur] \arrow[rr] & & \mathrm{Rep}(\check{M}) \arrow[ur] & \\
 & D(\mathrm{Bun}_G) \arrow[uu] \arrow[rr] & & QC^!(LS_{\check{G}}) \arrow[uu] \\
D(\mathrm{Bun}_M) \arrow[uu] \arrow[ur] \arrow[rr] & & QC^!(LS_{\check{M}}) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Need top and two side squares commute.

- Spectral side:

$$\begin{tikzcd}
\mathrm{Rep}(\check{M}) \arrow[r, dashed, "?"] & \mathrm{Rep}(\check{G}) \\
QC^!(LS_{\check{M}}) \arrow[u, "\Gamma_{spec}"] \arrow[r, "\mathrm{Eis}^-_{spec}"] & QC^!(LS_{\check{G}}) \arrow[u, "\Gamma_{spec}"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & LS_{\check{P}^-} \arrow[dl] \arrow[dr] & \\
LS_{\check{M}} & & LS_{\check{G}}
\end{tikzcd}$$

Paradigm: For $H_1 \to H_2$

$$\begin{tikzcd}
LS_{H_1} \arrow[r] \arrow[d] & (\mathbb{B}H_1)_x \arrow[d] \\
LS_{H_2} \arrow[r] & (\mathbb{B}H_2)_x
\end{tikzcd}$$

$$\begin{tikzcd}
QC^!(LS_{H_2}) \arrow[r, "\Gamma"] & \mathrm{Rep}(H_2)_x \\
QC^!(LS_{H_1}) \arrow[u, "push"] \arrow[r, "\Gamma"] & \mathrm{Rep}(H_1)_x \arrow[u, "inv"']
\end{tikzcd}$$

But no obvious diagram for pull. vs Γ.

Dually pull vs Loc

$$\begin{tikzcd}
QC^!(LS_{H_2}) \arrow[d, "pull"] & Rep(H_2)_x \arrow[l, "Loc"'] \arrow[d, "res"] \\
QC^!(LS_{H_1}) & Rep(H_1)_x \arrow[l, "Loc"']
\end{tikzcd}$$

However, there is a diagram for push vs Loc:

Lem: If $H_1 \twoheadrightarrow H_2$ has unipotent kernel,

$$\begin{tikzcd}
QC^!(LS_{H_2}) & Rep(H_2)_{Ran_{x\subset}} \arrow[l, "Loc"'] \\
QC^!(LS_{H_1}) \arrow[u, "push"] & Rep(H_1)_x \arrow[l, "Loc"'] \arrow[u, "inv^+"']
\end{tikzcd}$$

Here $Ran_{x\subset} = \{ I \in Ran \mid x \in I \}$

$$\begin{tikzcd}
inv^+ : Rep(H_1)_x \arrow[r, "ins_{\mathbb{1}}"] & Rep(H_1)_{Ran_{x\subset}} \arrow[d, "inv"] \\
 & Rep(H_2)_{Ran_{x\subset}}
\end{tikzcd}$$

$$ins(V_x)_{\{x, y_1, \dots, y_m\}} := V_x \otimes \mathbb{1}_{y_1} \otimes \cdots \otimes \mathbb{1}_{y_m}$$

More generally. $\mathcal{Y} \xrightarrow{\pi} X$ D-scheme

If $\mathcal{Y}$ is quasi-affine or coaffine, or more generally

$$QC(\mathcal{Y}) \simeq \pi_* \mathcal{O}_{\mathcal{Y}}\text{-mod}(QC(X)).$$

(Note that $\pi_* \mathcal{O}_{\mathcal{Y}}$ is a complex

$\underline{\text{Ex}}$: $\mathbb{A}^2 \setminus 0 \longrightarrow *$, $\mathbb{B}\mathbb{G}_a \longrightarrow *$ satisfy similar condition)

Prop:

$$\begin{tikzcd}
QC^!(\mathcal{Y}(\overset{\circ}{D}_x)) \arrow[r, "\mathrm{ins}"] \arrow[d] & QC^!(\mathcal{Y}(\overset{\circ}{D}_x))_{\mathrm{Ran}_{x,c}} \arrow[d, "\Gamma"] \\
QC^!(\mathcal{Y}(\overset{\circ}{X})) \arrow[r, "\Gamma"] & \mathrm{Vect}
\end{tikzcd}$$

where $\mathcal{Y}(\overset{\circ}{D}_x)_{\mathrm{Ran}_{x,c}}$ means at $(x, y_1, \dots, y_n) \in \mathrm{Ran}$

fiber is $\mathcal{Y}(\overset{\circ}{D}_x) \times \mathcal{Y}(D_{y_1}) \times \cdots \times \mathcal{Y}(D_{y_n})$

$\mathrm{ins}(M) = M \boxtimes \mathcal{O} \boxtimes \cdots \boxtimes \mathcal{O}$.

Prop $\Rightarrow$ Lem: case when M is supported on $\mathcal{Y}(D_x) \hookrightarrow \mathcal{Y}(\overset{\circ}{D}_x)$.

& use a parametrised version

($\mathbb{B}H_1 \longrightarrow \mathbb{B}H_2$ is coaffine).

Now Lem $\Rightarrow$

$$\begin{tikzcd}
\mathrm{QC}^!(\mathrm{LS}_{\check{P}}) \arrow[d, "\mathrm{push}"] & \mathrm{Rep}(\check{P})_x \arrow[l, "\mathrm{Loc}"'] \arrow[d, "\mathrm{inv}^*"] \\
\mathrm{QC}^!(\mathrm{LS}_{\check{M}}) & \mathrm{Rep}(\check{M})_{\mathrm{Ran}_{xc}} \arrow[l, "\mathrm{Loc}"']
\end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd}
\mathrm{QC}^!(\mathrm{LS}_{\check{G}}) \arrow[d, "\mathrm{CT}"] & \mathrm{Rep}(\check{G})_x \arrow[l, "\mathrm{Loc}"'] \arrow[d, "\mathrm{chev}^*"] \\
\mathrm{QC}^!(\mathrm{LS}_{\check{M}}) & \mathrm{Rep}(\check{M})_{\mathrm{Ran}_{xc}} \arrow[l, "\mathrm{Loc}"']
\end{tikzcd}$$

Explicitly,

$$\mathrm{chev}^*(\mathcal{M}_x)_{(x, y_1, \dots, y_m)} = \mathcal{M}_x^{u^-} \otimes \mathbb{1}_{y_1}^{u^-} \otimes \cdots \otimes \mathbb{1}_{y_m}^{u^-}$$

($\mathbb{1}^{u^-} = C^{\cdot}(u^-)$ chevally complex).

$$\Rightarrow \quad \begin{tikzcd}
\mathrm{QC}^!(\mathrm{LS}_{\check{G}}) \arrow[r, rightsquigarrow, "P"] & \mathrm{Rep}(\check{G})_x \\
\mathrm{QC}^!(\mathrm{LS}_{\check{M}}) \arrow[u, "\mathrm{Eis}"] \arrow[r, "P"] & \mathrm{Rep}(\check{M})_{\mathrm{Ran}_{xc}} \arrow[u, "\text{dual functor}"']
\end{tikzcd}$$

- Geometric side

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[r, "\mathrm{coeff}"] & \mathrm{Whit}(G)_x \\
D(\mathrm{Bun}_M) \arrow[u, "\mathrm{Eis}_!^-"] \arrow[r, "\mathrm{coeff}"] & \mathrm{Whit}(M)_{\mathrm{Ran}_{xc}} \arrow[u, dashed, "\text{dual functor.}"']
\end{tikzcd}$$

The dual functor

$$\mathrm{Whit}(G)_* \longrightarrow \mathrm{Whit}(M)_{\mathrm{Rm}_*c}$$

is given by inserting unit for a functor

$$\begin{tikzcd}
\mathrm{Whit}(G) \arrow[r, "J^-"] \arrow[d, equal] & \mathrm{Whit}(M) \arrow[d, equal] \\
D(\mathrm{Gr}_G)_r^{(\mathcal{L}N, \chi_G)} & D(\mathrm{Gr}_M)^{(\mathcal{L}N_M, \chi_M)}
\end{tikzcd}$$

Def 1:

$$SI_P^- := D(\mathrm{Gr}_G)^{\mathcal{L}N^- \cdot \mathcal{L}^+M} \qquad (P = B \quad N^-(K) \cdot T(O))$$

$$= (D(\mathrm{Gr}_G) \otimes D(\mathrm{Gr}_M))^{\mathcal{L}P^-}$$

$$= \mathrm{Fun}_{\mathcal{L}P^-}(D(\mathrm{Gr}_G), D(\mathrm{Gr}_M))$$

For any $F : D(\mathrm{Gr}_G) \longrightarrow D(\mathrm{Gr}_M)$ $\mathcal{L}P^-$-linear.

$$\begin{tikzcd}
\rightsquigarrow D(\mathrm{Gr}_G)^{(\mathcal{L}N, \chi_G)} \arrow[r, "\mathrm{oblv}"] & D(\mathrm{Gr}_G)^{(\mathcal{L}N_M, \chi_M)} \arrow[d, "F"] \\
 & D(\mathrm{Gr}_M)^{(\mathcal{L}N_M, \chi_M)}
\end{tikzcd}$$

Now consider F corresponds to the unit of SI_P^-, i.e., the standard object on the orbit $\mathcal{L}N^- \cdot 1 \cdot \mathcal{L}^+G / \mathcal{L}^+G \hookrightarrow \mathrm{Gr}_G$.

Def 2: characters of $\mathcal{L}N$

$$\underset{\text{(non-deg)}}{\chi_G} \rightsquigarrow \chi_P$$

(trivial on $\mathcal{L}U$
gives χ_M on $\mathcal{L}N_M$)

$$\begin{tikzcd}
D(\mathrm{Gr}_G)^{(\mathcal{L}N,\chi_G)} \arrow[r, squiggly, "\text{nearby cycles}"] \arrow[rddd, "J^-"'] & D(\mathrm{Gr}_{\tilde{G}})^{(\mathcal{L}N,\chi_P)} \arrow[d] \\
& D(\mathrm{Gr}_P)^{(\mathcal{L}N,\chi_P)} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\
& D(\mathrm{Gr}_M)^{(\mathcal{L}N_M,\chi_M)}
\end{tikzcd}$$

Why Def 1 uses P^-, Def 2 uses P?

2nd-adjointness. No time to explain.

The desired diagram

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[r] & \mathrm{Whit}(G)_{\mathrm{Ran}_x} \\
D(\mathrm{Bun}_M) \arrow[r] \arrow[u, "\mathrm{Eis}"] & \mathrm{Whit}(M)_x \arrow[u, "\text{dual factor}"']
\end{tikzcd}$$

can be proven pure geometrically.

- The top square, need

$$\begin{tikzcd}
\mathrm{Whit}(G) \arrow[r, "\sim"] \arrow[d, "J^-"] & \mathrm{Rep}(\check{G}) \arrow[d, "\mathrm{chev}."] \\
\mathrm{Whit}(M) \arrow[r, "\sim"] & \mathrm{Rep}(\check{M})
\end{tikzcd}$$

Can be proven using

Thm

$$SI_{P^-} = D(G_K)^{\mathcal{L}N^-\mathcal{L}^+M}$$

$$SI_{\check{P}^-,\mathrm{spec}}$$

$$\|$$

$$\simeq QC^!\left(LS_{\check{G}\times\check{M}}(D) \times_{LS_{\check{G}\times\check{M}}(\mathring{D})} LS_{\check{P}^-}(\mathring{D})\right)$$

(case $P = G$, this is derived Satake !)

Why expect this in local Langlands?

$$\begin{tikzcd}
\mathcal{L}G\text{-mod} \arrow[r] \arrow[d, "(\ )^{\mathcal{L}N}"'] & \mathrm{ShvCat}(LS_{\check{G}}(\mathring{D})) \arrow[d, "(\ )|_{LS_{\check{P}}(\mathring{D})}"] \\
\mathcal{L}M\text{-mod} \arrow[r] & \mathrm{ShvCat}(LS_{\check{M}}(\mathring{D}))
\end{tikzcd}$$

Start with

$$\begin{tikzcd}
D(G_K) \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & QC(LS_{\check{G}}(D)) \arrow[d, rightsquigarrow] \\
D(G_K)^{\mathcal{L}N} \arrow[r, rightsquigarrow] & QC(LS_{\check{G}}(D) \times_{LS_{\check{G}}(\mathring{D})} LS_{\check{P}}(\mathring{D})) \\
D(G_M) \arrow[r, rightsquigarrow] & QCoh(LS_{\check{M}}(\mathring{D}))
\end{tikzcd}$$

Calculate $\mathrm{Fun}(-,-)$ on both sides.

This give theorem E.b.

The same method does not gives Thm CT:

$$\begin{tikzcd}
 & \mathrm{Whit}(G) \arrow[rr, "\sim"] & & \mathrm{Rep}(\check{G}) \\
\mathrm{Whit}(M) \arrow[ur] \arrow[rr] & & \mathrm{Rep}(\check{M}) \arrow[ur] & \\
 & D(\mathrm{Bun}_G) \arrow[uu] \arrow[rr] & & QC^!(LS_{\check{G}}) \arrow[uu] \\
D(\mathrm{Bun}_M) \arrow[uu] \arrow[ur] \arrow[rr] & & QC^!(LS_{\check{M}}) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

no way to flip the arrows of direction ↗.

However: Whit ⇝ KM duality suggests

$$\begin{tikzcd}
 & D(\mathrm{Bun}_G) \arrow[dl] \arrow[rr] & & QC^!(LS_{\check{G}}) \arrow[dl] \\
D(\mathrm{Bun}_M) \arrow[rr] & & QC^!(LS_{\check{M}}) & \\
 & KL_G \arrow[uu] \arrow[dl] \arrow[rr, "\sim"] & & QC^!(Op^{unr}_{\check{G}}) \arrow[uu] \arrow[dl] \\
KL_M \arrow[uu] \arrow[rr, "\sim"] & & QC^!(Op^{unr}_{\check{M}}) \arrow[uu] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Whit}_\kappa(M) \arrow[r] & \mathrm{Whit}_\kappa(G) \\
D_\kappa(\mathrm{Bun}_M) \arrow[u] \arrow[r] & D_\kappa(\mathrm{Bun}_G) \arrow[u]
\end{tikzcd}$$

$\rightsquigarrow$ $\kappa \to \infty$

$$\begin{tikzcd}
QC^!(\mathrm{Op}_{\check{M}}^{u,\mathrm{unr}}) \arrow[r] & QC^!(\mathrm{Op}_{\check{G}}^{\mathrm{unr}}) \\
QC^!(LS_M) \arrow[u] \arrow[r] & QC^!(LS_G) \arrow[u]
\end{tikzcd}$$

passing to duality gives the right face.

The functors

- $KL_G \longrightarrow KL_M$

  $\mathcal{M} \longmapsto C^{\frac{\infty}{2}}(N^-((t)), \mathcal{M})$

  (proof is similar to spectral in Thm E6)

- $QC^!(\mathrm{Op}_{\check{G}}^{u,\mathrm{unr}}) \longrightarrow QC^!(\mathrm{Op}_{\check{M}}^{u,\mathrm{unr}})$

  acted by unit in $SI_{\check{P}^-,\mathrm{spec}}$.

Prove the top & side faces commute.

However, we haven't shown the front or back faces commute.

$$\begin{tikzcd}
D(\mathrm{Bun}_G) \arrow[r] & \mathrm{QC}^!(\mathrm{LS}_{\check{G}}) \\
\mathrm{KL}_G \arrow[r] \arrow[u] & \mathrm{QC}^!(\mathrm{Op}^{\mathrm{unr}}_{\check{G}}) \arrow[u]
\end{tikzcd}$$

Need
$$\begin{tikzcd}
\mathrm{Whit}_G \arrow[r] & \mathrm{Rep}(\check{G}) \\
D(\mathrm{Bun}_G) \arrow[u] & \mathrm{QC}^!(\mathrm{LS}_{\check{G}}) \arrow[u] \\
\mathrm{KL}_G \arrow[r] \arrow[u] & \mathrm{QC}^!(\mathrm{Op}^{\mathrm{unr}}_{\check{G}}) \arrow[u]
\end{tikzcd}$$

chiral method.

□ Thm CT.

<u>Lem</u>: $\mathbb{L}_G : D(\mathrm{Bun}_G) \longrightarrow \mathrm{QC}^!(\mathrm{LS}_{\check{G}})$

has a left adjoint. $\mathbb{L}^L_G$

<u>Fact</u>: $\mathrm{QC}^!(\mathrm{LS}_{\check{G}})$ is generated by $\mathrm{Eis}_{\mathrm{spec},!^*}(\mathrm{QC}(\mathrm{LS}_{\check{M}}))$

Only need existence of:

$$\begin{tikzcd}
 & & QC(LS_{\check{M}}) \arrow[d, "Eis_{spec}^{!}"] \\
D(Bun_G) & & QC^!(LS_{\check{G}}) \arrow[ll, "\mathbb{L}^L"']
\end{tikzcd}$$

i.e. The following has left adjoint.

$$\begin{tikzcd}
D(Bun_G) \arrow[r, "\mathbb{L}"] \arrow[d, "CT"] & QC^!(LS_{\check{G}}) \arrow[d, "CT"] \\
D(Bun_M) \arrow[r, "\mathbb{L}"] \arrow[dr, "\mathbb{L}^{coarse}"'] & QC^!(LS_{\check{M}}) \arrow[d] \\
 & QC(LS_{\check{M}})
\end{tikzcd}$$

But we know $\mathbb{L}^{coarse}$ has a left adjoint

$$\begin{tikzcd}
Rep(\check{M})_{Ran} \arrow[d] \arrow[dr, "\text{act on } W_!"] & \\
QC(LS_{\check{M}}) \arrow[r, dashed] & D(Bun_M)
\end{tikzcd}$$

Thm: $\mathbb{L}_{Eis}: D(Bun_G)_{Eis} \xrightarrow{\sim} QC^!(LS_{\check{G}})_{red}$

if $\mathbb{L}_M$ is invertible for all <u>proper</u> M.

reduce to show
$\rightsquigarrow$

$\mathbb{L}_{cusp}: D(Bun_G)_{cusp} \xrightarrow{\sim} QC(LS_{\check{G}, irr})$

$\boxed{\text{Next time}}$