Lecture 3: Bun_G and Hecke eigenproperty

<u>Last time</u> $\left\{ \begin{matrix} \text{rank 1 loc sys} \\ E \text{ on } X \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Hodge eigensheaf} \\ \text{on } \mathrm{Pic}_{X/k} \end{matrix} \right\}$

$E \longmapsto \mathrm{Aut}_E$ w/ eigenval $= E$.

§1 GLC

<u>Goal</u> $\left\{ \begin{matrix} \text{rank } n \text{ loc sys} \\ \mathcal{E} \text{ on } X \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Hecke eigensheaf} \\ \text{on } \mathrm{Bun}_n \end{matrix} \right\}$

"$G^\vee$-side" to be Bun_G for general G.

<u>Def'n</u> A <u>prestack</u> $/k$ is a functor

$$\mathcal{Y} : \mathrm{Aff}_k^{op} \longrightarrow \mathrm{Grpd} \quad \text{(2-cat of groupoids)}$$

(<u>Recall</u> A k-sheaf is $\mathcal{Y} : \mathrm{Aff}_k^{op} \to \mathrm{Set}$)

- For any S, $\mathcal{Y}(S) \in \mathrm{Grpd}$
- For any $S_1 \to S_2$, get $\mathcal{Y}(S_2) \to \mathcal{Y}(S_1)$

s.t.
$$\begin{tikzcd} S_1 \arrow[rr] \arrow[dr] & \circlearrowleft & S_2 \arrow[dl] \\ & S_3 & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{Y}(S_3) \arrow[rr] \arrow[dr] & \circlearrowleft & \mathcal{Y}(S_2) \arrow[dl] \\ & \mathcal{Y}(S_1) & \end{tikzcd} \text{ commutes.}$$

Let $\mathcal{T}$ Grothendieck topology of Aff_k.

e.g. $\mathcal{T} = \text{ét}, \text{Zar}, \text{fppf}, \text{fpqc}$, etc.

<u>Def'n</u> A $\mathcal{T}$-stack $/k$ is a functor

$$\mathcal{Y} : \mathrm{Aff}_k^{op} \longrightarrow \mathrm{Grpd}$$

satisfies τ-closed condition, i.e.

if $S^0 \to S'$ is a τ-cover

$$\cdots S^2 \substack{\to\\\to\\\to} S^1 \rightrightarrows S^0 \to S'$$

then $\mathcal{Y}(S') \sim \varprojlim (\mathcal{Y}(S^0) \rightrightarrows \mathcal{Y}(S^1) \substack{\to\\\to\\\to} \mathcal{Y}(S^2) \cdots)$.

<u>Example</u> For G/k grp sch, τ-classifying space

$$\mathbb{B}_\tau G := \operatorname{colim}[* \leftleftarrows G \substack{\leftarrow\\\leftarrow\\\leftarrow} G \times G \substack{\leftarrow\\\leftarrow\\\leftarrow} \cdots]$$

taken in Stk_k^{I}

s.t. $\mathbb{B}_\tau G(S) = \{\tau\text{-locally } G\text{-torsors on } S\}$

<u>Exercise</u> If $G \to k$ smooth, then étale classifying space

$$\mathbb{B}_{\acute{e}t} G = \mathbb{B}_{sm} G = \mathbb{B}_{fppf} G = \mathbb{B}_{fpqc} G.$$

<u>Def'n</u> G/k affine alg grp,

$p: X \to S$ proj & flat $\rightsquigarrow$ $\underset{\mathrm{Bun}_{G,X/S}}{\overset{\underline{\mathrm{Hom}}_S(X, \mathbb{B}_{\acute{e}t} G)}{\shortparallel}} \longrightarrow S$

For any $S' \to S$,

$$\mathrm{Bun}_{G,X/S}(S') = \mathrm{Hom}(X \times_S S', \mathbb{B}_{\acute{e}t} G)$$
$$= \{\text{étale } G\text{-torsors on } X \times_S S'\}$$

<u>Thm</u> $\mathrm{Bun}_{G,X/S}$ is an Artin stack, locally finitely presented $/S$ with affine diagonal.

- $\mathrm{Bun}_{G,X/S} \longrightarrow \mathrm{Bun}_{G,X/S} \times_S \mathrm{Bun}_{G,X/S}$ is affine
- $\mathrm{Bun}_{G,X/S} = \bigcup U_r$, $U_r \to \mathrm{Bun}_{G,X/S}$ is open embedding

  $\exists V_r \to U_r$ smooth s.t. $V_r \to S$ is fp.

note If G/k is smooth & $X \to S$ a (rel) curve,

$\mathrm{Bun}_{G,X/S} \to S$ is smooth ($V_r \to S$ are smooth).

E.g. When $G = GL_n$. fix $\mathcal{O}(1)$ ample line bundle on X

$$U_r = \left\{ \begin{array}{l} \text{rk } n \text{ vec bun } \mathcal{F} \text{ on } X \\ \text{s.t. } R^i p_*(\mathcal{F}(r)) = 0, \ i>0 \\ \qquad p^* p_*(\mathcal{F}(r)) \twoheadrightarrow \mathcal{F}(r) \end{array} \right\}$$

- Bun_G is not connected, not quasi-cpt neither.
- Bun_G^λ is not quasi-cpt.

Thm If any G-torsor on $\mathrm{Spec}\, k(x)$ and $\mathrm{Spec}\, k'$ ($[k':k] < \infty$) is trivial, then $G(\mathbb{O}) \backslash G(\mathbb{A}) / G(k) \xrightarrow{\sim} \mathrm{Bun}_G(k)$.

Rmk When k finite, this works for

G simply conn & semisimple (e.g. SL_2)

or $G = GL_n$.

When $k = \bar{k}$, char $k = 0$, valid for any G.

When $k = \bar{k}$, char $k = p$, valid for connected red grp G.

Write

$$\begin{tikzcd} & \mathrm{Hecke}_{G,X} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "(\mathrm{supp}, \overrightarrow{h})"] & & X \arrow[dl, "\mathrm{triv}"] \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G & \end{tikzcd}$$

$$\mathrm{Hecke}_{G,X}(S) = \left\{ \begin{array}{l} x: S \to X, \ \mathcal{P}, \mathcal{P}' \text{ on } X \times_k S \\ \qquad \mathcal{P}|_{X_S - \Gamma_x} \simeq \mathcal{P}'|_{X_S - \Gamma_x} \end{array} \right\}.$$

Thm $Gr_{G,x}$ is an ind-finite type ind-scheme.
If G is reductive,
then $Gr_{G,x}$ is ind-projective.

Morally "$Hecke_{G,x} = Gr_{G,x} \tilde{\times} Bun_G$".

- $\mathcal{L}G_x(S) = \{x : S \to X,\ D_x^\circ \to G\}$
  $L^+G_x(S) = \{x : S \to X,\ D_x \to G\}$.
- If $S = \operatorname{Spec} k$, $D_x = \operatorname{Spec} k[[t]]$,
  $D_x^\circ = \operatorname{Spec} k((t))$.

Fix $S \xrightarrow{x} X$. $\rightsquigarrow$ $\Gamma_x : S \to X \times S$
with $\hat{\Gamma}_x$ formal completion of Γ_x (formal sch).

$$\begin{array}{c} \hat{\Gamma}_x \\ \downarrow \\ D_x \end{array}$$

For any affine $Z \to S$ & $\begin{tikzcd} \hat{\Gamma}_x \arrow[rr] \arrow[dr] & & Z, \\ & D_x \arrow[ur, dashed] & \end{tikzcd}$
denote $D_x^\circ := D_x \setminus \Gamma_x$.

$Gr_{G,x} \simeq [\mathcal{L}G_x / \mathcal{L}^+G_x]_{\text{ét, sm}, \dots}$
$\rightsquigarrow$ $\mathbb{B}\mathcal{L}^+G = \{G\text{-torsors on } D_x\}$ & $\mathbb{B}\mathcal{L}G = \{G\text{-torsors on } D_x^\circ\}$.

Pass to infinite level:

$Bun_G^\infty(S) = \{x : S \to X,\ P \text{ on } X_S \text{ s.t. } P|_{D_x} \cong triv\}$.
$\hookrightarrow Bun_G$ $\mathcal{L}^+G_x$-torsor.
$\mathcal{L}^+G_x \curvearrowright Gr_{G,x}$
$\rightsquigarrow$ $Hecke_{G,x} \simeq Gr_{G,x} \tilde{\times} Bun_G$
$\Rightarrow$ $Hecke_{G,x}$ is an ind-Artin stack locally of finite type.

Define the local Hecke stack

$$\mathrm{Hecke}^{loc}_{G,x} := [\mathcal{L}^+G_x \backslash \mathcal{L}G_x / \mathcal{L}^+G_x]$$

$$\rightsquigarrow \quad \mathrm{Hecke}_{G,x} \longrightarrow \mathrm{Hecke}^{loc}_{G,x}.$$

Then: Geometric Satake says

$$\text{“} \mathrm{Perv}(\mathcal{L}^+G_x \backslash \mathcal{L}G_x / \mathcal{L}^+G_x) \simeq \mathrm{Rep}(\check{G}) \text{”}.$$

For any $V \in \mathrm{Rep}(\check{G})$, Sat_V sheaf on $\mathcal{L}^+G \backslash \mathcal{L}G / \mathcal{L}^+G$.

(correct track to define Hecke eigen-property).

$$\begin{tikzcd} & \mathrm{Hecke}_{G,x} \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "m"] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_G & \mathrm{Hecke}^{loc}_{G,x} & \mathrm{Bun}_G \end{tikzcd}$$

$$\rightsquigarrow \quad H_{V,x} : D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G)$$

via $\overrightarrow{h}_!(m^*(\mathrm{Sat}_V) \otimes \overleftarrow{h}^*(-))$

$$H_V : D(\mathrm{Bun}_G) \longrightarrow D$$

via $\mathcal{F} \longmapsto V_E \otimes \mathcal{F}$

Hecke eigensheaf.

E.g. $\mathcal{L}^+G \curvearrowright \mathrm{Gr}_G$, $G = GL_n$.

then $\exists$ exactly $(n+1)$ closed orbits Gr^i_G

s.t. $\mathrm{Hecke}^i_n \simeq \mathrm{Gr}^i_{G,x} \tilde{\times} \mathrm{Bun}_n$.

$$H^i_n : D(\mathrm{Bun}_n) \longrightarrow D(X \times \mathrm{Bun}_n)$$

$$\mathcal{F} \longmapsto \overleftarrow{h}_! \overrightarrow{h}^* \mathcal{F}\,[\text{shift}]$$

Def $\mathcal{F}$ is a Hecke eigensheaf w/ eigenval E if

$$H^1_n(\mathcal{F}) \simeq E \boxtimes \mathcal{F}.$$

Claim: $H^i_n(\mathcal{F}) = \Lambda^i E \boxtimes \mathcal{F}$. (see next time.)

Lecture 4: Hecke eigenproperty

<u>Starting with</u>

$$\begin{tikzcd} & \mathrm{Gr}_n^i \tilde{\times} \mathrm{Bun}_n = \mathrm{Hecke}_n^i \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\mathrm{supp}"] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_n & X & \mathrm{Bun}_n \end{tikzcd} \qquad (0 \le i \le n)$$

- Hecke_n^i classifies $(x \in X, V, V', V' \hookrightarrow V$ s.t. $V/V' \simeq \mathcal{O}_x^{\oplus i})$
  or $V' \hookrightarrow V \hookrightarrow V'(x)$ s.t. length $(V/V') = i$.

<u>Def'n</u> Define the ith Hecke eigensheaf to be

$$H_n^i(\mathcal{F}) \simeq (\mathrm{supp} \times \overrightarrow{h})_* \circ \overleftarrow{h}^* \mathcal{F}\left(\tfrac{i(n-i)}{2}\right)[i(n-i)].$$

rel dim$(\mathrm{Hecke}^i_{n,x}, \mathrm{Bun}_n) = i(n-i)$

<u>Fact</u> It is an eigensheaf b/c:

ith Hecke eigenproperty (with eigenval $\mathcal{E}$)

$$H_n^i \mathcal{F} \simeq \Lambda^i E \boxtimes \mathcal{F}.$$

$\mathrm{Hecke}_G \simeq \mathrm{Gr}_G \tilde{\times} \mathrm{Bun}_G$ $\quad \left[\begin{array}{l} \mathcal{L}^+G \circlearrowright \mathrm{Gr}_G \\ \mathrm{Bun}_G^\infty \to \mathrm{Bun}_G \;\; \mathcal{L}^+G\text{-action} \end{array}\right.$

$\uparrow$ ∞-level

$\Rightarrow$ for any $\mathcal{L}^+G$-orbit Gr_G^λ on Gr_G

$\rightsquigarrow$ $\mathrm{Hecke}_G^\lambda = \mathrm{Gr}_G^\lambda \tilde{\times} \mathrm{Bun}_G$.

<u>Combinatorics</u> $G = GL_n$, $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \ge \dots \ge \lambda_n$

$\rightsquigarrow$ $\mathrm{Gr}_n^\lambda := \mathcal{L}^+G \cdot t^\lambda \cdot \mathcal{L}^+G / \mathcal{L}^+G$, $\; t^\lambda = \mathrm{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$.

<u>Fact</u> $\mathrm{Gr}_n^\lambda \subset \overline{\mathrm{Gr}_n^\mu} \iff \mu - \lambda > 0$.

__Notation__ Denote

$$\mathrm{Gr}^i_n = \mathcal{L}^+G \cdot \begin{pmatrix} t & & & \\ & \ddots & & \\ & & t & \\ & & & 1 \end{pmatrix} \cdot \mathcal{L}^+G \,/\, \mathcal{L}^+G .$$

Automorphic analog of $\mathcal{H}^i_n \mathcal{F} \simeq \Lambda^i E \boxtimes \mathcal{F}$:

$$T^i_n f(x) = q^{-\frac{i(n-i)}{2}} \int_{\mathrm{Gr}^i_n} f(xg)\, dg, \qquad f \in C^\infty_c(\mathcal{L}^+G \backslash \mathcal{L}G / \mathcal{L}^+G).$$

$$\underset{\uparrow}{=} \mathrm{tr}(\mathrm{Fr}_x ; \Lambda^i \rho) \otimes f.$$

$$\left(\begin{array}{l} \text{take } E \leftrightarrow W_{F/K} \xrightarrow{\rho} GL_n(\overline{\mathbb{Q}}_\ell) \text{ Weil rep} \\ \quad \text{w/ } \det(1 - \rho(\mathrm{Fr}_x)\cdot s) = 1 - \mathrm{tr}(\rho(\mathrm{Fr}_x))\cdot s + \mathrm{tr}(\Lambda^2 \rho(\mathrm{Fr}_x))\, s^2 + \cdots \end{array} \right).$$

__Prop__ If $\mathcal{F}$ is a perverse sheaf equipped w/ $\mathcal{H}^1_n \mathcal{F} \simeq E \boxtimes \mathcal{F}$
s.t. $\mathcal{H}^1_n \circ \mathcal{H}^1_n \mathcal{F}|_{X^2 - \Delta} \longrightarrow E \boxtimes E \boxtimes \mathcal{F}|_{X^2-\Delta}$ is S_2-equiv.
Then $\mathcal{H}^i_n \mathcal{F} \xrightarrow{\sim} \Lambda^i E \boxtimes \mathcal{F}$.

$$\begin{tikzcd}[column sep=small]
 & & S_2 \circlearrowright \{x \neq x', V, V', V'' \text{ s.t. } V/V' \simeq V'/V'' \simeq \mathcal{O}_x,\ V'' \hookrightarrow V' \hookrightarrow V\} \arrow[dl, "2"'] \arrow[dr] & & \\
 & \mathrm{Hecke}^1_n \arrow[dl] \arrow[dr] & & \mathrm{Hecke}^1_n \arrow[dl] \arrow[dr] & \\
\mathrm{Bun} & & \mathrm{Bun} & & \mathrm{Bun}_n
\end{tikzcd}$$

__Recall__ $\mathrm{Perv}(\mathcal{L}^+G \backslash \mathcal{L}G / \mathcal{L}^+G) \simeq \mathrm{Rep}(G^\vee)$ geom Satake.

$$IC_{\overline{\mathrm{Gr}^\lambda_G}} \longmapsto V^\lambda$$

Take $\mathrm{Std} \leftrightarrow$ cowt $(1, 0, \cdots, 0)$,

$\mathrm{Std}^{\otimes i} \longleftrightarrow$ cowt $(i, 0, \cdots, 0)$.

$$\begin{array}{c} \leq \\ (\underbrace{1, \cdots, 1}_{i}, \underbrace{0, \cdots, 0}_{n-i}) \end{array}, \quad \forall\, 0 \leq i \leq n.$$

$$\begin{tikzcd}
\left\{ \begin{array}{l} V' \hookrightarrow V \text{ s.t. } V/V' \\ \text{is a torsor of length } i \end{array} \right\} \arrow[r] & \mathrm{Mod}_n^{-i} \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\mathrm{supp}"] \arrow[dr, "\overrightarrow{h}"] & \\
\mathrm{Bun}_n & X^{(i)} & \mathrm{Bun}_n
\end{tikzcd}$$

<u>Compatible def'n</u>

$$\begin{tikzcd}
H_n^i \arrow[dr, hook] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& H_n^{i,+} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Mod}_n^{-i} \arrow[d, "\mathrm{supp}"] \\
& X \arrow[r, "\Delta"] & X^{(i)}
\end{tikzcd}$$

<u>Note</u> $\mathrm{Gr}_{G,X^I} \tilde{\times} \mathrm{Bun}_G = \mathrm{Hecke}_{G,X^I}$

$$\begin{tikzcd}
\mathrm{Gr}_{G,X^I}^{+,\leq(1,\cdots,1)} \tilde{\times} \mathrm{Bun}_n = \mathrm{Hecke}_{G,X^I}^{+,\leq(1,\cdots,1)} \arrow[r] \arrow[d] & \mathrm{Mod}_n^{-i} \arrow[d] \\
X^i \arrow[r] & X^{(i)}
\end{tikzcd}$$

$$\mathrm{Gr}_{G,X^I}^{\leq(1,\cdots,1)}\Big|_{X^I \setminus \Delta} = \left(\mathrm{Gr}_{G,X}^1 \times \mathrm{Gr}_{G,X}^1 \times \cdots \times \mathrm{Gr}_{G,X}^1\right)\Big|_{X^I \setminus \Delta}.$$

$$\hookrightarrow \mathrm{Gr}_{G,X^I}^{+,\leq(1,\cdots,1)}$$

$\widetilde{\mathrm{Mod}}_n^{-i} := \{ V_0 \hookrightarrow \cdots \hookrightarrow V_i,\ \mathrm{length}(V_{j+1}/V_j) = 1 \}$.

Also define

$$\begin{tikzcd}
\widetilde{\mathrm{Mod}}_n^{-i} \arrow[rr, "p", "\text{proper}"'] \arrow[dr, "\theta"] \arrow[dd] & & \mathrm{Mod}_n^{-i} \arrow[dd] \\
& ? \arrow[ur] \arrow[dl] \arrow[r, phantom, "\ulcorner"] & \\
X^i \arrow[rr] & & X^{(i)}
\end{tikzcd}$$

θ isom on $X^i - \Delta$.

<u>Claim</u> (1) $p_! \mathrm{IC} =: \mathrm{Spr}$ is perverse

(2) It is a $!*$-ext'n from $\mathrm{Mod}_n^{-i}|_{X^{(i)} - \Delta} =: \mathrm{Spr}^\circ \circlearrowleft S_i$.

(3) $\mathrm{Hom}_{S_i}(\mathrm{Sign}, \mathrm{Spr}) = \underline{K}\left(\frac{i(n-i)}{2}\right)[i(n-i)]$.

$\uparrow$ const sheaf

singularity resolved

$$\begin{tikzcd}
& \widetilde{\mathrm{Mod}}_n^{-i} \arrow[rr, "\widetilde{\mathrm{supp}}"] \arrow[d] \arrow[ddl, bend right, "\tilde{\overleftarrow{h}}"'] \arrow[ddr, bend left, "\tilde{\overrightarrow{h}}"] & & X^i \arrow[d, "\mathrm{Sym}"] \\
& \mathrm{Mod}_n^{-i} \arrow[rr, "\mathrm{supp}"] \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & & X^{(i)} \\
\mathrm{Bun}_n & & \mathrm{Bun}_n &
\end{tikzcd}$$

Calculate $(\mathrm{supp}\times\overrightarrow{h})_!\, \overleftarrow{h}^* \mathcal{F} \otimes \mathrm{Spr}\left(\frac{i(n-i)}{2}\right)[i(n-i)]$

$$\Big\downarrow \wr \quad \text{projection formula}$$

$$(\mathrm{supp}\circ\widetilde{\mathrm{supp}}\times\tilde{\overrightarrow{h}}_!)\circ \tilde{\overleftarrow{h}}^*\mathcal{F}\left(\frac{i(n-i)}{2}\right)[i(n-i)]$$

$$\Big\downarrow \wr$$

$$\mathrm{Sym}_!(\underbrace{H_n^i\circ\cdots\circ H_n^i}_{i \text{ times}}(\mathcal{F})) = \mathrm{Sym}_!(E^{\boxtimes i}\boxtimes\mathcal{F})$$

Lecture 5: Whittaker

<u>Last time</u> Hecke eigensheaf

$$\begin{tikzcd} \mathrm{Bun}_n & \mathcal{H}_n^d \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] \arrow[d, "\mathrm{supp}"] & \mathrm{Bun}_n \\ & X & \end{tikzcd}$$

rk n v.b.s $\to \mathrm{Bun}_n$

$\mathcal{H}_n^d \ni (x, M, M', M' \subset M \subset M'(x)$ with $\mathrm{length}(M/M') = d)$.

$\mathcal{H}_n^d(K) = (\mathrm{supp} \times \overrightarrow{h})_! \overleftarrow{h}^*(K) \left(\frac{(n-i)i}{2}\right)[(n-i)i]$.

<u>Def</u> Fix local system E on X.

Say E satisfies d-th Hecke-eigenproperty for K if

- $\mathcal{H}_n^d(K) \simeq \wedge^d E \boxtimes K$.

<u>Prop</u> If K is perverse and satisfies 1st eigenproperty then K satisfies dth eigenproperty.

<u>Thm</u> [FGV] If E is geometrically irred, then there exists Aut_E, which is a cuspidal Hecke-eigensheaf. Also, $\mathrm{Aut}_E|_{\mathrm{Bun}_n^d}$ is irred.

<u>Def</u> For any G red grp

$\cup$

P proper parabolic subgrp

$\cup$

M Levi

we have $\mathrm{Bun}_G \xleftarrow{p} \mathrm{Bun}_P \xrightarrow{q} \mathrm{Bun}_M$.

Define $CT_! = q_! \circ p^*$

Say K is <u>cuspidal</u> if $CT_!(K) = 0$ for any p.

<u>Classical Theory</u>

$\sigma : Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_\ell)$

f function on $GL_n(F) \backslash GL_n(\mathbb{A}) / GL_n(\mathcal{O})$

$$T_x^d f(g) = \int_{M_n^d(\mathcal{O}_x)} f(gh)\,dh$$

where $M_n^d(\mathcal{O}_x) = GL_n(\mathcal{O}_x) \cdot \pi_x^{(1,\dots,1,0,\dots,0)} GL_n(\mathcal{O}_x)$

$\subset GL_n(K_x) \subset GL_n(\mathbb{A})$.

Def f satisfies dth Hecke eigenproperty

$$T_x^d f = q_x^{d(n-d)/2}\, Tr(\underbrace{\Lambda^d \sigma(Fr_x)}_{\text{acting on v.s. of dim } \binom{n}{d}})f \quad , \ \forall x \in |X|$$

$(q_x = \# k(x))$

$$= q_x^{-d(d-1)/2}\, Tr(\Lambda^d \sigma(\tfrac{n-1}{2})(Fr_x))f.$$

Def f is cuspidal if $\forall g \in GL_n(\mathbb{A})$,

$$\int_{U(\mathbb{A})/U(F)} f(ug)\,du = 0 . \quad U = \ker(P \to M).$$

Conj (Langlands, theorem by Lafforgue)

For σ, there exists a unique cuspidal Hecke eigenfunction f_σ.

Fix a character $\Psi : N(K_x) \to \bar{\mathbb{Q}}_\ell^\times$

Def A function f on $G(K_x)$ is $(N(K_x), \Psi)$-invariant if

$$f(ug) = \Psi(u) \cdot f(g), \ \forall g \in G(K_x),\ u \in N(K_x).$$

$$
\begin{tikzcd}
\dot N(K_x) \arrow[r] & \dot N/[\dot N,\dot N](K_x) \simeq \bigoplus_x K_x\Omega_x \arrow[d, "\mathrm{res}"] \\
\bar{\mathbb{Q}}_\ell^\times & k \arrow[l, "\psi"'] & \bigoplus k \arrow[l, "\Sigma"'] & \bigoplus_x k_x \arrow[l, "tr"']
\end{tikzcd}
$$

(Ω_x canonical bundle)

fix this ψ

$\dot G = \rho$-shift on G

e.g. $\dot{GL}_n = \{(a_{ij}) : a_{ij} \in \Omega_x^{j-i}\}$

$$
\begin{pmatrix}
K_x & K_x\Omega_x & \cdots & K_x\Omega_x^{1-n} \\
K_x\Omega_x & K_x & \ddots & \vdots \\
\vdots & \ddots & \ddots & K_x\Omega_x \\
K_x\Omega_x^{n-1} & \cdots & K_x\Omega_x & K_x
\end{pmatrix}
$$

Def A <u>Whittaker</u> function on $\dot G(K_x)$ is an $(\dot N(K_x), \Psi)$-invariant fcn.

<u>**Thm**</u> For any $\gamma \in GL_n(\bar{\mathbb{Q}}_\ell)$ (e.g. $\gamma = \sigma(Fr_x)$),
there exists a unique Hecke-eigen Whittake function W
on $\dot G(K_x)/\dot G(\mathcal{O}_x)$,

i.e. (0) $W(0) = 1$,

(1) $W(gh) = W(g)$, $h \in \dot G(\mathcal{O}_x)$

(2) $W(ug) = \Psi_x(u) W(g)$, $u \in \dot N(K_x)$

(3) $T_x^d W = q_x^{-\frac{d(d-1)}{2}} \mathrm{Tr}(\Lambda^d \gamma) W$.

For any λ coweight,

$$
W(\varpi_x^\lambda) = \begin{cases} q_x^{\Sigma(i-1)\lambda_i} \cdot \mathrm{Tr}(\gamma, V^\lambda), & \text{if } \lambda \text{ dominant}, \\ 0, & \text{otherwise}. \end{cases}
$$

$$N(K_x)\backslash G(K_x)/G(\mathcal{O}_x) \xleftrightarrow{\sim} \Lambda$$

$$N(K_x)\cdot \pi_x^{\lambda}\cdot G(\mathcal{O}_x) \longleftarrow\!\shortmid\; \lambda$$

(Spoiler: $D(N(K_x), \Psi_x)\backslash G(K_x)/G(\mathcal{O}_x)) \simeq \mathrm{Rep}(\check{G})$.)

<u>Cor</u> There is a unique unramified Hecke-eigen Whittaker function on $G(\mathbb{A})$

(1) $W(gh) = W(g)$, $h \in G(\mathcal{O})$

(2) $W(ug) = \Psi(u) g$, $u \in N(\mathbb{A})$, $\Psi = \prod' \Psi_x$.

(3) $T_x^d W = q_x^{d(n-d)/2}\, \mathrm{Tr}(\Lambda^d \cdot \sigma(\mathrm{Fr}_x))\, W$.

Define $W_\sigma := \prod' W_{\sigma(\frac{n-1}{2})(\mathrm{Fr}_x), x}$ on $(N(\mathbb{A}), \Psi)\backslash G(\mathbb{A})/G(\mathcal{O})$.

<u>Thm</u> $\exists$ canonical isom

$$\phi : C^\infty((\dot{N}(\mathbb{A}), \Psi)\backslash \dot{G}(\mathbb{A})) \xrightarrow{\sim} C^\infty(\dot{Q}(F)\backslash \dot{G}(\mathbb{A}))_{cusp}$$

$$W \longmapsto \int_{\dot{Q}(F)/\dot{Q}\cap\dot{N}(F)} W(hg)\, dh$$

$$W_\sigma \longmapsto f_\sigma.$$

$$Q = \begin{pmatrix} \boxed{GL_{n-1}} & \boxed{V_{n-1}} \\ \boxed{\quad 0 \quad} & 1 \end{pmatrix}.$$

Next spoiler: $\mathrm{Whit}(\mathrm{Gr}_{G,x}) \simeq \mathrm{Rep}(\check{G})$

$$\Updownarrow$$

$$\begin{tikzcd}
\int_x^{ch} \mathrm{Whit}(\mathrm{Gr}_{G,x}) \arrow[r, phantom, "\simeq"] \arrow[d, equal] & \int_x^{ch} \mathrm{Rep}(\check{G}) \\
\mathrm{Shv}((N(\mathbb{A}), \chi)\backslash G(\mathbb{A})/G(\mathcal{O})) \arrow[d, phantom, "\text{if } G = GL_n \ \text{IS}"] & \mathrm{QCoh}(LS_{\check{G}}) \arrow[u, hook, "\subset"'] \\
\mathrm{Shv}(\dot{Q}(F)\backslash \dot{G}(\mathbb{A})/\dot{G}(\mathcal{O}))_{cusp}. & \\
\mathrm{Shv}(\mathrm{Bun}_G)_{cusp} \arrow[u, hook] \arrow[r, dashed, leftrightarrow, "\text{Conj}"] & \mathrm{QCoh}(LS^{irred}_{\check{G}}) \arrow[uu, hook]
\end{tikzcd}$$

Lecture 6: Whittaker II

Last time Set $G = GL_n$, $Q_1 = \begin{pmatrix} GL_{n-1} & V \\ 0 & 1 \end{pmatrix}$ mirabolic.
$\qquad\qquad\qquad\qquad\qquad\qquad \cup$
$\qquad\qquad\qquad\qquad\qquad\qquad N$

Fix $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$.

Goals

$$\begin{tikzcd}
\mathrm{Av}: C^\infty((N(\mathbb{A}), \Psi)\backslash G(\mathbb{A})) \arrow[r, "\simeq", "\text{Thm 1}"'] & C^\infty(Q_1(K)\backslash G(\mathbb{A}))_{cusp} \\
W_\sigma \arrow[u, phantom, "\in"] \arrow[r, mapsto] \arrow[d, leftrightarrow] & f_\sigma \\
C^\infty((N(K_x), \Psi_x)\backslash G(K_x)) \ni W_{\sigma(Fr_x),x} & \\
\text{Cor of "Thm 2"} \arrow[u] &
\end{tikzcd}$$

Thm 3 f_σ is $G(K)$-inv.

Will see

$$\begin{tikzcd}
\mathrm{Bun}_G^{N\text{-gen}} \arrow[r] & \mathrm{Bun}_G^{B\text{-gen}} \\
\mathrm{Bun}_B^{N\text{-gen}} \arrow[r] \arrow[u, "str"] \arrow[ur, phantom, "\llcorner"] & \mathrm{Bun}_B \arrow[u, "str"']
\end{tikzcd}$$

- $\mathrm{Bun}_B^{N\text{-gen}} = \mathrm{Bun}_B \times_{\mathrm{Bun}_T} \mathrm{Bun}_T^{N\text{-gen}} = \mathrm{Bun}_B \times_{\mathrm{Bun}_T} \mathrm{Div}_\Lambda$.
- $\mathrm{Bun}_N = \{ 0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n,\ \mathcal{F}_i/\mathcal{F}_{i-1} = \Omega^{n-i} \}$

  & $\mathrm{Bun}_N \longrightarrow \prod \mathbb{G}_a \xrightarrow{\Sigma} \mathbb{G}_a$

  $\qquad\qquad\qquad \int^{(N(\mathbb{A}), \Psi)}$

Have

$$\begin{tikzcd}
\mathrm{Av}_!: D(\mathrm{Bun}_G^{N\text{-gen}})^{whit} \arrow[r, "\sim", "\text{Thm 1}'"'] \arrow[d, phantom, "\wr\|"] \arrow[ddr, Rightarrow] & D(\mathrm{Bun}_G^{Q_1\text{-gen}})_{cusp} \arrow[d, phantom, "\wr\| \ \text{Thm 2}'"] \\
D(\mathrm{Bun}_G^{Q\text{-gen}})_{cusp} \simeq \int_x^{ch} D(Gr_{G,x})^{whit} \arrow[d] & \int_x^{ch} QCoh(LS_{\check{G},x}) \arrow[d, shift right] \\
\mathrm{Aut}_E \in D(\mathrm{Bun}_G)_{cusp} \arrow[r, leftrightarrow, "?", "\text{Conj}"'] & QCoh(LS_{\check{G}})_{irred} \ni \delta_E \arrow[u, dashed, shift right, "c"']
\end{tikzcd}$$

$D(\mathrm{Bun}_G^{Q\text{-gen}})_{cusp} \ni \mathrm{Aut}_E$

$T(K) \circlearrowright N(K)^\perp \simeq Ch(N(\mathbb{A})/N(K)) \simeq K^{\oplus(n-1)} \simeq T_{ad}(K) \ni T(K)$.

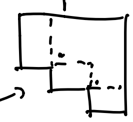
$\rightsquigarrow$ orbits of $T(K) \circlearrowright N(K)^\perp$ $\xleftrightarrow{1-1}$ $\{J \mid J \subset \text{Dykin diagram}\}$

(2^{n-1}- orbits) $\qquad\qquad\qquad\qquad$ $\parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad$ $\{P \mid B \subset P \subset G\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ parabolic.

So $P \rightsquigarrow \Psi_P \in N(K)^\perp$

$\qquad (\Psi_P : N(\mathbb{A}) \to \bar{\mathbb{Q}}_\ell^\times) \leftrightarrow$ 

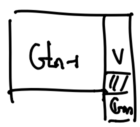
Denote $\Psi = \Psi_G$.

Thm 1 (Extended, by Piatetski-Shapiro, Shalika)

$$Q = \begin{pmatrix} GL_{n-1} & V \\ 0 & \mathbb{G}_m \end{pmatrix}. \text{ Then}$$

$$C^\infty(Q(F)\backslash G(\mathbb{A})) \hookrightarrow \prod_P C^\infty((N(\mathbb{A}), \Psi_P) Z_M(K)\backslash G(\mathbb{A})).$$

sending cuspidal part onto the G-factor.

Levi decomp $\quad Q = L \ltimes V =$ 

$\leftarrow$ corner entry

$V(K)^\perp = Ch(V(\mathbb{A})/V(K))$.

$\circlearrowright$

$L(K)$

There are two orbits:

(1) open one: η_{n-1} stabilizer $Q'(K)$.

(2) closed one: C stabilizer $L(K)$.

By Fourier trans:

$$C^\infty(Q(F)\backslash G(\mathbb{A})) \simeq C^\infty((Q'(K) \ltimes V(\mathbb{A}), \eta_{n-1})\backslash G(\mathbb{A}))$$
$$\times\, C^\infty(L(K) \ltimes V(\mathbb{A}) \backslash G(\mathbb{A})).$$

If $n \geq 3$, $Q' = L' \ltimes V'$

(1) $Q'(\mathbb{A}) \circlearrowright C^\infty((V(\mathbb{A}), \eta_{n-1})\backslash G(\mathbb{A}))$

(2) 2nd factor $\hookrightarrow C^{\infty}(Q'(K) \ltimes V(\mathbb{A}) \backslash G(\mathbb{A}))$.

<u>Thm 1'</u> (Extended)

$$D(\mathrm{Bun}_G^{\dot{Q}\text{-gen}}) \xhookrightarrow{f.f.} \text{"Glue"}\ D(\mathrm{Bun}_G^{Z_M\text{-gen}, \mathrm{Whit}_P}).$$

<u>Variant</u>

$$\begin{tikzcd}
 & \mathrm{Bun}_G^{Q\text{-gen}} = \{(M, \Omega^{\otimes(n-1)} \underset{\neq 0}{\dashrightarrow} M)\} & \\
\mathrm{Bun}_n & \mathrm{Bun}'_n = \{(M, \Omega^{\otimes(n-1)} \xrightarrow{\neq 0} M)\} \arrow[l] \arrow[u, hook] \arrow[dl] \arrow[dd] & \\
{}^{\neq 0}\mathrm{Coh}'_n \arrow[dr] & & \\
 & \mathrm{Coh}'_n = \{(M \text{ coh}, \Omega^{\otimes(n-1)} \to M)\} & \\
 & {}^{\neq}\mathrm{Coh}'_n = \{(M \text{ coh}, \Omega^{\otimes(n-1)} \xrightarrow{\neq 0} M)\}. \arrow[u, hook] &
\end{tikzcd}$$

Caveat: generic v.b. (not v.b.)

$$\left(\begin{array}{l} 0 \to \Omega^{\otimes(n-1)} \hookrightarrow M \twoheadrightarrow M' \to 0 \\ \rightsquigarrow \mathrm{Ext}^1(M', \Omega^{\otimes(n-1)})^* \simeq \mathrm{Hom}(\Omega^{\otimes(n-2)}, M') \end{array} \right)$$

Have the correspondences

$$\begin{tikzcd}
{}^{\neq}\mathrm{Coh}'_n \arrow[dr, "\mathrm{Ext}^1(M', \Omega^{\otimes(n-1)})"'] & & \mathrm{Coh}'_{n-1} \supset {}^{\neq 0}\mathrm{Coh}'_{n-1} \arrow[dl, "H^0(\Omega^{\otimes(n-2)}, M')"] \arrow[dr] & & \mathrm{Coh}'_1 \supset {}^{\neq 0}\mathrm{Coh}'_{n-1} \arrow[dl] \\
 & \mathrm{Coh}_{n-1} \ni M' & & \cdots &
\end{tikzcd}$$

$$F : D({}^{\neq 0}\mathrm{Coh}'_n) \longrightarrow D(\mathrm{Coh}'_{n-1})$$

<u>Rmk</u> Starting from E-Hecke eigensheaf on ${}^{\neq 0}\mathrm{Coh}'_n$,
can produce a sheaf on ${}^{\neq 0}\mathrm{Coh}'_1$,
which has to be a "Whittaker" sheaf.

Lecture 7: Laumon's sheaf

<u>Construction</u> E rk n local system on X.
Tor stack of torsion shv on X.

Will explain: $\mathcal{L}_E \in \mathrm{Perv}(\mathrm{Tor})$.

<u>Motivation</u> Recall the favorate picture:

$$\begin{tikzcd}
& \int \mathrm{Whit}(\mathrm{Gr}_{G,x}) \arrow[r, phantom, "\simeq"] \arrow[d, equal] & \int \mathrm{Rep}(G^\vee) & \\
\text{Whit sheaf } W_E \in & \mathrm{Whit}(\mathrm{Bun}_G^{N\text{-gen}}) \arrow[d, "\mathrm{Av}_!"] \arrow[r, phantom, "\circlearrowleft"] & \mathrm{QCoh}(\mathrm{LS}_{G^\vee}) \arrow[u, hook] \ni \delta_E & \\
& D(\mathrm{Bun}_G^{Q\text{-gen}}) \arrow[d] & & \\
\mathrm{Aut}_E \in & D(\mathrm{Bun}_G) & &
\end{tikzcd}$$

with $W_E \longmapsto \mathrm{Aut}_E$.

<u>Hope</u> $W_E = \delta_E \star W_0$

$$\begin{tikzcd}
\mathrm{pt} \arrow[r, "E"] \arrow[rr, bend right, "\mathrm{ev}_x"'] & \mathrm{LS}_{\check{G}} \arrow[r] & \mathrm{LS}_{G^\vee}(\mathcal{O}_x) \simeq \mathbb{B}G^\vee.
\end{tikzcd}$$

<u>Thm 1</u> (Geometric Satake)

$$\mathrm{Perv}(G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)) \xleftrightarrow[\otimes]{\sim} \mathrm{Rep}(G^\vee)^\heartsuit$$

$$\mathrm{IC}_{\overline{\mathrm{Gr}}_G^\lambda} \longleftrightarrow V^\lambda \qquad (\lambda \in \Lambda^+)$$

<u>note</u>

$$\begin{tikzcd}
\mathrm{Gr}_G \arrow[r, phantom, "\simeq"] & G(K_x)/G(\mathcal{O}_x) \\
\mathrm{Gr}_G^\lambda \arrow[u, hook] \arrow[r, phantom, "\simeq"] & \pi_x^\lambda G(\mathcal{O}_x)/G(\mathcal{O}_x) \arrow[u, hook]
\end{tikzcd}$$

<u>Thm 2</u> (Geometric Casselman-Shalika)

$$\mathrm{Whit}(\mathrm{Gr}_{G,x}) \simeq \mathrm{Rep}(G^\vee)$$

$$W_\lambda \longleftrightarrow V^\lambda$$

Compatible w/ action of Sat.

$N(K_x)$-orbits on $Gr_{G,x}$

$$S^\lambda := N(K_x)\, \pi_x^\lambda G(\mathbb{O}_x)/G(\mathbb{O}_x) \quad (\lambda \in \Lambda)$$

<u>Fact</u> There is an $(N(K_x), \Psi_x)$-invariant sheaf on S^λ

$\Leftrightarrow$ λ is dominant.

<u>Proof of fact</u> $\mathrm{Stab}_{N(K_x)}(\pi_x^\lambda) \subset \ker(\Psi_x) \Leftrightarrow \lambda$ dominant

$\Leftrightarrow \exists\, (N(K_x), \Psi_x)$-sheaf on S^λ. $\square$

<u>Claim</u> $(N(K_x), \Psi_x)$-sheaf on S^λ has clean ext'n to Gr_G.

<u>Ex</u> $\mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1 \overset{i_\infty}{\hookleftarrow} \{\infty\}$

$\mathbb{A}^1 \ni \exp_{\psi}$, $\exp_\psi$ has a clean ext'n along $\mathbb{P}^1$.

$\Rightarrow i_\infty^! j_!(\exp_\psi) = H_c(\exp_\psi(\mathbb{G}_m)) = 0$.

Consider $\mathrm{Rep}(G^\vee)^\heartsuit \xrightarrow{\mathrm{Sat}} \mathrm{Perv}(G(\mathbb{O}_x)\backslash G(K_x)/G(\mathbb{O}_x))$.

$\mathcal{O}_{G^\vee} \longmapsto \mathrm{Sat}(\mathcal{O}_{G^\vee})$

$\rightsquigarrow \int \mathrm{Sat}(\mathcal{O}_{G^\vee}) \in G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}) = \{(M, M', M \overset{\sim}{\dashrightarrow} M' \text{ generically})\}$

$(S_E)^{reg} \in G(\mathbb{O})\backslash G(\mathbb{A})^{reg}/G(\mathbb{O}) \xrightarrow{f} \mathrm{Tor}$.

$\overset{\wr\!\mid}{p^{\otimes}}(\mathcal{L}_E) \qquad \overset{\shortparallel}{\{}(M, M', M \overset{reg}{\hookrightarrow} M' \text{ generically})\}$

<u>Why switch to reg locus</u>

$Gr_G = G(K_x)/G(\mathbb{O}_x)$, $\mathbb{C}^n \overset{reg}{\dashrightarrow} \mathcal{E}$ def'd on unit disc

$Gr_G^{reg} = \bigcup_\lambda Gr_G^\lambda = G(K_x)^{reg}/G(\mathbb{O}_x)$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$

$\rightsquigarrow \ Gr_G^{reg} = \mathrm{Mat}_{n\times n}(\mathcal{O}_x) \cap G(K_x)$.

Can recover Gr_G from Gr_G^{reg}.

$$\begin{array}{ccc} Gr_G^{reg} & \longrightarrow & Gr_G^{reg} \\ (\mathcal{O}^n \xrightarrow{\alpha} \mathcal{E}) & \longmapsto & (\mathcal{O}^n(-x) \to \mathcal{O}^n \xrightarrow{\alpha} \mathcal{E}). \end{array}$$

__Definition of Tor stack__

$\mathrm{Tor}(S) = \{ \mathcal{M} \in \mathrm{Coh}(X\times S),\ \mathcal{M} \text{ is } S\text{-flat},\ \mathcal{M}|_s \text{ is torsion} \}$

$\mathrm{Tor} = \coprod \mathrm{Tor}^d$

Want to show each Tor^d is a stack.

$\mathrm{Quot}_{\mathcal{O}_x^d}^{tor, deg = d} = \{ \mathcal{O}_x^{\oplus d} \overset{p}{\twoheadrightarrow} \mathcal{M},\ \mathcal{M} \text{ torsion of deg } d \}$

$\overset{\circ}{\mathrm{Quot}}{}_{\mathcal{O}_x^d}^{d} \overset{\cup}{} = \{ p : p \text{ bijective on } H^0(X, \mathcal{O}_x^{\oplus d}) \}$.

__Fact__ $\overset{\circ}{\mathrm{Quot}}{}_{\mathcal{O}_x^d}^{d} / GL_d \simeq \mathrm{Tor}^d$.

__Ex__ If $X = \mathbb{A}^1$, $\mathrm{Tor}^d \simeq \mathfrak{gl}_d / GL_d$.

__Stratification of Tor^d__

It is stratified by $\mathrm{Part}^d = \{ (d_1 \geq d_2 \geq \cdots \geq d_s) : d_1 + \cdots + d_s = d \}$

$\rightsquigarrow \mathrm{Tor}^{(d_1, \cdots, d_s)}$ is the image of

$$\begin{array}{ccc} X^{(d_1 - d_2)} \times X^{(d_2 - d_3)} \times \cdots \times X^{(d_s)} & \longrightarrow & \mathrm{Tor}^d \\ (D_1, \cdots, D_s) & \longmapsto & \mathcal{O}_{D_1 + \cdots + D_s} \oplus \mathcal{O}_{D_2 + \cdots + D_s} \oplus \cdots \oplus \mathcal{O}_{D_s}. \end{array}$$

$\rightsquigarrow X^{(d)} \to \mathrm{Tor}^d$ gives the open strata.

__Rmk__ $\mathrm{Tor}^{(d)} = \mathfrak{gl}_d^{reg} / GL_d$.

$\mathrm{Tor}^{(d)}|_{X^{(d)} - diag} = \mathfrak{gl}_d^{rss} / GL_d$.

$X^{(d)} \simeq \mathfrak{gl}_d /\!/ GL_d = \mathfrak{t}_d /\!/ W$.

$$\begin{tikzcd}
X^{(d)} \arrow[r] & \mathrm{Tor}^{(d)} \arrow[r, "q"] \arrow[d, phantom, "\cap"] & X^{(d)} \\
\underbrace{\mathring{\mathrm{Quot}}^d_{\mathcal{O}_X^{\oplus d}}}_{=\{\mathcal{O}_X^{\oplus d} \twoheadrightarrow \mathcal{M}\}} / GL_d \simeq & \mathrm{Tor}^d \arrow[ur] & \\
& \widetilde{\mathrm{Tor}}^d \arrow[u, "\pi"] \arrow[r, "\rho"] & X^d \arrow[uu, "r"']
\end{tikzcd}$$

$E^{\boxtimes d}$ loc sys

$$\widetilde{\mathrm{Tor}}^d = \{0 \subset M_1 \subset \cdots \subset M_d,\ \mathrm{length}(M_i/M_{i-1}) = 1\}.$$

$$\begin{tikzcd}
G \arrow[r, hook, "s"] & \mathfrak{gl}_d^{reg}/GL_d \arrow[r] \arrow[d, phantom, "\cap"] & G \\
& \mathfrak{gl}_d/GL_d \arrow[ur] & \\
& \widetilde{\mathfrak{gl}}_d/GL_d \arrow[u] \arrow[r] & t_d \arrow[uu]
\end{tikzcd}$$

<u>Thm</u> $\underset{\substack{\| \\ \mathrm{Spr}^d}}{\pi_* \rho^* (E^{\boxtimes d})} = i^{(d)}_{!*} q^* r_* (E^{\boxtimes d}) \underbrace{[\mathrm{shift}]}_{\text{to make it perv.}}$

$L_E^d = \mathrm{Hom}_{S_d}(\mathrm{triv}, \mathrm{Spr}^d)$
$\in \mathrm{Perv}(\mathrm{Tor}^d)$.

Define L_E s.t. $L_E|_{\mathrm{Tor}^d}$ is L_E^d.