LECTURE 10

Last time, for a diagram $u : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{C}$, we constructed a canonical morphism

$$\operatorname*{colim}_{y \in \mathsf{K}_2} \lim_{\mathsf{K}_1} u(-,y) \to \lim_{x \in \mathsf{K}_1} \operatorname*{colim}_{\mathsf{K}_2} u(x,-)$$

when assuming both sides exist. In this lecture, we discuss several cases when the above morphism is invertible.

1. Finite diagrams

Definition 1.1. *We say a simplicial set K is* ***finite*** *if it has finitely many* non-degenerate *simplexes.*

We say an ∞-category K is ***essentially finite*** *if it can be represented by a quasi-category which is a finite simplicial set.*

1.2. Recall we asy an *ordinary* category K is essentially finite iff the set of equivalence classes of objects and the set of morphisms in K are finite.

Exercise 1.3. *Let K be an essentially finite ∞-category. Show that hK is an essentially finite ordinary category.*

Exercise 1.4. *Let K be an ordinary category. If K is essentially finite as an ∞-category, then it is essentially finite as an ordinary category.*

Warning 1.5. *The converse is not true: even a finite ordinary category may fail to be essentially finite as an ∞-category.*

Exercise 1.6. *Let Idem be the ordinary category defined by*

- *There is an unique object $*$;*
- $\mathsf{Hom}(*,*) := \{\mathsf{id}, e\}$*, with* $e \circ e = e$.

Show that $\mathsf{N}_\bullet(\mathsf{Idem})$ is not equivalent to a finite quasi-category.

Exercise 1.7. *Let $\mathbb{B}(\mathbb{Z}/2\mathbb{Z})$ be the ordinary category defined by*

- *There is an unique object $*$;*
- $\mathsf{Hom}(*,*) := \{\mathsf{id}, f\}$*, with* $f \circ f = \mathsf{id}$.

Show that $\mathsf{N}_\bullet(\mathbb{B}(\mathbb{Z}/2\mathbb{Z}))$ is not equivalent to a finite quasi-category.

1.8. Nevertheless, for finite *partially ordered set*, the above abnormality does not appear.

Exercise 1.9. *Let J be a finite partially ordered set, viewed as an ordinary category. Show that $\mathsf{N}_\bullet(\mathsf{J})$ is a finite simplicial set.*

Proposition 1.10 (Ker.02NB)**.** *For any simplicial set K, there exists a partially ordered set J and an initial morphism $\mathsf{N}_\bullet(\mathsf{J}) \to K$. If K is finite, we can take J to be finite.*

Corollary-Definition 1.11. *Let C be an ∞-category. The following are equivalent:*

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- C *admits limits indexed by finite simplicial sets.*
- C *admits limits indexed by essentially finite ∞-categories.*
- C *admits limits indexed by finite partially ordered sets.*

*We say C **admits finite limits** if it satisfies the above conditions.*

Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between ∞-categories such that C admits finite limits. The following are equivalent:

- *F preserves limits indexed by finite simplicial sets.*
- *F perserves limits indexed by essentially finite ∞-categories.*
- *F preserves limits indexed by finite parially ordered sets.*

*We say F is **left exact** if it satisfies the above conditions.*

*Dually, for a functor $F : \mathsf{C} \to \mathsf{D}$ such that C admits finite colimits, we say F is **right exact** if it preserves finite colimits.*

Warning 1.12. *Let C be an ∞-category that admits finite limits. It may not admit limits indexed by finite* ordinary *categories. For example, C may fail to be idempotent complete.*

Remark 1.13. *In future lectures, we will define left/right exact functors between general ∞-categories.*

Proposition 1.14. *Let C be an ∞-category. The following are equivalent:*

(i) C *admits finite limits.*
(ii) C *admits fiber products and a final object.*
(iii) C *admits finite products and equalizers.*

We also have similar results for left exact functors.

Sketch. The implications (i)⇒(ii)⇔(iii) are standard. To show (ii)⇒(i), let $u : K \to \mathsf{C}$ be a finite diagram. We need to show $\lim u$ exsits. When $K = \varnothing$, this is the final object of C. If K is nonempty, we can find a pushout square in Set_Δ

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & K' \\ \big\downarrow{\scriptstyle\subset} & & \big\downarrow{\scriptstyle\subset} \\ \Delta^n & \xrightarrow{\subset} & K. \end{array}$$

Note that this is also a homotopy pushout sqaure in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$. By the theorem on decomposition of diagrams (Ker.03DB), we have

$$\lim u \xrightarrow{\simeq} \lim u|_{K'} \underset{\lim u|_{\partial\Delta^n}}{\times} \lim u|_{\Delta^n}$$

where the source exists if the target does. By induction, we reduce to prove the claim when $\mathsf{K} = \Delta^n$, which follows from the fact that $\Delta^0 \xrightarrow{0} \Delta^n$ is initial. □

1.15. In fact, an elaboration of the above argument shows:

Proposition 1.16 (HTT.4.4.2.6, 4.4.3.2)**.** *Let C be an ∞-category. The following are equivalent:*

(i) C *admits small limits.*
(ii) C *admits fiber products and small products*
(iii) C *admits small products and equalizers.*

We also have similar results for functors preserving small limits.

2. Filtered ∞-categories

Definition 2.1. *We say an ∞-category C is **filtered** if any finite diagram $u : K \to \mathsf{C}$ can be extended to a diagram $K^{\triangleright} \to \mathsf{C}$.*

*Dually, we say C is **cofiltered** if for any finite diagram $u : K \to \mathsf{C}$ can be extended to a diagram $K^{\triangleleft} \to \mathsf{C}$.*

2.2. Note that C is filtered iff C^{op} is cofiltered. Hence we will focus on filtered ∞-categories.

Exercise 2.3. *Being filtered is invariant under equivalences.*

2.4. An induction argument, which is similar to but more elaborate than that in the proof of Proposition 1.14, gives the following result:

Lemma 2.5 (Ker.02Q0)**.** *An ∞-category C is filtered iff any diagram $u : \partial\Delta^n \to \mathsf{C}$ with $n \geq 0$[1] can be extended to a diagram $(\partial\Delta^n)^{\triangleright} \to \mathsf{C}$.*

Remark 2.6. *Note that $(\partial\Delta^n)^{\triangleright} \simeq \Lambda^{n+1}_{n+1}$.*

2.7. Recall we say an ordinary category C is filtered if

(0) It is nonempty.
(1) For any objects x_1, x_2 in C, there exists an object x equipped with morphisms $x_i \to x$.
(2) For any diagram $x \rightrightarrows y$ in C, there exists $y \to z$ such that the two compositions $x \rightrightarrows y \to z$ are equal.

Note that condition (i) corresponds to the extension problem for $K = \partial\Delta^i$.

Proposition 2.8 (Ker.02PS)**.** *An ∞-category C is filtered iff it satisfies the following conditions:*

(0) *It is nonempty.*
(1) *For any objects x_1, x_2 in C, there exists an object x equipped with morphisms $x_i \to x$.*
(2) *For any objects x, y in C and any diagram $\partial\Delta^n \to \mathsf{Maps}(x,y)$ with $n \geq 1$, there exists a morphism $y \to z$ such that the composition*
$$\partial\Delta^n \to \mathsf{Maps}(x,y) \to \mathsf{Maps}(x,z)$$
is null-homotopic, i.e., equivalent to a constant diagram[2].

Remark 2.9. *Roughly speaking, for $n \geq 2$, the extension problem*
$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{u} & \mathsf{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$
corresponds to an extension problem
$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \mathsf{Maps}(x,y) \\ \downarrow & \nearrow & \\ \Delta^{n-1} & & \end{array}$$

[1]By definition, $\partial\Delta^0 = \varnothing$.

[2]Exercise: for a Kan complex K, a morphism $\partial\Delta^n \to K$ is null-homotopic iff it factors through Δ^n.

such that $x = u(0)$ and $y = u(n)$.

Exercise 2.10. *Let C be a filtered ∞-category. Show that hC is a filtered ordinary category.*

Exercise 2.11. *Let C be an ordinary category. Show that C is filtered as an ordinary category iff it is filtered as an ∞-category.*

Example 2.12. *An ∞-category is filtered if it admits a final object.*

Example 2.13. *An ∞-category is filtered if it admits finite colimits.*

Exercise 2.14. *The category Idem is filtered.*

3. Filtered colimits and finite limits

Theorem 3.1 (Ker.05XS)**.** *Let C be a small ∞-category. Then C is filtered iff the functor*

$$\mathsf{colim} : \mathsf{Fun}(\mathsf{C}, \mathsf{Grpd}_\infty) \to \mathsf{Grpd}_\infty$$

preserves finite limits.

Warning 3.2. *The claim may be false if Grpd_∞ is replaced by general ∞-categories.*

Exercise 3.3. *Consider the discrete topological space $\mathbb{Z}$ and its one-point-compactification $\mathbb{Z} \cup \{\infty\}$*[3] *Consider the partially ordered set P of closed subsets of $\mathbb{Z} \cup \{\infty\}$. Show that filtered colimits in P may not commute with finite limits.*

Corollary 3.4. *Let $F : \mathsf{C} \to \mathsf{D}$ be a final functor between ∞-categories. If C is filtered, so is D.*

Challenge 3.5. *Can you find a proof of the above corollary without using Theorem 3.1?*

3.6. Recall a partially ordered set J is *directed* if any finite subset of it admits an upper bound in J.

Proposition 3.7 (Ker.02QA)**.** *Let C be an ∞-category. The following are equivalent:*

- *C is filtered.*
- *There exists a directed partially ordered set J and a final functor $\mathsf{J} \to \mathsf{C}$.*

Proposition 3.8. *Let C be a filtered ∞-category and $u : K \to \mathsf{C}$ be any finite diagram. Then the forgetful functor $\mathsf{C}_{u/} \to \mathsf{C}$ is final.*

Proof. By Quillen's Theorem A, we only need to show $\mathsf{C}_{u/} \times_{\mathsf{C}} \mathsf{C}_{x/}$ is weakly contractible for any $x \in \mathsf{C}$. We have

$$\mathsf{C}_{u/} \underset{\mathsf{C}}{\times} \mathsf{C}_{x/} \simeq \mathsf{C}_{u \sqcup x/}$$

where $u \sqcup x : K \sqcup \Delta^0 \to \mathsf{C}$ is the disjoint union of u and x. Now the claim follows from the following two exercises. □

Exercise 3.9. *A filtered ∞-category is weakly contractible. Hint: $K \times \Delta^1 \to K^{\triangleright}$.*

Proposition 3.10. *Let C be an ∞-category. The following are equivalent:*

(i) *C is filtered.*

[3] An closed subset of $\mathbb{Z} \cup \{\infty\}$ is either finite or contains ∞.

(ii) *For any finite diagram* $u : K \to \mathsf{C}$, *the* ∞*-category* $\mathsf{C}_{u/}$ *is filtered.*
(iii) *For any finite diagram* $u : K \to \mathsf{C}$, *the* ∞*-category* $\mathsf{C}_{u/}$ *is weakly contractible.*
(iv) *For any finite* $K \in \mathsf{Set}_\Delta$, *the diagonal functor* $\mathsf{C} \to \mathsf{Fun}(K, \mathsf{C})$, $x \mapsto \underline{x}$ *is final.*

Sketch. The implications (i)⇒(ii)⇒(iii)⇒(i) are left as exercises. We will prove (ii)⇔(iii). By Quillen's Theorem A, (ii) is equivalent to:

$$\mathsf{C} \underset{\mathsf{Fun}(K,\mathsf{C})}{\times} \mathsf{Fun}(K, \mathsf{C})^{u/}$$

is weakly contractible for any $u : K \to \mathsf{C}$. It is easy to see the above fiber product is equivalent to $\mathsf{C}^{u/}$, which implies the claim. □

4. Sifted diagram

Definition 4.1. *Let* K *be a simplicial set. We say* K *is* ***sifted*** *if for any finite set* I, *the diagonal morphism* $K \to K^I$ *is final.*

Dually, we say K *is* ***cosifted*** *if for any finite set* I, *the diagonal morphism* $K \to K^I$ *is initial.*

4.2. Note that C is sifted iff C^{op} is cosifted. Hence we will focus on sifted ∞-categories.

Exercise 4.3. *Being filtered is invariant under equivalences.*

Example 4.4. *Any filtered* ∞*-category is sifted.*

Exercise 4.5. *Show that a simplicial set* K *is sifted iff it is nonempty and* $K \to K \times K$ *is final. Hint: If* $K \to K \times K$ *is a weak homotopy equivalent and* $K \neq \varnothing$, *then* K *is weakly contractible.*

Corollary 4.6. *An* ∞*-category* C *is sifted iff it is nonempty and* $\mathsf{C}_{x/} \times_{\mathsf{C}} \mathsf{C}_{y/}$ *is weakly contractible for any pair of objects* $x, y \in \mathsf{C}$.

Warning 4.7. *In classical category theory, an ordinary category is called sifted if it is nonempty and* $\mathsf{C}_{x/} \times_{\mathsf{C}} \mathsf{C}_{y/}$ *is connected for any pair of objects* $x, y \in \mathsf{C}$. *A sifted ordinary category may fail to be sifted as an* ∞*-category.*

Exercise 4.8. *Let* $\Delta_{\leq 1} \subset \Delta$ *be the full subcategory consisting of* $[0]$ *and* $[1]$. *Show that* $\Delta_{\leq 1}^{\mathsf{op}}$ *is sifted as an ordinary category but not as an* ∞*-category.*

Exercise 4.9. *Show that* Δ^{op} *is sifted as an* ∞*-category.*

Definition 4.10. *The colimit of a* Δ^{op}*-indexed diagram is called the* ***geometric realization*** *of such diagram. The limit of a* Δ*-indexed diagram is called the* ***totalization*** *of such diagram.*

4.11. It is known that geometric realizations and filtered colimits *almost* generate all sifted diagrams. See §A.10.

Proposition 4.12. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a final functor between* ∞*-categories. If* C *is sifted, so is* D.

Proof. For any finite set I, consider the commutative diagram

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{F} & \mathsf{D} \\ {\scriptstyle\delta_1}\downarrow & \searrow & \downarrow{\scriptstyle\delta_2} \\ \mathsf{C}^I & \xrightarrow[F^I]{} & \mathsf{D}^I. \end{array}$$

Since F is final, so is F^I ([Lecture 7, Proposition 3.16]). Since C is sifted, δ_1 is final. It follows that δ_2 is also final as desired ([Lecture 7, Exercise 3.13]). □

Exercise 4.13. *Let K be a sifted simplicial set. Suppose C, D and E are ∞-categories that admit K-indexed colimits, and $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is a functor that preserves K-indexed colimits in each variable*[4]. *Show that F preserves K-indexed colimits.*

5. Sifted colimits and finite products

Theorem 5.1 (HTT.5.5.8.11, Ker.05XM)**.** *Let C be a small ∞-category. Then C is sifted iff the functor*

$$\mathsf{colim} : \mathsf{Fun}(\mathsf{C}, \mathsf{Grpd}_\infty) \to \mathsf{Grpd}_\infty$$

preserves finite products.

6. Universality of colimits

Construction 6.1. *Let C be an ∞-category that admits fiber products. Let $\mathsf{X} \to \mathsf{Y}$ be a functor in Grpd_∞. As in classical category theory, we have an adjunction*

$$(\mathsf{Grpd}_\infty)_{/\mathsf{X}} \rightleftarrows (\mathsf{Grpd}_\infty)_{/\mathsf{Y}} \tag{6.1}$$

where:

- *The left adjoint is compatible with the forgetful functors;*
- *The right adjoint is given by $- \times_{\mathsf{Y}} \mathsf{X}$.*

Remark 6.2. *Alternatively, the above right adjoint can be constructed via the simplicial model category $\mathsf{Set}_\Delta^{\mathsf{KQ}}$.*

Theorem 6.3 (Ker.05V5)**.** *The functor*

$$- \underset{\mathsf{Y}}{\times} \mathsf{X} : (\mathsf{Grpd}_\infty)_{/\mathsf{Y}} \to (\mathsf{Grpd}_\infty)_{/\mathsf{X}}$$

preserves small colimits.

Exercise 6.4. *Let $u : \mathsf{K} \to \mathsf{Grpd}_\infty$, $i \mapsto \mathsf{X}_i$ be a small diagram and write $X := \mathsf{colim}\, u$. Show that there is a canonical isomorphism*

$$\underset{i \in \mathsf{K}}{\mathsf{colim}} \, (X_i \underset{X}{\times} Y) \simeq Y.$$

Appendix A. Filtered colimits of ∞-categories

Proposition A.1. *Let C be a small filtered ordinary category. Then any colimit diagram $\overline{u} : \mathsf{C}^{\triangleright} \to \mathsf{Set}_\Delta$ is also a homotopy colimit diagram in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.*

[4] This means $F(c, -)$ and $F(-, d)$ preserve K-indexed colimits for any $c \in \mathsf{C}$ and $d \in \mathsf{D}$.

A.2. We introduce two applications of Proposition A.1. The first application is a description of filtered colimits of ∞-categories.

Exercise A.3. *The functor* $\mathsf{QCat} \to \mathsf{Cat}_\infty$ *preserves small filtered* ∞*-colimits.*

Exercise A.4. *Let* K *be a small filtered* ∞*-category and* $\mathsf{K} \to \mathsf{Cat}_\infty$, $i \mapsto \mathsf{C}_i$ *be a diagram. Show that any object in* $\mathsf{C} := \mathsf{colim}_{\mathsf{K}} \mathsf{C}_i$ *is equivalent to* $\mathsf{ins}_i(x)$ *for some* $i \in \mathsf{K}$ *and* $x \in \mathsf{C}_i$.

Exercise A.5. *Let* K *be a small filtered* ∞*-category and* $\mathsf{K} \to \mathsf{Cat}_\infty$, $i \mapsto \mathsf{C}_i$ *be a diagram. For* $i, j \in \mathsf{K}$ *and* $x \in \mathsf{C}_i$, $y \in \mathsf{C}_j$, *show that*

$$\mathsf{Maps}_{\mathsf{C}}(\mathsf{ins}_i(x), \mathsf{ins}_j(y)) \xleftarrow{\simeq} \underset{k \in \mathsf{K}_{(i,j)/}}{\mathsf{colim}} \mathsf{Maps}_{\mathsf{C}_k}(x_k, y_k),$$

where

- $\mathsf{K}_{(i,j)/}$ *is the coslice* ∞*-category for* $\Delta^0 \sqcup \Delta^0 \xrightarrow{(i,j)} \mathsf{K}$;
- *For* $k \in \mathsf{K}_{(i,j)/}$, x_k *is the image of* x *under the functor* $\mathsf{C}_i \to \mathsf{C}_k$, *and similarly for* y_k.

A.6. The second application is to decompose general colimits into filtered colimits of finite colimits.

Exercise A.7. *Let* K *be any simplicial set. Show that the partially ordered set* $\mathsf{Fin}(K)$ *of finite simplicial subsets of* K *is filtered, and*

$$K \simeq \underset{J \in \mathsf{Fin}(K)}{\mathsf{hocolim}}\, J.$$

Exercise A.8. *For any diagram* $u : K \to \mathsf{C}$, *there is a canonical equivalence*

$$\underset{J \in \mathsf{Fin}(K)}{\mathsf{colim}}\ \underset{J}{\mathsf{colim}}\ u|_J \xrightarrow{\simeq} \underset{K}{\mathsf{colim}}\, u,$$

where the RHS exists if the LHS does.

Corollary A.9. *An* ∞*-category* C *admits small colimits iff it admits small filtered colimits and finite colimits. For such* C, *a functor* $F : \mathsf{C} \to \mathsf{D}$ *perserves small colimits iff it preserves filtered colimits and finite colimits.*

A.10. The above result says

$$\text{small colimits} = \text{small filtered colimits} + \text{finite colimits}.$$

In fact, we also have:

small colimits = small sifted colimits + finite coproducts .

small colimits = small filtered colimits + geometric realization + finite coproducts .

See HA.1.3.3.10.

A.11. **Suggested readings.** Ker.03DD.

Appendix B. Strong universality of colimits

Theorem B.1. *Let* $K \to \mathsf{Grpd}_\infty$, $i \mapsto \mathsf{X}_i$ *be a small diagram. Consider* $\mathsf{X} := \mathsf{colim}_{i \in K} \mathsf{X}_i$. *Then there is a canonical equivalence*

$$(\mathsf{Grpd}_\infty)_{/\mathsf{X}} \to \lim_{i \in K} (\mathsf{Grpd}_\infty)_{/\mathsf{X}_i}, \tag{B.1}$$

where each evaluation functor is

$$- \underset{X}{\times} \mathsf{X}_i : (\mathsf{Grpd}_\infty)_{/\mathsf{X}} \to (\mathsf{Grpd}_\infty)_{/\mathsf{X}_i}.$$

Exercise B.2. *Use Theorem 6.3 to show* (B.1) *is fully faithful.*

Exercise B.3. *Show that Theorem 6.3 remains true if* Grpd_∞ *is replaced by* Set*, but Theorem B.1 would fail.*

B.4. **Suggested readings.** Ker.05SB.