Laumon's sheaf II.

Last time, defined

$$\mathcal{L}_E \in \mathrm{Perv}(\mathrm{Tor}).$$

$$\begin{tikzcd}
X^d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathrm{Tor}}^{(d)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{\mathrm{Tor}}^d \arrow[r, "p"] \arrow[d, "\pi"] & X^d \arrow[d, "r"] \\
\underline{X}^{(d)} \arrow[r] & \underline{\mathrm{Tor}}^{(d)} \arrow[r, hook, "j"] & \mathrm{Tor}^d \arrow[r] & \underline{X}^{(d)}
\end{tikzcd}$$

Claim: Outer is Cartesian

$$\begin{tikzcd}
& \widetilde{\mathfrak{gl}_d}/GL_d \arrow[r] \arrow[d] & \mathfrak{t}_d \arrow[d] \\
\mathfrak{gl}_d^{reg}/GL_d \subset & \mathfrak{gl}_d/GL_d \arrow[r] & \mathfrak{c}_d
\end{tikzcd}$$

<u>Thm</u>: $\mathrm{Spr}_E^d := \pi_! p^*(E^{\boxtimes d})[d]$

is a perverse sheaf, $!*$-extended from $\mathrm{Tor}^{d,rss}$. (and therefore $\mathrm{Tor}^{d,reg}$).

$$\mathrm{Spr}_E^d = j_{!*}\left(\underline{r_\circ (E^{\boxtimes d})} \big|_{\mathrm{Tor}^{(d)}} \right)$$

<u>Cor</u>: $(W =) S_d \curvearrowright \mathrm{Spr}_E^d$.

$$\mathcal{L}_E^d := (\mathrm{Spr}_E^d)^W$$

$$= j_{!*}\left(r_\cdot(E^{\boxtimes d})^W \right)\Big|_{\mathrm{Tor}^d}$$

$$= j_{!*} E^{(d)}\Big|_{\mathrm{Tor}^d}$$

<u>Lem</u>: If E is geometrically irreducible, then $\mathcal{L}_E^d \in \mathrm{Perv}(\mathrm{Coh}^d)$ is so.

<u>Lem</u>:

$$\mathrm{Tor}_x \times \mathrm{Tor}_y \xrightarrow{\sim} \mathrm{Tor}_{x\cup y} \qquad (x\neq y)$$

$$\mathcal{L}_{E,x\cup y} \simeq \mathcal{L}_{E,x} \boxtimes \mathcal{L}_{E,y}$$

$\Rightarrow$ Reduce to understand

$$\mathcal{L}^d_{E,x}[-d] \in \mathrm{Perv}(\mathrm{Tor}^d_x)$$

Stratification on Tor_x.

$d = \lambda_1+\lambda_2+\cdots+\lambda_s \qquad \underline{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0}$

$\lambda = (\lambda_1, \ldots, \lambda_s)$

$$\begin{tikzcd}
\mathrm{Tor}^\lambda \arrow[r, hook] & \mathrm{Tor}^d \\
X^{(\lambda_1-\lambda_2)} \times X^{(\lambda_2-\lambda_3)} \times \cdots \times X^{(\lambda_s)} \arrow[u] \arrow[ur] & \\
(D_1, D_2, \ldots, D_s) \arrow[r, mapsto] & \mathcal{O}_{D_1+\cdots+D_s} \oplus \mathcal{O}_{D_2+\cdots+D_s} \oplus \cdots \oplus \mathcal{O}_{D_s}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Tor}^\lambda_x \arrow[r, hook] & \mathrm{Tor}^d_x \\
pt \arrow[u] \arrow[r] & \mathcal{O}/\mathcal{O}(-d_1 x) \oplus \cdots \oplus \mathcal{O}/\mathcal{O}(-d_s x)
\end{tikzcd}$$

$\dim = -d$

Def: $B_{\lambda,x} := IC_{\overline{\mathrm{Tor}_x^\lambda}}$

Prop: $\mathcal{L}^d_{E,x}[-d] \simeq \bigoplus_{\substack{s\leq n \\ |\lambda|=d}} B_{\lambda,x} \otimes E_x^\lambda$ (twists)

Rmk: E_x^λ highest weight rep of $GL(E_x)$ $(= GL_n)$

$\lambda = (d_1 \geq d_2 \geq \cdots \geq d_s > 0)$ $s \leq n$

It is more convenient to extend it to

$\lambda = (d_1 \geq d_2 \geq \cdots \geq d_s \geq \underbrace{\cdots \geq d_n}_{=0})$

λ is a dominant weight of GL_n.

or

$E_x^\lambda := (E_x^d \otimes \chi^\lambda)^{S_d}$, $\chi^\lambda \in \mathrm{Irr}(S_d)$

(If $s > \dim(E_x) = n$, $R^\lambda(E_x) = 0$)

(Tate twist $= -\sum (i-1)\lambda_i$)

$|\lambda| = d$

$$\begin{tikzcd}
 & \{\text{regular modifications of degree } d\} \arrow[d, phantom, "\parallel"] \\
\mathrm{Gr}_{n,x}^{\lambda} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_{n,x}^{d,\mathrm{reg}} \arrow[d, "q"', "\underline{\text{smooth}}"] \\
\mathrm{Tor}_x^{\lambda} \arrow[r, hook] & \mathrm{Tor}_x^{d}
\end{tikzcd}$$

$$A_\lambda = IC_{\overline{\mathrm{Gr}_{n,x}^{\lambda}}}$$

$$q^* B_\lambda = A_\lambda[\text{shift}](\text{twist})$$

$$\text{shift} = -d \cdot n$$

$$\text{twist} = -(\lambda, \rho)$$

Proof: Springer theory. $\quad X = \mathbb{A}^1_k$, $x \in X$, $0 \in \mathbb{A}^1_k$

$$\mathrm{Tor}^d_x = \underline{\mathcal{N}_d / GL_d}$$

$$\int$$

$$\mathrm{Tor}^d \simeq \mathfrak{gl}_d / GL_d.$$

$$\mathrm{Spr}^d_E \in \mathrm{Perv}(\mathrm{Tor}^d)$$

Base change to $\bar{\mathbb{F}}_q$ (ignore Tate twist)

Recall: $\quad \mathrm{Spr}^d_{\bar{\mathbb{Q}}_\ell} = \bigoplus_{\chi \in \mathrm{Irr}(W)} \chi \otimes IC_{\mathcal{O}_\chi}$

$\mathrm{Irr}(W) \xleftrightarrow{1-1} \{\text{Partition of } d\} \xleftrightarrow{1-1}$

$\{\text{nilpotent orbit of } \mathfrak{gl}_d\}$

$\chi \longmapsto \mathcal{O}_\chi$

(triv $\longmapsto$ regular nilp. orbit)

$$\mathrm{Spr}^d_{E,x} = \bigoplus_{\chi \in \mathrm{Irr}(W)} \left(\chi \otimes E_x^{\otimes d} \otimes IC_{\bar{O}_x} \right)$$

$$\left(\mathrm{Spr}^d_{E,x}\right)^{\mathrm{triv}} = \bigoplus_{\chi} \left(\underline{\chi \otimes E_x^{\otimes d}} \right)^{S_d} \otimes IC_{\bar{O}_x}$$

Now $\underline{\left(\chi \otimes E_x^{\otimes d}\right)^{S_d}} = 0$ if $\underline{s > \dim E_x = n}$

$d = \lambda_1 + \cdots + \lambda_s \quad \lambda_1 \geq \cdots \geq \lambda_s > 0$

$\lambda = (\underbrace{\lambda_1, \ldots \lambda_s, 0, \ldots 0}_{n})$

$\therefore \left(\chi \otimes E_x^{\otimes d}\right)^{S_d} \simeq E_x^{\lambda}$

$\cdot\ IC_{\bar{O}_x}$ is $B_{\lambda, x}$.