

Math-Thms:

$$\mathbb{L}_G : \text{DMod}_{\text{crit}}(\text{Bun}_G) \longrightarrow \text{IndChNtp}(LS\check{G})$$

compatible with derived Satake, Eisenstein series, duality etc.

Recall:

Generalized vanishing conjecture \Rightarrow

$$\begin{array}{ccc} \text{DMod}_{\text{crit}}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G^0} & \text{QCoh}(LS\check{G}) \\ \text{coeff}_G \downarrow & & \downarrow \Gamma_{\check{G}}^{\text{Spec}} \\ \text{Whf}_{\text{crit}}(G_{\text{an}})_{\text{an}} & \simeq & \text{Rep}(\check{G})_{\text{an}} \end{array}$$

(\mathbb{L}_G^0 will be $\Psi \circ \mathbb{L}_G$, $\Psi: \text{IndChNtp} \rightarrow \text{QCoh}$)

$$\begin{array}{ccc} \text{DMod}_{\text{crit}}(\text{Bun}_G)^c & \xrightarrow{\mathbb{L}_G^0} & \text{QCoh}(LS\check{G})^{>-\infty} \quad (\text{by [FR]}) \\ & \searrow & \downarrow \text{Is} \\ & & \text{IndChNtp}(LS\check{G})^{>-\infty} \\ & & \downarrow \\ & & \text{IndCh}_{\text{Ntp}}(LS\check{G}) \end{array}$$

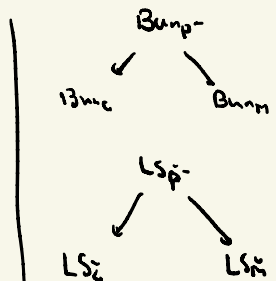
\mathbb{L}_G is obtained by Ind-completion.

Shorthand:

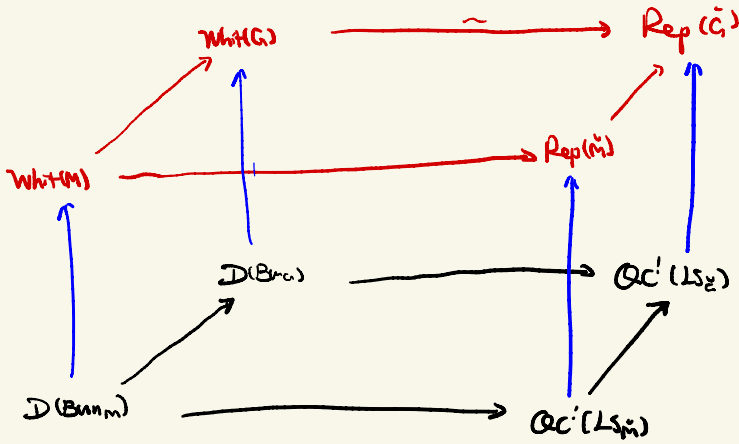
$$\mathcal{D}(\text{Bun}_G) \xrightarrow{\mathbb{L}_G} \text{QC}'(LS\check{G})$$

$$\begin{array}{ccc} \text{Thm:} & \mathcal{D}(\text{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \text{QC}'(LS\check{G}) \\ (E_{\check{G}}, \text{CT}) & \uparrow \text{E}_{\check{G}} \downarrow \text{CT}_{\check{G}} & & \uparrow \text{E}_{\check{G}}^{\text{Spec}} \downarrow \text{CT}_{\check{G}}^{\text{Spec}} \\ & \mathcal{D}(\text{Bun}_m) & \xrightarrow{\mathbb{L}_m} & \text{QC}'(LS\check{M}) \end{array}$$

By construction, compatible with derived Satake

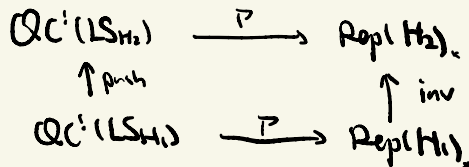
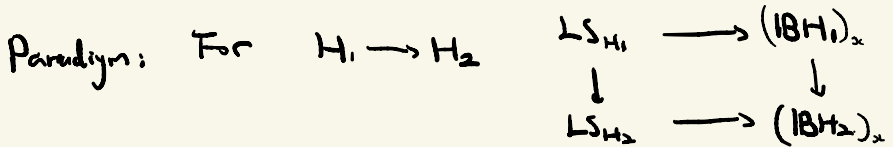
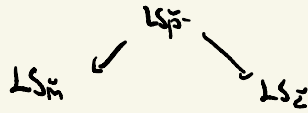
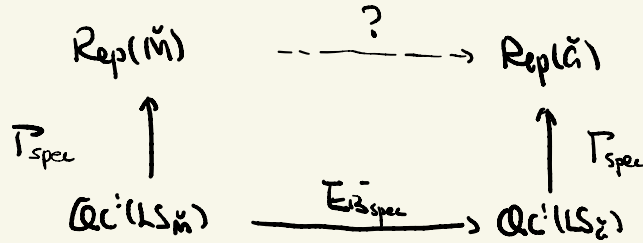


Proof of Eis :



Need top and two side squares commute.

① Spectral side:



But no obvious diagram for pull vs P .

Dually pull vs Loc

$$\begin{array}{ccc}
 \mathcal{O}C'(LSH_1) & \xleftarrow{\text{Loc}} & \text{Rep}(H_2)_x \\
 \downarrow \text{pull} & & \downarrow \text{res} \\
 \mathcal{O}C'(LSH_1) & \xleftarrow{\text{Loc}} & \text{Rep}(H_1)_x
 \end{array}$$

However, here is a diagram for
push vs Loc:

Lemma: If $H_1 \twoheadrightarrow H_2$ has unipotent kernel,

$$\begin{array}{ccc}
 \mathcal{O}C'(LSH_2) & \xleftarrow{\text{Loc}} & \text{Rep}(H_2)_{\text{Ran}_x} \\
 \uparrow \text{push} & & \uparrow \text{inv}^+ \\
 \mathcal{O}C'(LSH_1) & \xleftarrow{\text{Loc}} & \text{Rep}(H_1)_x
 \end{array}$$

$$\text{Here } \text{Ran}_x = \{ I \in \text{Ran} \mid x \in I \}$$

$$\begin{array}{ccc}
 \text{inv}^+ : \text{Rep}(H_1)_x & \xrightarrow{\text{ins}_x} & \text{Rep}(H_1)_{\text{Ran}_x} \\
 & & \downarrow \text{inv} \\
 & & \text{Rep}(H_2)_{\text{Ran}_x}
 \end{array}$$

$$\text{ins}(V_x)_{\{x, y_1, \dots, y_m\}} := V_x \otimes \mathbb{1}_{y_1} \otimes \dots \otimes \mathbb{1}_{y_m}$$

More generally, $Y \xrightarrow{\pi} X$ D -scheme

If Y is quasi-affine or affine, or more generally

$$QC(Y) \cong \pi_* \mathcal{O}_Y\text{-mod}(QC(X)).$$

(Note that $\pi_* \mathcal{O}_Y$ is a complex

$$\underline{\text{Ex}}: \mathbb{A}^2 \setminus 0 \rightarrow *, \mathbb{B}G_m \rightarrow *$$

Satisfy similar condition

Prop:

$$\begin{array}{ccc} QC^i(Y(\overset{\circ}{D}_x)) & \xrightarrow{\text{ins}} & QC^i(Y(\overset{\circ}{D}_x))_{\text{Fiber}} \\ \downarrow & & \downarrow \mathbb{P} \\ QC^i(Y(\overset{\circ}{X})) & \xrightarrow{\mathbb{P}} & \text{Vect} \end{array}$$

where $Y(\overset{\circ}{D}_x)_{\text{Fiber}}$ means at $(x, y_1, \dots, y_n) \in \text{Fiber}$

fiber is $Y(\overset{\circ}{D}_x) \times Y(D_{y_1}) \times \dots \times Y(D_{y_n})$

$$\text{ins}(M) = M \boxtimes \mathcal{O} \boxtimes \dots \boxtimes \mathcal{O}.$$

Prop \Rightarrow Lem: case when M is supported on

$$Y(\overset{\circ}{D}_x) \hookrightarrow Y(\overset{\circ}{D}_x).$$

& use a parametrised version

($\mathbb{B}G_m \rightarrow \mathbb{B}G_m$ is affine).

Now Lem \Rightarrow

$$\begin{array}{ccc} \mathcal{QC}'(LS\check{\alpha}) & \xleftarrow{Loc} & \text{Rep}(\check{N})_x \\ \downarrow \text{push} & & \downarrow \text{inv} \\ \mathcal{QC}'(LS\check{\alpha}) & \xleftarrow{Loc} & \text{Rep}(\check{M})_{\text{Perm}_x c} \end{array}$$

$$\Rightarrow \begin{array}{ccc} \mathcal{QC}'(LS\check{\alpha}) & \xleftarrow{Loc} & \text{Rep}(\check{N})_x \\ \downarrow \text{CT} & & \downarrow \text{chev}^+ \\ \mathcal{QC}'(LS\check{\alpha}) & \xleftarrow{Loc} & \text{Rep}(\check{M})_{\text{Perm}_x c} \end{array}$$

Explicitly,

$$\text{chev}^+(M_{(x,y_1,\dots,y_m)}) = M_x^U \otimes \mathbb{1}_{y_1}^U \otimes \dots \otimes \mathbb{1}_{y_m}^U$$

($\mathbb{1}^U = C^+(U)$ Chevalley complex).

$$\Rightarrow \begin{array}{ccc} \mathcal{QC}'(LS\check{\alpha}) & \xrightarrow{P} & \text{Rep}(\check{N})_x \\ \uparrow E_i & & \uparrow \text{dual functor} \\ \mathcal{QC}'(LS\check{\alpha}) & \xrightarrow{P} & \text{Rep}(\check{M})_{\text{Perm}_x c} \end{array}$$

• Geometric side

$$\begin{array}{ccc} D(\mathcal{B}_n) & \xrightarrow{\text{coeff}} & \text{Whit}(G)_x \\ \uparrow E_i & & \uparrow \text{dual functor} \\ D(\mathcal{B}_{m,n}) & \xrightarrow{\text{coeff}} & \text{Whit}(M)_{\text{Perm}_x c} \end{array}$$

The dual functor

$$\text{Whit}(G)_* \longrightarrow \text{Whit}(M)_{\text{Ran } c}$$

is given by inserting unit for a functor

$$\text{Whit}(G) \xrightarrow{J^-} \text{Whit}(M)$$

$$D(G_{G_0})_{\mathbb{R}}$$

$$D(G_M)_{\mathbb{R}}$$

Def 1:

$$SI_{\mathbb{R}}^- := D(G_{G_0})_{\mathbb{R}} \cdot \mathbb{R}U^- \cdot \mathbb{R}^+M \quad (\mu = \beta \quad N(K) = T(0))$$

$$= (D(G_{G_0}) \oplus D(G_M))_{\mathbb{R}} \cdot \mathbb{R}^+M$$

$$= \text{Fun}_{\mathbb{R}^+M} (D(G_{G_0}), D(G_M))$$

For any $F: D(G_{G_0}) \longrightarrow D(G_M)$ \mathbb{R}^+M -linear.

$$\rightsquigarrow D(G_{G_0})_{\mathbb{R}} \xrightarrow{\text{oblv}} D(G_{G_0})_{\mathbb{R}} \xrightarrow{F} D(G_M)_{\mathbb{R}}$$

$$\downarrow F$$

Now consider F corresponds to the unit of $SI_{\mathbb{R}}^-$, i.e., the standard object on the orbit $\mathbb{R}U^- \cdot 1 \cdot \mathbb{R}^+G / \mathbb{R}^+G \hookrightarrow G_{G_0}$.

Def 2: characters of $\mathbb{R}U$

$$\chi_{\mathbb{R}U} \rightsquigarrow \chi_{\mathbb{R}NM}$$

(non-deg)

(trivial on $\mathbb{R}U$
gives $\chi_{\mathbb{R}NM}$ on $\mathbb{R}NM$)

$$\begin{array}{ccc}
 D(G_{\mathbb{C}}) \xrightarrow{(\mathbb{Z}N, \chi_0) \text{ nearby cycles}} & D(G_{\mathbb{R}}) \xrightarrow{(\mathbb{Z}N, \chi_p)} \\
 \searrow \mathcal{J}^- & \downarrow \\
 & D(G_p) \xrightarrow{(\mathbb{Z}N, \chi_p)} \\
 & \downarrow \text{is} \\
 & D(G_m) \xrightarrow{(\mathbb{Z}N_m, \chi_m)}
 \end{array}$$

Why Def 1 uses p^- , Def 2 uses p ?

Ind-adjointness. No time to explain.

The derived diagram

$$\begin{array}{ccc}
 D(\text{Ban}_G) & \longrightarrow & \text{Whit}(G)_{\text{formal}} \\
 \uparrow E_3 & & \uparrow \text{dual factor} \\
 D(\text{Ban}_m) & \longrightarrow & \text{Whit}(M)
 \end{array}$$

can be proven pure geometrically.

- The top square, need

$$\begin{array}{ccc}
 \text{Whit}(G) & \xrightarrow{\sim} & \text{Rep}(k^{\vee}) \\
 \downarrow \mathcal{J}^- & & \downarrow \text{chev.} \\
 \text{Whit}(M) & \xrightarrow{\sim} & \text{Rep}(k^{\vee})
 \end{array}$$

Can be proven using

$$\begin{aligned}
 \text{This } S\tilde{I}p^- &= D(G_w) \xrightarrow{\text{Lun-Lun}} S\tilde{I}p^-, \text{ spec} \\
 &\cong \mathbb{C}c' \left(\underset{LS_{\tilde{G} \times \tilde{M}}(\tilde{D})}{LS_{\tilde{G}}(D)} \times \underset{LS_{\tilde{M}}(\tilde{D})}{LS_{\tilde{P}}(\tilde{D})} \right)
 \end{aligned}$$

(Case $p = G$, this is derived Satake!)

Why expect this in local Langlands?

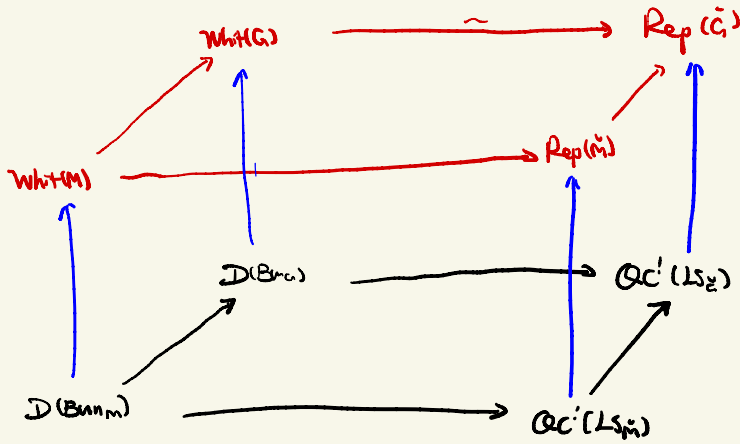
$$\begin{array}{ccc}
 \mathcal{L}G\text{-mod} & \xrightarrow{\quad} & \text{ShuCat}(LS_{\tilde{G}}(\tilde{D})) \\
 \downarrow \text{Lun} & & \downarrow \text{Lun} \\
 \mathcal{L}M\text{-mod} & \xrightarrow{\quad} & \text{ShuCat}(LS_{\tilde{M}}(\tilde{D}))
 \end{array}$$

$$\begin{array}{ccc}
 \text{Start with } D(G_w) & \rightsquigarrow & \mathbb{C}c'(LS_{\tilde{G}}(D)) \\
 \downarrow & & \downarrow \\
 D(G_w) \xrightarrow{\text{Lun}} & \rightsquigarrow & \mathbb{C}c'(LS_{\tilde{G}}(D) \times \underset{LS_{\tilde{M}}(\tilde{D})}{LS_{\tilde{P}}(\tilde{D})}) \\
 D(G_M) & \rightsquigarrow & \mathbb{C}Gh(LS_{\tilde{M}}(\tilde{D}))
 \end{array}$$

Calculate $\tilde{I}m(-, -)$ on both sides.

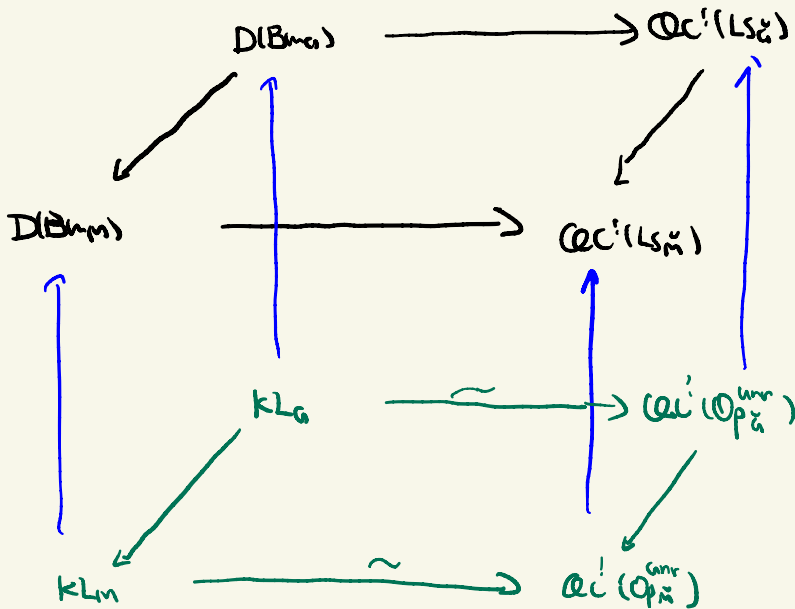
This gives theorem E.5.

The same method does not give them CT:



no way to flip the arrows of direction

However: $Whit(G) \leftrightarrow KM$ duality suggests



$$\text{Whit}_k(M) \longrightarrow \text{Whit}_k(G)$$

$$\uparrow$$

$$\uparrow$$

$$D_k(B_{\text{un}}) \longrightarrow D_k(B_{\text{un}})$$

\rightsquigarrow

$$k \rightarrow \infty \quad \mathcal{O}_G^i(\mathcal{O}_{\mathbb{P}^n}^{\text{un}}) \longrightarrow \mathcal{O}_G^i(\mathcal{O}_{\mathbb{P}^n}^{\text{un}})$$

$$\uparrow$$

$$\uparrow$$

$$\mathcal{O}_G^i(L_{\mathbb{P}^n}) \longrightarrow \mathcal{O}_G^i(L_{\mathbb{P}^n})$$

passing to duality gives the right face.

The functors

$$\bullet \quad K L_G \longrightarrow K L_M$$

$$M \longmapsto \left(\sum_{i=0}^{\infty} H^i(L(i)), M \right)$$

(proof is similar to spectral in Thm E6)

$$\bullet \quad \mathcal{O}_G^i(\mathcal{O}_{\mathbb{P}^n}^{\text{un}}) \longrightarrow \mathcal{O}_G^i(\mathcal{O}_{\mathbb{P}^n}^{\text{un}})$$

acted by unit is $\text{SI}_{\mathbb{P}^n, \text{spec}}$.

Prove the top & side faces commute.

However, we haven't shown the front or back faces commute.

$$\begin{array}{ccc}
 D(B_{\text{loc}}) & \longrightarrow & \mathcal{O}_C^!(L_{S_{\tilde{G}}}) \\
 \uparrow & & \uparrow \\
 kL_G & \longrightarrow & \mathcal{O}_C^!(\mathcal{O}_{P_{\tilde{G}}}^{\text{unr}})
 \end{array}$$

Need

$$\begin{array}{ccc}
 \text{Whit}_G & \longrightarrow & \text{Rep}(\tilde{G}) \\
 \uparrow & & \uparrow \\
 D(B_{\text{loc}}) & & \mathcal{O}_C^!(L_{S_{\tilde{G}}}) \\
 \uparrow & & \uparrow \\
 kL_G & \longrightarrow & \mathcal{O}_C^!(\mathcal{O}_{P_{\tilde{G}}}^{\text{unr}})
 \end{array}$$

chiral method.

□ Thm CT.

Len: $\mathbb{L}_{\tilde{G}}^L D(B_{\text{loc}}) \longrightarrow \mathcal{O}_C^!(L_{S_{\tilde{G}}})$
 has a left adjoint. $\mathbb{L}_{\tilde{G}}^L$

Fact: $\mathcal{O}_C^!(L_{S_{\tilde{G}}})$ is generated by $E_{\text{spec}, p^*}(\mathcal{O}_C(L_{S_{\tilde{G}}}))$

Only need existence of.

$$\begin{array}{ccc}
 & & \mathcal{O}_C(L_{S_i^{\text{red}}}) \\
 & & \downarrow \text{Eis spec} \\
 D(B_{\text{red}}) & \xleftarrow{\mathbb{H}^L} & \mathcal{O}_C^i(L_{S_i^{\text{red}}})
 \end{array}$$

i.e. The following has left adjoint.

$$\begin{array}{ccc}
 D(B_{\text{red}}) & \xrightarrow{\mathbb{H}} & \mathcal{O}_C^i(L_{S_i^{\text{red}}}) \\
 \downarrow \text{CT} & & \downarrow \text{CT} \\
 D(B_{\text{reg}}) & \xrightarrow{\mathbb{H}} & \mathcal{O}_C^i(L_{S_i^{\text{reg}}}) \\
 & \searrow \mathbb{H}^{\text{coral}} & \downarrow \\
 & & \mathcal{O}_C(L_{S_i^{\text{reg}}})
 \end{array}$$

But we know $\mathbb{H}^{\text{coral}}$ has a left adjoint

$$\begin{array}{ccc}
 \text{Rep}(\tilde{M})_{\text{reg}} & \xrightarrow{\text{act on } W_i} & \\
 \downarrow & & \\
 \mathcal{O}_C(L_{S_i^{\text{reg}}}) & \dashrightarrow & D(A_{\text{reg}})
 \end{array}$$

Thm: $\mathbb{H}_{\text{Eis}}^i D(B_{\text{red}})_{\text{Eis}} \xrightarrow{\sim} \mathcal{O}_C^i(L_{S_i^{\text{red}}})_{\text{red}}$

if \mathbb{H}_M is invertible for all proper M .

reduce to show
 \rightsquigarrow

$$\mathbb{H}_{\text{comp}} : D(B_{\text{red}})_{\text{comp}} \xrightarrow{\sim} \mathcal{O}_C(L_{S_i^{\text{red}}})_{\text{red}}$$

Next time