

LECTURE 11

In this lecture, we discuss the ∞ -categorical Yoneda lemma.

1. REPRESENTABLE FUNCTORS

Definition 1.1. Let \mathcal{C} be an ∞ -category and $F : \mathcal{C}^{\text{op}} \rightarrow \text{Grpd}_{\infty}$ be a functor. We say an object $\eta \in F(x)$ **exhibits F as represented by an object $x \in \mathcal{C}$** if, for any object y , the composition

$$(1.1) \quad \text{Maps}_{\mathcal{C}^{\text{op}}}(x, y) \rightarrow \text{Maps}_{\text{Grpd}_{\infty}}(F(x), F(y)) \xrightarrow{\text{ev}_{\eta}} F(y)$$

is an equivalence between ∞ -groupoids. We say F is **representable** if such x and η exist.

Dually, for a functor $G : \mathcal{C} \rightarrow \text{Grpd}_{\infty}$, we say an object $\eta \in G(x)$ **exhibits G as corepresented by an object $x \in \mathcal{C}$** if, for any object y , the composition

$$\text{Maps}_{\mathcal{C}}(x, y) \rightarrow \text{Maps}_{\text{Grpd}_{\infty}}(G(x), G(y)) \xrightarrow{\text{ev}_{\eta}} G(y)$$

is an equivalence between ∞ -groupoids. We say G is **corepresentable** if such x and η exist.

1.2. Note that $\eta \in F(x)$ exhibits $F : \mathcal{C}^{\text{op}} \rightarrow \text{Grpd}_{\infty}$ as represented by $x \in \mathcal{C}$ iff the same object exhibits F as corepresented by the corresponding object $x \in \mathcal{C}^{\text{op}}$. Hence we will focus on representable functors.

Remark 1.3. The composition (1.1) is defined up to homotopy. Note however that equivalences are invariant under homotopy.

Remark 1.4. The composition (1.1) sends id_x to $\eta \in F(x)$.

1.5. It is easy to see Definition 1.1 is invariant under equivalences. In particular:

- (1) Suppose $\eta_1, \eta_2 \in F(x)$ are isomorphic, then η_1 exhibits F as represented by x iff η_2 does so.
- (2) Let $f : x \rightarrow x'$ be an isomorphism, then $\eta \in F(x)$ exhibits F as represented by x iff $F(f)(\eta) \in F(x')$ exhibits F as represented by x' .

Warning 1.6. It may happen that both η_1 and η_2 exhibit F as represented by x , but they are not isomorphic. Namely, if η_1 exhibits F as represented by x , then any automorphism $f : x \rightarrow x$ produces an object $F(f)(\eta_1)$ satisfying the same property. One can show η_1 and $F(f)(\eta_1)$ are isomorphic iff f is homotopic to id_x .

Exercise 1.7. Let $F : \mathcal{C}^{\text{op}} \rightarrow \text{Grpd}_{\infty}$ be any functor. Show that there is a functor $\pi_0 F : \mathbf{h}\mathcal{C}^{\text{op}} \rightarrow \text{Set}$, unique up to unique equivalence, that fits into the following

commutative diagram

$$\begin{array}{ccc} \mathbf{C}^{\text{op}} & \xrightarrow{F} & \mathbf{Grpd}_{\infty} \\ \downarrow & & \downarrow \pi_0 \\ \mathbf{hC}^{\text{op}} & \xrightarrow{\pi_0 F} & \mathbf{Set}. \end{array}$$

Show that if F is representable, then $\pi_0 F$ is representable as a functor between ordinary categories.

Warning 1.8. The converse of Exercise 1.7 is false. In other words, in Definition 1.1, we cannot replace (1.1) by the map

$$\pi_0 \mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y) \rightarrow \pi_0 \mathbf{Maps}_{\mathbf{Grpd}_{\infty}}(F(x), F(y)) \xrightarrow{\text{ev}_{\eta}} \pi_0 F(y).$$

Exercise 1.9. Consider the constant functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_{\infty}$ with value $\{*\}$. The unique object in $F(x)$ exhibits F as represented by x iff x is final in \mathbf{C} . Note however that $\pi_0 F$ is representable iff \mathbf{hC} has a final object.

1.10. However, we also have the following result:

Proposition 1.11. Let $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_{\infty}$ be a representable functor. Then $\eta \in F(x)$ exhibits F as represented by $x \in \mathbf{C}$ iff for any object y , η induces a bijection

$$\pi_0 \mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y) \rightarrow \pi_0 F(y)$$

between sets.

Proof. By assumption, there exists $\eta' \in F(x')$ that exhibits F as represented by x' . Hence by definition, the object $\eta \in F(x)$ corresponds to a morphism $x' \xleftarrow{\alpha} x$ in \mathbf{C} such that $F(x') \rightarrow F(x)$ sends η' to an object isomorphic to η . It follows that we have a commutative diagram

$$\begin{array}{ccc} \pi_0 \mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y) & \longrightarrow & \pi_0 F(y) \\ \downarrow -\circ \alpha & & \parallel \\ \pi_0 \mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x', y) & \xrightarrow{\cong} & \pi_0 F(y). \end{array}$$

Unwinding the definitions, we have:

- η exhibits F as represented by x
- $\Leftrightarrow \alpha$ is an isomorphism in \mathbf{C}^{op}
- $\Leftrightarrow \alpha$ is an isomorphism in \mathbf{hC}^{op}
- \Leftrightarrow For any y , the left vertical map is bijective
- \Leftrightarrow For any y , the top horizontal map is bijective.

□

Remark 1.12. The ordinary category \mathbf{hC} has a natural (strict) enrichment over the ordinary symmetric monoidal category \mathbf{hGrpd}_{∞} (see [Lecture 4, §10]), at least when \mathbf{C} is locally small. Denote this \mathbf{hGrpd}_{∞} -enriched category¹ by \mathbf{hC} . One can similarly define the notion of representable \mathbf{hGrpd}_{∞} -enriched functors $\mathbf{hC}^{\text{op}} \rightarrow \mathbf{hGrpd}_{\infty}$. Then tautologically, a functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_{\infty}$ is representable iff the enriched functor $\mathbf{h}F$ is so.

¹Warning: in HTT, Lurie denoted it just by \mathbf{hC} , the same symbol as the ordinary homotopy category.

2. REPRESENTABLE FUNCTORS AS LEFT KAN EXTENSIONS

2.1. Note that a pair (F, η) can be viewed as a diagram

$$(2.1) \quad \begin{array}{ccc} & \mathbf{C}^{\text{op}} & \\ & \uparrow \eta & \searrow F \\ \Delta^0 & \xrightarrow{x} & \mathbf{Grpd}_\infty \\ & \{*\} & \end{array}$$

Proposition 2.2. *Let \mathbf{C} be an ∞ -category and (F, η) as above. Then η exhibits F as represented by x iff*

- (i) *For each $y \in \mathbf{C}^{\text{op}}$, the ∞ -groupoid $\mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y)$ is essentially small;*
- (ii) *In (2.1), η exhibits F as a left Kan extension of the constant functor $\{*\} : \Delta^0 \rightarrow \mathbf{Grpd}_\infty$ along $x : \Delta^0 \rightarrow \mathbf{C}^{\text{op}}$.*

Sketch. First note that if F is represented by x , then each $\mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y)$ is equivalent to $F(y) \in \mathbf{Grpd}_\infty$ and therefore essentially small. Hence we restrict to the case when (i) is true. By definition,

$$(\mathbf{C}^{\text{op}})_{/y} \times_{\mathbf{C}^{\text{op}}} \Delta^0 \simeq \mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y).$$

It follows that

$$(2.2) \quad (\mathbf{LKE}_x \{*\})(y) \simeq \operatorname{colim}_{\mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y)} \{*\} \simeq \mathbf{Maps}_{\mathbf{C}^{\text{op}}}(x, y).$$

Now (ii) is equivalent to:

- The natural transformation $\mathbf{LKE}_x \{*\} \rightarrow F$ induced by η is invertible.

This is equivalent to

- For any $y \in \mathbf{C}^{\text{op}}$, the morphism $(\mathbf{LKE}_x \{*\})(y) \rightarrow F(y)$ induced by η is invertible.

Unwinding the definitions, the above morphism is homotopic to the composition of (1.1) and (2.2), which implies the desired claim. \square

Corollary 2.3. *Let \mathbf{C} be an ∞ -category and $x \in \mathbf{C}$ be a fixed object satisfying the size condition (i). Then the pairs $(F, \eta \in F(x))$ such that η exhibits F as represented by x are essentially unique.*

Remark 2.4. *The precise meaning of the above corollary is as follows. The pairs $(F, \eta \in F(x))$ are classified by the fiber product of the following ∞ -categories:*

$$\begin{array}{ccc} & (\mathbf{Grpd}_\infty)_{\{*\}/} & \\ & \downarrow & \\ \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty) & \xrightarrow{\text{ev}_x} & \mathbf{Grpd}_\infty. \end{array}$$

Now the proposition claims the full sub- ∞ -category of this fiber product consisting of those pairs satisfying the representability condition, is equivalent to $[0]$.

2.5. By the above corollary, after enlarging the size bound of objects in \mathbf{Grpd}_∞ , we can talk about *the* functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_\infty$ represented by an object $x \in \mathbf{C}$, as long as we incorporate the object η as part of the data in its definition. We *denote* this functor by h_x . When using these notation, we always view it as an object in $\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)$ equipped with a *canonical* object in $h_x(x)$. We often denote this object by id_x because of Remark 1.4. Now Proposition 2.2 can be informally write as

$$h_x \xleftarrow{\simeq} \text{LKE}_x\{*\}$$

Dually, the functor corepresented by x is denoted by h^x .

Exercise 2.6. Describe the corepresentable functor h^x as a left Kan extension.

3. YONEDA LEMMA

Theorem 3.1. Let \mathbf{C} be an ∞ -category and $h_x : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_\infty$ be the functor represented by x . Then for any functor $G : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_\infty$, the composition

$$\mathbf{Maps}_{\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)}(h_x, G) \rightarrow \mathbf{Maps}_{\mathbf{Grpd}_\infty}(h_x(x), G(x)) \xrightarrow{\text{ev}_{\text{id}_x}} G(x)$$

is an equivalence between ∞ -categories.

Proof. Via the equivalence $h_x \xleftarrow{\simeq} \text{LKE}_x\{*\}$, this composition is

$$\begin{aligned} \mathbf{Maps}_{\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)}(\text{LKE}_x\{*\}, G) &\rightarrow \mathbf{Maps}_{\mathbf{Fun}(\Delta^0, \mathbf{Grpd}_\infty)}((\text{LKE}_x\{*\})|_x, G|_x) \rightarrow \\ &\rightarrow \mathbf{Maps}_{\mathbf{Fun}(\Delta^0, \mathbf{Grpd}_\infty)}(\{*\}, G|_x) \simeq G(x), \end{aligned}$$

which is an equivalence by the definition of LKE. \square

Exercise 3.2. Prove the last claim in Warning 1.6.

3.3. By Proposition 2.2, the functor h_x is characterized by the conclusion of Theorem 2.2. In fact, the following stronger result is true.

Proposition 3.4. Let \mathbf{C} be an ∞ -category and $x \in \mathbf{C}$ be an object satisfying the size condition (i). For a functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_\infty$ and $\eta \in F(x)$, the following are equivalent:

- (1) η exhibits F as a functor represented by x ;
- (2) For any functor $G : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_\infty$, η induces an equivalence

$$\mathbf{Maps}_{\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)}(F, G) \rightarrow G(x)$$

between ∞ -categories.

- (3) For any functor $G : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_\infty$, η induces an equivalence

$$\pi_0 \mathbf{Maps}_{\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)}(F, G) \rightarrow \pi_0 G(x)$$

between sets.

Proof. Only (3) \Rightarrow (1) requires proof. The size assumption implies h_x exists. By Theorem 3.1, there is a natural transformation $\alpha : h_x \rightarrow F$ sending $\text{id}_x \in h_x(x)$ to an object isomorphic to $\eta \in F(x)$. It follows that we have a commutative diagram

$$\begin{array}{ccc} \pi_0 \mathbf{Maps}_{\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)}(F, G) & \xrightarrow{\simeq} & \pi_0 G(x) \\ \downarrow -\circ \alpha & & \parallel \\ \pi_0 \mathbf{Maps}_{\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_\infty)}(h_x, G) & \xrightarrow{\simeq} & \pi_0 G(x). \end{array}$$

This implies α is an isomorphism in $\mathbf{hFun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_{\infty})$ and therefore an isomorphism in $\mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_{\infty})$, which implies (1). \square

Remark 3.5. *In this remark, we compare Proposition 1.11 and Proposition 3.4.*

Once we have developed the theory of partially defined adjoint functors. We can rephrase the conditions in Definition 1.1 and Proposition 1.11 respectively as

(a) η exhibits x as the potential value of $\{*\} \in \mathbf{Grpd}_{\infty}$ under the left adjoint functor of

$$F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Grpd}_{\infty}.$$

(a') η exhibits x as the potential value of $\{*\} \in \mathbf{hGrpd}_{\infty}$ under the left adjoint functor of

$$\mathbf{h}F : \mathbf{hC}^{\text{op}} \rightarrow \mathbf{hGrpd}_{\infty}.$$

Proposition 1.11 says if the partially defined left adjoint of F is defined at $\{*\}$, then its value can be detected by the homotopy categories.

Similarly, we can rephrase (2) and (3) in Proposition 3.4 respectively as:

(b) η exhibits F as the potential value of $\{*\} \in \mathbf{Grpd}_{\infty}$ under the left adjoint functor of

$$\mathbf{ev}_x : \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_{\infty}) \rightarrow \mathbf{Grpd}_{\infty},$$

(b') η exhibits F as the potential value of $\{*\} \in \mathbf{hGrpd}_{\infty}$ under the left adjoint functor of

$$\mathbf{h}(\mathbf{ev}_x) : \mathbf{hFun}(\mathbf{C}^{\text{op}}, \mathbf{Grpd}_{\infty}) \rightarrow \mathbf{hGrpd}_{\infty},$$

Proposition 3.4 says if the partially defined left adjoint of \mathbf{ev}_x is defined at $\{*\}$, then its value can be detected by the homotopy categories.

In fact, similar claim is true for the partially defined adjoint functor of any functor.

Exercise 3.6. *Apply the above paradigm to final objects in an ∞ -category. Compare with [Lecture 6, Exercise 2.8].*

Exercise 3.7. *Find what is wrong with the following argument.*

- *False claim: If the limit of a diagram $u : K \rightarrow \mathbf{C}$ exists, then it can be calculated in the homotopy category \mathbf{hC} .*
- *Fake proof: the limit can be written as a LKE, which is the value of a partially defined left adjoint.*

4. YONEDA EMBEDDING

4.1. In this section, we provide a construction of the functor

$$\mathbf{Maps}_{\mathcal{C}}(-, -) : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Grpd}_{\infty}$$

via simplicial categories. In future lectures, we will provide a pure quasi-categorical construction using the theory of left fibrations.

Construction 4.2. *Let \mathcal{C} be a locally small² quasi-category and consider the simplicial category $\mathfrak{C} := \mathfrak{C}(\mathcal{C})$. Let $\mathfrak{C} \rightarrow \mathfrak{C}'$ be a fibrant replacement. Consider the composition of simplicial functors:*

$$\mathfrak{C}(\mathbf{C}^{\text{op}} \times \mathbf{C}) \rightarrow \mathfrak{C}(\mathbf{C}^{\text{op}}) \times \mathfrak{C}(\mathbf{C}) \simeq \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow (\mathfrak{C}')^{\text{op}} \times \mathfrak{C}' \rightarrow \mathbf{Kon},$$

²This means $\mathbf{Maps}_{\mathcal{C}}(x, y)$ is small for any objects x and y .

where the last functor is given by $\text{Hom}_{\mathcal{C}'}(-, -)$. By adjunction, we obtain a functor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathfrak{N}_{\bullet}(\mathcal{K}\text{an}) =: \mathcal{K}\text{an}.$$

Exercise 4.3. The equivalence class of the obtained functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{K}\text{an}$ does not depend on the choice of \mathcal{C}' . In other words, we obtain a well-defined element in

$$\pi_0(\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}\text{an})^{\simeq}).$$

Exercise 4.4. The above equivalence class is invariant under equivalences in \mathcal{C} . In other words, for an equivalence $\mathcal{C} \rightarrow \mathcal{D}$, the map

$$\pi_0(\text{Fun}(\mathcal{D}^{\text{op}} \times \mathcal{D}, \mathcal{K}\text{an})^{\simeq}) \rightarrow \pi_0(\text{Fun}(\mathcal{C}^{\text{op}} \times \mathcal{C}, \mathcal{K}\text{an})^{\simeq}).$$

preserves the corresponding elements.

4.5. As a consequence, for a locally small ∞ -category \mathcal{C} , we obtain a functor

$$\text{Maps}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Grpd}_{\infty}$$

whose equivalence class does not depend on any choice.

Exercise 4.6. Up to equivalence, the above functor lifts the functor $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{hGrpd}_{\infty}$ constructed in [Lecture 4, §10].

Exercise 4.7. For any object $x \in \mathcal{C}$, a morphism $f \in \text{Maps}_{\mathcal{C}}(y, x)$ exhibits the functor

$$\text{Maps}_{\mathcal{C}}(-, x) : \mathcal{C}^{\text{op}} \rightarrow \text{Grpd}_{\infty}$$

as represented by y iff f is invertible. In particular, the morphism id_x provides a canonical equivalence

$$\text{Maps}_{\mathcal{C}}(-, x) \simeq \mathbf{h}_x.$$

Dually, there is a canonical equivalence

$$\text{Maps}_{\mathcal{C}}(x, -) \simeq \mathbf{h}^x.$$

Corollary 4.8. Let \mathcal{C} be a locally small ∞ -category. The functor

$$\mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd}_{\infty}), x \mapsto \text{Maps}_{\mathcal{C}}(-, x)$$

is fully faithful and its essential image consists of representable functors.

Definition 4.9. Let \mathcal{C} be a locally small ∞ -category.

- **A (covariant) Yoneda embedding** is a functor $\mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd}_{\infty})$ equivalent to $x \mapsto \text{Maps}_{\mathcal{C}}(-, x)$.
- **A (contravariant) Yoneda embedding** is a functor $\mathcal{C}^{\text{op}} \rightarrow \text{Fun}(\mathcal{C}, \text{Grpd}_{\infty})$ equivalent to $x \mapsto \text{Maps}_{\mathcal{C}}(x, -)$.

Definition 4.10. Let \mathcal{C} be an essentially small ∞ -category. We write

$$\text{PShv}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Grpd}_{\infty})$$

and call it the ∞ -category of presheaves on \mathcal{C} .

Remark 4.11. In the above definition, we require \mathcal{C} to be essentially small rather than locally small to guarantee $\text{PShv}(\mathcal{C})$ is locally small. Note however that $\text{PShv}(\mathcal{C})$ is almost never small.

4.12. Let \mathcal{C} be an essentially small ∞ -category. Recall $\mathbf{PShv}(\mathcal{C}) := \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Grpd}_{\infty})$ admits all small limits and colimits, which can be calculated pointwisely ([Lecture 7, Theorem 2.14]).

The following result is equivalent to [Lecture 7, Theorem 2.11]. See HTT.5.1.3.2 for a proof using simplicial categories.

Theorem 4.13. *Let \mathcal{C} be an essentially small ∞ -category. Then a Yoneda embedding $\mathcal{C} \rightarrow \mathbf{PShv}(\mathcal{C})$ preserves and detects all limits.*

Warning 4.14. *The limit of a small diagram $u : K \rightarrow \mathcal{C}$ might not exist. This happens iff the limit of $K \rightarrow \mathcal{C} \rightarrow \mathbf{PShv}(\mathcal{C})$ is not a representable presheaf.*

5. COCOMPLETION

Definition 5.1. *We say a functor $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a cocompletion of \mathcal{C} if the following conditions are satisfied:*

- The ∞ -category $\widehat{\mathcal{C}}$ admits small colimits.
- For any ∞ -category \mathcal{D} that admits small colimits, the restriction functor

$$\mathbf{LFun}(\widehat{\mathcal{C}}, \mathcal{D}) \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence, where $\mathbf{LFun}(\widehat{\mathcal{C}}, \mathcal{D}) \subset \mathbf{Fun}(\widehat{\mathcal{C}}, \mathcal{D})$ is the full sub- ∞ -category consisting of functors that preserve small colimits.

Theorem 5.2 (Ker.04BE). *Let \mathcal{C} be an essentially small ∞ -category. A Yoneda embedding $\iota : \mathcal{C} \rightarrow \mathbf{PShv}(\mathcal{C})$ exhibits $\mathbf{PShv}(\mathcal{C})$ as a cocompletion of \mathcal{C} .*

Idea of the proof. We only need to show the restriction functor

$$(5.1) \quad - \circ \iota : \mathbf{LFun}(\mathbf{PShv}(\mathcal{C}), \mathcal{D}) \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence. We claim \mathbf{LKE}_{ι} exists and induces a functor

$$(5.2) \quad \mathbf{LKE}_{\iota} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathbf{LFun}(\mathbf{PShv}(\mathcal{C}), \mathcal{D})$$

that is inverse to the above restriction functor.

In fact, for any $F : \mathcal{C} \rightarrow \mathcal{D}$, the *pointwise* $\mathbf{LKE}_{\iota} F : \mathbf{PShv}(\mathcal{C}) \rightarrow \mathcal{D}$ exists because

$$(5.3) \quad \mathcal{C}_{/\mathcal{M}} := \mathbf{PShv}(\mathcal{C})_{/\mathcal{M}} \times_{\mathbf{PShv}(\mathcal{C})} \mathcal{C}$$

is essentially small³ and therefore

$$(\mathbf{LKE}_{\iota} F)(\mathcal{M}) \simeq \operatorname{colim}_{(h_x \rightarrow \mathcal{M}) \in \mathcal{C}_{/\mathcal{M}}} F(x).$$

One can show $\mathcal{C}_{/\operatorname{colim} \mathcal{M}_i} \simeq \operatorname{colim} \mathcal{C}_{/\mathcal{M}_i}$ and use decomposition of diagrams to show the functor $\mathbf{LKE}_{\iota} F$ preserves small colimits.

It remains to show the obtained functor (5.2) is an inverse to (5.1). It is a right inverse because ι is fully faithful. To show it is a left inverse, we need to show any colimit-preserving $G : \mathbf{PShv}(\mathcal{C}) \rightarrow \mathcal{D}$ satisfies

$$(5.4) \quad \mathbf{LKE}_{\iota}(G \circ \iota) \xrightarrow{\simeq} G.$$

³It is in this step that we use \mathcal{C} is essentially small rather than just locally small (Ker.03WG). The theorem would fail under the latter weaker assumption (Ker.03WF).

Since G is colimit-preserving and the above LKE is pointwise, we only need to treat the universal case when $G = \text{Id}$. Since small colimits in $\text{PShv}(\mathcal{C})$ are preserved and detected by $\mathbf{h}_{\mathcal{M}}^{\text{op}} : \text{PShv}(\mathcal{C}) \rightarrow \text{Grpd}_{\infty}^{\text{op}}$, we only need to treat the cases when $G = \mathbf{h}_{\mathcal{M}}^{\text{op}}$.

In other words, we need to show the commutative diagram

$$\begin{array}{ccc} & \text{PShv}(\mathcal{C})^{\text{op}} & \\ \iota^{\text{op}} \nearrow & \text{---} & \text{h}_{\mathcal{M}} \searrow \\ \mathcal{C}^{\text{op}} & \xrightarrow{\mathcal{M}} & \text{Grpd}_{\infty} \end{array}$$

induces

$$\mathbf{h}_{\mathcal{M}} \xrightarrow{\simeq} \text{RKE}_{\iota^{\text{op}}} \mathcal{M}.$$

Here the equivalence $\mathcal{M} \simeq \mathbf{h}_{\mathcal{M}} \circ \iota^{\text{op}}$ follows from the Yoneda lemma for \mathcal{C} . We only need to show for any $\mathcal{N} \in \text{PShv}(\mathcal{C})^{\text{op}}$,

$$\mathbf{h}_{\mathcal{M}}(\mathcal{N}) \xrightarrow{\simeq} (\text{RKE}_{\iota^{\text{op}}} \mathcal{M})(\mathcal{N}).$$

Applying the Yoneda lemma to $\text{PShv}(\mathcal{C})$, we only need to show⁴

$$\mathbf{h}_{\mathcal{M}}(\mathcal{N}) \xrightarrow{\simeq} \text{Maps}(\mathbf{h}_{\mathcal{N}}, \text{RKE}_{\iota^{\text{op}}} \mathcal{M})$$

Unwinding the definitions, this functor can be identified with

$$\mathbf{h}_{\mathcal{M}}(\mathcal{N}) \simeq \text{Maps}(\mathcal{N}, \mathcal{M}) \simeq \text{Maps}(\mathbf{h}_{\mathcal{N}} \circ \iota^{\text{op}}, \mathcal{M}) \simeq \text{Maps}(\mathbf{h}_{\mathcal{N}}, \text{RKE}_{\iota^{\text{op}}} \mathcal{M}),$$

where the last equivalence uses the universal property of the RKE. \square

5.3. In the above proof, we actually proved the following:

Corollary 5.4. *Let \mathcal{C} be an essentially small ∞ -category. Then $\text{PShv}(\mathcal{C})$ is generated by the image of $\iota : \mathcal{C} \rightarrow \text{PShv}(\mathcal{C})$ under small colimits.*

Remark 5.5. *The precise meaning of the above corollary is: any object in $\mathcal{M} \in \text{PShv}(\mathcal{C})$ is isomorphic to a small colimit of objects contained in the image of ι . Indeed, we have*

$$\mathcal{M} \simeq \text{colim}_{(\mathbf{h}_x \rightarrow \mathcal{M}) \in \mathcal{C}_{/\mathcal{M}}} \iota(x).$$

by applying (5.4) to $G := \text{Id}$.

APPENDIX A. COMPLETELY COMPACT OBJECTS

Definition A.1. *Let \mathcal{D} be a locally small ∞ -category which admits small colimits. We say an object $d \in \mathcal{D}$ is **completely compact** if the functor*

$$\text{Maps}_{\mathcal{D}}(d, -) : \mathcal{D} \rightarrow \text{Grpd}_{\infty}$$

preserves small colimits.

Exercise A.2. *Suppose d is completely compact in \mathcal{D} . Show that any retract of d is also completely compact. Here recall we say c is a retract of d if id_c factors as $c \rightarrow d \rightarrow c$.*

⁴This is where the magic happens. In the general setting, to check a functor $T : \mathcal{U}^{\text{op}} \rightarrow \mathcal{V}$ is a RKE of $\mathcal{U}_0^{\text{op}} \rightarrow \mathcal{V}$, one needs to use *all* functors $S : \mathcal{U}^{\text{op}} \rightarrow \mathcal{V}$ as testing functors and calculate $\text{Maps}(S, T)$. However, when $\mathcal{V} = \text{Grpd}_{\infty}$, the above argument says it is enough to use representable functors as testing functors.

Exercise A.3. *Let \mathcal{C} be an essentially small ∞ -category. Show that representable presheaves $h_x \in \mathbf{PShv}(\mathcal{C})$ are completely compact.*

Theorem A.4 (HTT.5.1.6.8). *Let \mathcal{C} be an essentially small ∞ -category. Then a presheaf $\mathcal{M} \in \mathbf{PShv}(\mathcal{C})$ is completely compact iff it is a retract of some representable presheaf.*

A.5. Suggested readings. : HTT.5.1.6.