

Laumon's sheaf II.

Last time, defined

$$L_E \in \text{Perv}(\text{Tor}).$$

$$\begin{array}{ccccccc}
 X^d & \longrightarrow & \widetilde{\text{Tor}}^{(d)} & \longrightarrow & \widetilde{\text{Tor}}^d & \xrightarrow{p} & X^d \\
 \downarrow \perp & & \downarrow \perp & & \downarrow \pi & & \downarrow \Gamma \\
 \underline{X}^{(d)} & \longrightarrow & \underline{\text{Tor}}^{(d)} & \xrightarrow{j} & \text{Tor}^d & \longrightarrow & \underline{X}^{(d)}
 \end{array}$$

Claim:
Outer is Cartesian

$$\begin{array}{ccc}
 \widehat{\mathfrak{gl}}_d / \mathfrak{gl}_d & \longrightarrow & \mathfrak{td} \\
 \downarrow & & \downarrow \\
 \mathfrak{gl}_d^{\text{reg}} / \mathfrak{kl}_d \subset \widehat{\mathfrak{gl}}_d / \mathfrak{gl}_d & \longrightarrow & \mathfrak{cd}
 \end{array}$$

Thm: $\text{Spr}_E^d := \pi_* \rho^*(E^{\otimes d})[d]$

is a perverse sheaf, $!*$ -extended from

$\text{Tor}^{d, \text{reg}}$ (and therefore $\text{Tor}^{d, \text{reg}}$).

$$\text{Spr}_E^d = j_! * \left(\underline{\rho_* (E^{\otimes d})} \Big|_{\underline{\text{Tor}}^{(d)}} \right)$$

Cor: $(W \rightarrow S_d) \rightsquigarrow \text{Spr}_E^d.$

$$L_E^d := (S_{pr_E^d})^W$$

$$= \text{div} \left(r \cdot (E^{\otimes d})^W \right) \Big|_{\text{Tor}^d}$$

$$= \text{div} E^{(d)} \Big|_{\text{Tor}^d}$$

Lemma: If E is geometrically irreducible,

then $L_E^d \in \text{Peru}(G_h^d)$ is so.

Lem: $\text{Tor}_x^* \otimes \text{Tor}_y^* \xrightarrow{\sim} \text{Tor}_{x \otimes y}^*$ (XFY)

$$\mathcal{L}_{E, x \otimes y} \cong \mathcal{L}_{E, x} \otimes \mathcal{L}_{E, y}$$

\Rightarrow Reduce to understand

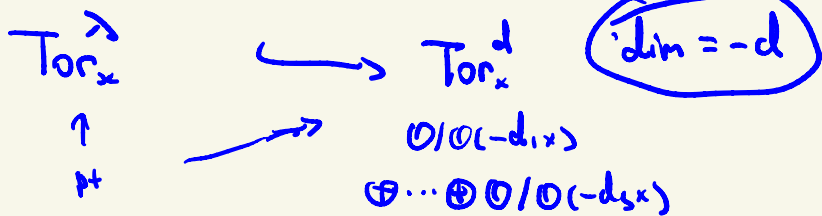
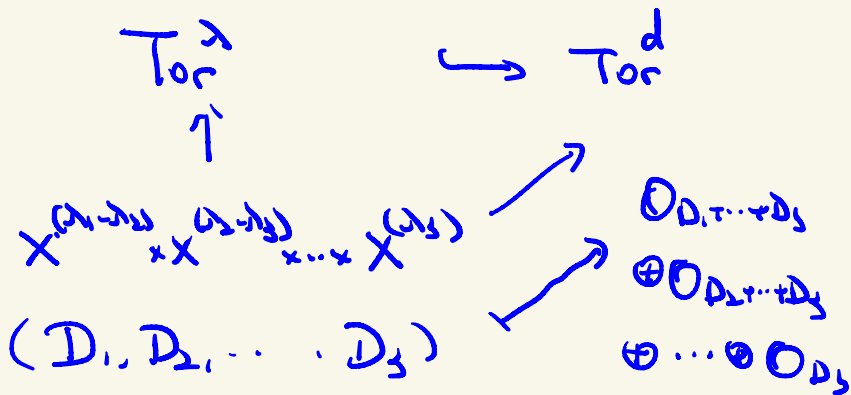
$$\mathcal{L}_{E, x}^d[-d] \in \text{Per}_v(\text{Tor}_x^d)$$

Stratification of Tor_x^* .

$$d = \lambda_1 + \lambda_2 + \dots + \lambda_s$$

$$\lambda = (\lambda_1, \dots, \lambda_s)$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0$$



Def: $B_{\lambda, x} := \text{IC} \overline{\text{Tor}}_{\mathbb{Z}}^{\lambda}$

Prop: $\bigoplus_{\substack{\lambda \in \mathbb{N} \\ |\lambda|=d}} L_{E, x}^d[-d] \simeq \bigoplus_{\substack{\lambda \in \mathbb{N} \\ |\lambda|=d}} B_{\lambda, x} \otimes E_x^{\lambda} \text{ (twists)}$

Rank:

E_x^{λ} highest weight rep of $GL(E_x)$ ($= GL_n$)

$\lambda = (d_1 \geq d_2 \geq \dots \geq d_s > 0)$

$s \leq n$

It is more convenient to extend it to

$\lambda = (d_1 \geq d_2 \geq \dots \geq d_s \geq \underbrace{\dots}_{=0} \geq d_{n+1})$

λ is a dominant weight of GL_n .

or

$E_x^{\lambda} := (E_x^d \otimes \chi^{\lambda})^{S_d}$, $\chi^{\lambda} \in \text{Irr}(S_d)$

(If $s > \dim(E_x) = n$, $R^{\lambda}(E_x) = 0$)

(Total twist = $-\sum (i-1) \lambda_i$)

$$(\lambda) = d$$

regular modification
of degree d
//

$$\begin{array}{ccc} G_{n,x}^\lambda & \longrightarrow & G_{n,x}^{d, \text{reg}} \\ \downarrow \perp & & \downarrow \text{Smooth} \\ \text{Tor}_x^\lambda & \longrightarrow & \text{Tor}_x^d \end{array}$$

$$A_\lambda = \text{IC} \overrightarrow{G_{n,x}^\lambda}$$

$$q^* B_\lambda = A_\lambda[\text{shift}] (\text{twist})$$

$$\text{shift} = -d \cdot n$$

$$\text{twist} = -(\lambda, p)$$

Proof: Springer theory.

$$X = \mathbb{A}^1_k$$

$$\mathrm{Tor}_x^d = \underline{N_d / \mathrm{GL}_d}$$

\downarrow

$$\mathrm{Tor}^d \cong \mathfrak{gl}_d / \mathrm{GL}_d.$$

$$\mathrm{Spr}_E^d \in \mathrm{Peru}(\mathrm{Tor}^d)$$

Base change to $\bar{\mathbb{A}}$ (ignore Tate twist)

$$\text{Recall: } \mathrm{Spr}_{\bar{\mathbb{A}}}^d = \bigoplus_{\chi \in \mathrm{Irr}(W)} \chi \otimes \mathrm{IC}_{\mathbb{O}_\chi}$$

$$\mathrm{Irr}(W) \xleftrightarrow{1-1} \{ \text{Partitions of } d \} \xleftrightarrow{1-1}$$

$\{ \text{nilpotent orbit of } \mathfrak{gl}_d \}$

$$\chi \longmapsto \mathbb{O}_\chi$$

(triv \longmapsto regular nilp. orbit)

$$\mathrm{Sp}_{E,x}^{\mathrm{nd}} = \bigoplus_{\lambda \in \mathrm{Im}(W)} (\lambda \otimes E_x^{\otimes d} \otimes \mathcal{I}_{\mathcal{O}_x})$$

$$\left(\mathrm{Sp}_{E,x}^{\mathrm{d}} \right)^{\mathrm{triv}} = \bigoplus_{\lambda} (\lambda \otimes E_x^{\otimes d})^{\mathrm{Sd}} \otimes \mathcal{I}_{\mathcal{O}_x}$$

Now $(\lambda \otimes E_x^{\otimes d})^{\mathrm{Sd}} = 0$ if $\sum \mathrm{dir}_i = n$

$$d = \lambda_1 + \dots + \lambda_s \quad \lambda_1, 2, \dots = \lambda_s = 0$$

$$\lambda = (\underbrace{\lambda_1, \dots, \lambda_s, 0, \dots, 0}_s)$$

$$\cdot (\lambda \otimes E_x^{\otimes d})^{\mathrm{Sd}} \simeq E_x^{\lambda}$$

$$\cdot \mathcal{I}_{\mathcal{O}_x} \text{ is } B_{\lambda,x}.$$