Proof II

Last time:

- We constructed

$$\mathbb{L}_G : D(\mathrm{Bun}_G) \longrightarrow \mathrm{QC}^!(\mathrm{LS}_{\check{G}})$$

and its left adjoint $\mathbb{L}_G^L$.

- We proved they give equivalence

$$D(\mathrm{Bun}_G)_{\mathrm{Eis}} \simeq \mathrm{QC}^!(\mathrm{LS}_{\check{G}})_{\mathrm{red}}$$

It remains to show

$$\mathbb{L}_{G,\mathrm{cusp}} : D(\mathrm{Bun}_G)_{\mathrm{cusp}} \simeq \mathrm{QC}(\mathrm{LS}_{\check{G}}^{\mathrm{irre}})$$

Summerize what we know about $\mathbb{L}_G^{\mathrm{cusp}}$.

- <u>Thm</u>: $\mathbb{L}_{G,\mathrm{cusp}}$ is conservative.

<u>Prf</u>:
$$\begin{array}{ccc} D(\mathrm{Bun}_G)_{\mathrm{cusp}} & \longrightarrow & \mathrm{QC}(\mathrm{LS}_{\check{G}}^{\mathrm{irre}}) \\ \big\downarrow \mathrm{coeff}_G & & \big\downarrow \\ \mathrm{Whit}(G)_{\mathrm{Ran}} & \simeq & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \end{array}$$

<u>Lem</u>: [Ber]: $D(\mathrm{Bun}_G)_{\mathrm{cusp}} \subset D(\mathrm{Bun}_G)_{\mathrm{temp}}$.

Recall [FR]: $\mathrm{coeff}_G|_{D(\mathrm{Bun}_G)_{\mathrm{temp}}}$ is conservative □.

- $\mathbb{L}_{G,\mathrm{cusp}}$ has a left adjoint $\mathbb{L}_{G,\mathrm{cusp}}^L$

Only need to show

$$\mathbb{L}_{G,\mathrm{cusp}} \circ \mathbb{L}_{G,\mathrm{cusp}}^L \in \mathrm{End}(\mathrm{QC}(\mathrm{LS}_{\check{G}}^{\mathrm{irre}}))$$

is

Id.

$\mathbb{L}_G \circ \mathbb{L}_G^L$ is a monad on $\mathrm{QCoh}^!(LS_{\check{G}})$.

It is $\mathrm{QCoh}(LS_{\check{G}})$-linear.

<u>Claim</u>: $\mathbb{L}_{G,\mathrm{cusp}} \circ \mathbb{L}_{G,\mathrm{cusp}}^L$ is given by tensoring with

$$A^{\mathrm{irred}} \in \mathrm{Alg}(\mathrm{QCoh}(LS_{\check{G}}^{\mathrm{irred}})).$$

$\mathbb{L}_G \circ \mathbb{L}_G^L(\mathcal{O}) =: A \in \mathrm{Alg}(\mathrm{QCoh}(LS_{\check{G}}))$

Then $A^{\mathrm{irred}} = A|_{LS_{\check{G}}^{\mathrm{irred}}}$

Note: $\mathrm{QCoh}^! \neq \mathrm{QCoh}$, so it's not obvious $\mathbb{L}_G \circ \mathbb{L}_G^L$ is $- \otimes A$.

Only need to show $\mathcal{O} \xrightarrow{\varepsilon} A$ is iso.

(Last time $\Rightarrow$ $\mathcal{O}|_{\mathrm{red}} \xrightarrow{\sim} A|_{\mathrm{red}}$).

<u>Thm 1</u>: $\mathbb{L}_{G,\mathrm{cusp}}^L \simeq \mathbb{L}_{G,\mathrm{cusp}}^R$ canonically.

(Ambidexterity)

$$\mathbb{L}_{G,\mathrm{cusp}}^L \simeq \mathbb{L}_{G,\mathrm{cusp}}^\vee \simeq \mathbb{L}_{G,\mathrm{cusp}}^R$$

($D(\mathrm{Bun}_G)_{\mathrm{cusp}}$, $\mathrm{QCoh}(LS_{\check{G}}^{\mathrm{irred}})$ are self-dual)

<u>Proof</u>: Last time

$$\begin{tikzcd}
KL(G)_{\mathrm{Ran}} \arrow[r, "\sim"] \arrow[d, "\mathrm{Loc}_G"'] & \mathrm{QCoh}^!(\mathrm{Op}_{\check{G}}^{\mathrm{unr}})_{\mathrm{Ran}} \arrow[d, "\mathrm{Poinc}_{\check{G}}^{\mathrm{spec}}"] \\
D(\mathrm{Bun}_G)_{\mathrm{cusp}} \arrow[r, "\mathbb{L}"] \arrow[d, "\mathrm{coeff}_G"'] & \mathrm{QCoh}(LS_{\check{G}}^{\mathrm{irred}}) \arrow[d, "\Gamma_{\check{G}}^{\mathrm{spec}}"] \\
\mathrm{Whit}(G)_{\mathrm{Ran}} \arrow[r, "\sim"] & \mathrm{Rep}(\check{G})_{\mathrm{Ran}}
\end{tikzcd}$$

$\mathrm{coeff}_G^L \simeq \mathrm{Poinc}_G = \mathrm{coeff}_G^\vee$

$(\Gamma_{\check{G}}^{\mathrm{spec}})^L \simeq \mathrm{Loc}_{\check{G}}^{\mathrm{spec}} = (\Gamma_{\check{G}}^{\mathrm{spec}})^\vee$

$\Rightarrow \mathbb{L}^L \simeq \mathbb{L}^\vee$.

$\mathrm{Loc}_G^R = \Gamma_G = \mathrm{Loc}_G^\vee$

$\ldots$

$\Rightarrow \mathbb{L}^R \simeq \mathbb{L}^\vee$.

<u>Cor</u>: A_{irred} is a perfect complex on $LS_{\check{G}}^{\mathrm{irred}}$

& canonically self dual.

B_{irred} : view A_{irred} as a coalgebra.

$(B_{\mathrm{irred}} := \mathbb{L}_{\mathrm{irred}} \circ \mathbb{L}^R_{\mathrm{irred}}(\mathcal{O})$, comonad $)$

$$A_{\mathrm{irred}} \simeq B_{\mathrm{irred}} \quad \text{in} \quad QC(LS^{\mathrm{irred}}_{\check{G}}).$$

Since $KL(G)_{\mathrm{Ran}} \twoheadrightarrow D(\mathrm{Bun}_G)_{\mathrm{cusp}}$

$\Rightarrow \mathbb{L}_{\mathrm{irred}} \circ \mathbb{L}^R_{\mathrm{irred}} = \mathrm{Poinc}^{\mathrm{spec}}_{\check{G},\mathrm{irred}} \circ \mathrm{coeff}^{\mathrm{spec}}_{\check{G},\mathrm{irred}}$

$$\begin{tikzcd}
\mathrm{Op}^{\mathrm{unr},\circ,\mathrm{irred}}_{\check{G}}(\overset{\circ}{X})_{\mathrm{Ran}} \arrow[r] \arrow[d, "\pi"'] & \mathrm{Op}^{\mathrm{unr}}_{\check{G}}(\overset{\circ}{X})_{\mathrm{Ran}} \arrow[r] \arrow[d] & \mathrm{Op}^{\mathrm{unr}}_{\check{G}}(\overset{\circ}{D})_{\mathrm{Ran}} \\
LS_{\check{G}}(X)^{\mathrm{irred}} \arrow[r, hook] & LS_{\check{G}}(X) &
\end{tikzcd}$$

π is "proper".

Some chiral stuff
$\Longrightarrow$

<u>Thm</u>: $\pi_!(\omega) \in \mathrm{DMod}(LS_{\check{G}}(X))$.
co-comm, coalg.

$$B_{\mathrm{irred}} \simeq \underline{\pi_!(\omega)} \quad \text{as coalg in } QC(LS^{\mathrm{irred}}_{\check{G}}(X))$$

B_{irred} is cocomm & ≤ 0, w/ connection

$\Rightarrow A_{\mathrm{irred}}$ is comm & ≥ 0. w/ connection

$\Rightarrow A_{\mathrm{irred}}$ is comm & in the heart
w/ connection.

Moreover. $\underline{\pi_!(\omega)}$ has finite monodromy.
(trivial on finite étale cover of $LS^{\mathrm{irred}}_{\check{G}}$)

<u>Thm 2</u>: A_{irred} is a finite étale commutative $\mathcal{O}$-alg.

Rmk: In ABCD type, the fiber of π is known to be connected. hence win.

[Beraldo-Kazhdan-Schlank]

Rmk: Ignore $QC^! \neq QC$.

[Ber]: $QC(LS_{\check{G}}(D) \times_{LS_{\check{G}}(B)} LS_{\check{G}}(D))_{Ran}$-linear endo-functor

Any on $QC(LS_{\check{G}})$ is given by tensoring with an object in $Dmod(LS_{\check{G}})$.

Fact: If G is of adjoint type, $g > 1$

then $LS^{irred}_{\check{G}}$ is simply connected.

except $g = 2$ $\check{G} = SL_2$.

Restrict to this general case. (other case can be reduced to it).

$\Rightarrow$ $A_{irred} \simeq \mathcal{O}^{\oplus r}$. Need $r = 1$.

(some small technical things).

Only need to show

$$\Gamma(LS^{irred}_{\check{G}}, \mathcal{O}) \xrightarrow{\sim} \Gamma(LS^{irred}_{\check{G}}, A)$$

Only need

$$\begin{array}{ccc} \Gamma(LS_{\check{G}}, \mathcal{O}) & \simeq & \Gamma(LS_{\check{G}}, A) \\ \wr\!\!\mid \ \text{[Frenkel-Teleman]} & & \wr\!\!\mid \ \text{By definition} \\ k & \overset{?}{\simeq} & End(W_!) \end{array}$$

W:

baby Whittaker

$$\begin{tikzcd}
\mathrm{Bun}_{N,\rho} \arrow[r] \arrow[d] & \mathrm{Bun}_N/T \arrow[r, hook, "\text{loc. closed. embedding}"] & \mathrm{Bun}_G \\
\widehat{\mathbb{G}_a} \arrow[r] \arrow[d] & \mathbb{G}_a/\widehat{\mathbb{G}_a}\,. & \\
\mathbb{G}_a & &
\end{tikzcd}$$

$$\mathrm{DMod}(\mathrm{Bun}_{N,\rho}) = \mathrm{DMod}(\widehat{\mathbb{G}_a})$$

$$(\mathrm{Bun}_{N,\rho} \text{ “}=\text{” } \widehat{\mathbb{G}_a} \times \mathbb{B}(\text{unipotent})).$$

direct calculation works.