Fourier

Last time:

For any rank n local system E,

studied

$$\mathcal{L}_E \in \mathrm{Perv}(\mathrm{Tor}).$$

Use $\mathcal{L}_E$ to construct Aut_E.

$$\begin{tikzcd}
\mathrm{Gr}_G^{reg} \tilde{\times} \mathrm{Bun}_N = \left\{ \begin{array}{c} 0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \\ M_i/M_{i-1} \simeq \Omega_x^{n-i} \\ M_n \hookrightarrow M \text{, generic iso} \end{array} \right\} \arrow[d, "\pi"] \\
\mathrm{Bun}'_G := \mathrm{Bun}_G^{Q_1\text{-reg}} = \left\{ \begin{array}{c} \Omega_x^{n-1} \hookrightarrow M \\ \text{injective} \end{array} \right\}
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathrm{Gr}_G^{reg} \tilde{\times} \mathrm{Bun}_N \arrow[dl, "p_1"'] \arrow[d] \arrow[dr, "p_2"] & \\
\mathrm{Tor} & \mathrm{Bun}_N \arrow[r] & \mathbb{G}_a
\end{tikzcd}$$

$$\mathrm{Aut}'_E := \pi_! \left(p_1^*(\mathcal{L}_E) \otimes p_2^*(\exp_\psi) \, [\dim] \left(\tfrac{\dim}{2}\right) \right)$$

$\mathrm{Aut}_E^{\prime, d}$ on deg. d component.

Hope :

① If E is irre. .

$\mathrm{Aut}_E^{\prime, d}$ is perverse and irreducible for $d \gg 0$

② $\mathrm{Aut}_E^{\prime, d}$ descents to Bun_G^d with $d \gg 0$

$\rightsquigarrow$ $\mathrm{Aut}_E^d \in \mathrm{Perv}(\mathrm{Bun}_G^d)$

③ $\mathrm{Aut}_E^{\bullet}$ satisfies "truncated" n-th

Hecke :

$$\begin{aligned} \mathrm{Bun}_G^{d+n} &\xrightarrow{\sim} \mathrm{Bun}_G^d \\ M &\longmapsto M(-x) \end{aligned}$$

sends Aut_E^{d+n} to $\Lambda^n E_x \otimes \mathrm{Aut}_E^d$.

There is no obvious reason to believe $\pi_!$ preserves perversity (and irreducibility)

But remember Fourier.

$G = GL_2$

$$\begin{array}{ccc}
\left\{ \begin{array}{c} 0 \to \Omega_x \to M' \to \mathcal{O}_x \to 0 \\ \cap \\ M \end{array} \right\} & \xrightarrow{\pi} & \{ \Omega_x \hookrightarrow M \} = \mathrm{Bun}_2' \\
\Big\downarrow {\scriptstyle \mathcal{L} = M \amalg_{M'} \mathcal{O}_x} & & \Big\downarrow {\scriptstyle \mathcal{L} = M/\Omega_x} \\
\mathrm{Coh}_1' = \{ \mathcal{O}_x \hookrightarrow \mathcal{L} \} & \longrightarrow & \{\mathcal{L}\} = \mathrm{Coh}_1
\end{array}$$

This is Cartesian:

Given $0 \to \Omega_x \to M \to \mathcal{L} \to 0$, let M' be the pullback

$$\begin{array}{c} \mathcal{L} \\ \uparrow \\ \mathcal{O}_x \end{array}$$

- $\mathrm{Bun}'_2 \longrightarrow \mathrm{Coh}_1$ is a vector bundle with fiber

$$\mathrm{Ext}^1(\mathcal{L}, \Omega_x) \simeq \mathrm{Hom}(\mathcal{O}_x, \mathcal{L})^\vee$$

(Serre duality)

as long as $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{L}) = \mathrm{Ext}^1(\mathcal{O}_x, \mathcal{L}^{tf}) = 0$

This is an <u>open</u> condition on Coh_1.

<u>Warn</u>: Not Coh_1' because $\mathcal{O}_x \to \mathcal{L}$ can be 0.

$$
\begin{tikzcd}
 & \square \arrow[dl] \arrow[dr] & \supset & \mathrm{Gr}_G^{reg} \tilde{\times} \mathrm{Bun}_P \arrow[dr] & \\
\mathrm{Bun}_2' \arrow[dr] & & \mathrm{Coh}_1^{',+} \arrow[dl] & \overset{j}{\supset} & \mathrm{Coh}_1' \arrow[d] \\
 & \mathrm{Coh}_1 & & & \mathrm{Tor}
\end{tikzcd}
$$

$\square = \left\{ \begin{array}{l} 0 \to \Omega_x \to M \to \mathcal{L} \to 0 \\ \mathcal{O}_x \to \mathcal{L} \end{array} \right\}$

$\square \to \mathrm{Bun}_N$ by pullback.

$$\square \longrightarrow \mathrm{Bun}_N \longrightarrow \mathbb{G}_a$$

is just the pairing for the two bundles.

(shift to make it perverse)

$$\Longrightarrow \quad \mathrm{Aut}_E' \simeq \mathrm{Four} \circ j_!\left(\mathcal{L}_E|_{\mathrm{Coh}_1'}\right)$$

at least over the open substack of Coh_1 where $\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{L}) = 0$.

$\Rightarrow$). If $j_! = j_{!*} = j_*$ (clean extension)

then Aut'_E is perverse and irreducible at least on some open of Bun'_2.

$\mathrm{Bun}_1 \subset \mathrm{Coh}_1$ open.

<u>Thm</u> (Drinfeld)

$$\mathrm{Coh}_1' \xhookrightarrow{j} \mathrm{Coh}_1'^{,+}$$

$j_!(\mathcal{L}_E|_{\mathrm{Coh}_1'})$ is clean when restricted to

$\mathrm{Bun}_1^d \subset \mathrm{Coh}_1$ with $d > 4(g-1)$.

<u>Proof</u>:

$$\begin{tikzcd} \mathrm{Bun}_1' \arrow[r, hook, "j"] & \mathrm{Bun}_1'^{,+} \arrow[d, "\pi"] & \mathrm{Bun}_1 \arrow[l, hook', "i"'] \\ & \mathrm{Bun}_1 & \end{tikzcd}$$

$i(\mathcal{L}) = (\mathcal{O}_X \xrightarrow{0} \mathcal{L})$

(maps between line bundles are either injective or 0)

Need $i^! j_! (\mathcal{L}_E|_{Bun_1'}) = 0$

Contraction principle $\pi_!(\_) = i^!(\_)$

($\mathcal{L}_E|_{Bun_1'}$ is $\mathbb{G}_m$-monodromic for the dilation $\mathbb{G}_m$-action)

Need $Bun_1' \longrightarrow Bun_1$

kills $\mathcal{L}_E|_{Bun_1'}$ for $d > 4(g-1)$.

modulo stacky part, this map is

$$X^{(d)} \xrightarrow{AJ^d} Pic^d$$

<u>Thm</u> (Deligne's vanishing)

If E is rank n irreducible, then for $d > n(2g-2)$,

$AJ_!^d$ kills $E^{(d)}$.

For GL_n:

Naive:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{F}_n & & & & \mathcal{F}_2 & & & \mathcal{F}_1 = \mathbb{1}_E|_{Coh_1'} & \\
\mathrm{Coh}_n^{1,+} \arrow[dl] & \supseteq^{j_n} & \mathrm{Coh}_n' \arrow[dr] & \cdots & \mathrm{Coh}_2^{1,+} \arrow[dl] & \supseteq^{j_2} & \mathrm{Coh}_2' \arrow[dr] & \mathrm{Coh}_1^{1,+} \arrow[d] & \supseteq^{j_1} & \mathrm{Coh}_1' \arrow[d] \\
\mathrm{Coh}_n & & & \mathrm{Coh}_{n-1} \;\; \mathrm{Coh}_2 & & & & \mathrm{Coh}_1 & & \mathrm{Tor}
\end{tikzcd}$$

$\mathcal{F}_{k+1} = \mathrm{Four}(j_{k,!}\,\mathcal{F}_k)$

But need to restrict to certain opens

$C_k \subseteq \mathrm{Coh}_k$ s.t.

1. $$\begin{tikzcd}
\mathrm{Coh}_{k+1}' \arrow[dr] & & \mathrm{Coh}_k^{1,+} \arrow[dl] \\
 & \mathrm{Coh}_k &
\end{tikzcd}$$

are indeed vector bundles

2. $j_{k,!}(\mathcal{F}_k)$ is clean over such opens.

But unlike the case $G = GL_2$, we cannot make $C_k \subset \mathrm{Bun}_k$ because otherwise we only detect part of $\mathrm{Gr}_G^{rg} \tilde{\times} \mathrm{Bun}_{n} \longrightarrow \mathrm{Bun}_n'$.

when $\left| \begin{array}{ll} 0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subseteq M & M_k \hookrightarrow M \text{ is maximal } k < n \end{array} \right|$.

We will define C_k later.

But now: If $M_k \in C_k$, then

$\boxed{\text{Condition (1)}}$:

$$\mathrm{Ext}^1(\Omega^{k-1}, M_k) \simeq \mathrm{Ext}^1(\Omega^{k-1}, M_k^{tf}) = 0$$

$\boxed{\text{Condition (2)}}$

$$\dot{\deg}(M_k^{tf})$$
$$:= \deg(M_k^{tf}) - \deg(\mathcal{O} \oplus \Omega \oplus \cdots \oplus \Omega^{k-1})$$
$$> nk(2g-2)$$

($\dot{\deg}$ is the normalized degree corresponding to p-shift.

From now on, all degrees should be $\dot{\deg}$.

I might forget to put the dot.)

Thm (Cleanness): $j_{k,!}\mathcal{F}_k$ is clean (and thereby irreducible) at least over

$$\mathrm{Bun}_k \cap \mathcal{C}_k \subset \mathrm{Coh}_k .$$

(More conditions on $\mathcal{C}_k$ will guarantee the claim over $\mathcal{C}_k$)

Thm (vanishing)

$$\begin{tikzcd} & & \mathrm{Tor}^d \\ \mathrm{Bun}_k & \mathrm{Mod}_k^d \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] \arrow[ur, "\pi"] & \mathrm{Bun}_k \end{tikzcd}$$

$\mathrm{Bun}_k = \{M\}$, $\quad \mathrm{Mod}_k^d = \{M \hookrightarrow M'\}$, length $M'/M = d$, $\quad \mathrm{Bun}_k = \{M'\}$

$$\mathrm{Av}_E^d := \overrightarrow{h}_!\left(\overleftarrow{h}^*(-) \otimes \pi^*(\mathcal{L}_E^d)\right)[dn]\left(\tfrac{dn}{2}\right)$$

If E is rank n irreducible, and $d > nk(2g-2)$, then

$$\mathrm{Av}_E^d = 0.$$

Next time:

we will show

Thm V. $\Rightarrow$ Thm. C.