LECTURE 12

In this lecture, we introduce adjoint functors between ∞-categories.

1. Definition

1.1. Recall the following definition in ordinary category theory.

Definition 1.2. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be functors between ordinary categories. An* ***adjunction*** *between F and G is a collection of bijections*

$$\rho_{x,y} : \mathsf{Hom}_{\mathsf{D}}(F(x), y) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{C}}(x, G(y))$$

which are functorial on $x \in \mathsf{C}^{\mathsf{op}}$ and $y \in \mathsf{D}$. We also say the construction $(x,y) \mapsto \rho_{x,y}$ ***exhibits F as a left adjoint of G****.*

1.3. Dually, we also say $(x,y) \mapsto \rho_{x,y}$ **exhibits G as a right adjoint of F.**

1.4. In above, the bijections $\rho_{x,y}$ can be encoded as an invertible natural transformation from the clockwise arc to the counterclockwise arc in the following diagram:

$$\begin{array}{ccc} \mathsf{C}^{\mathsf{op}} \times \mathsf{D} & \xrightarrow{(F^{\mathsf{op}}, \mathsf{Id})} & \mathsf{D}^{\mathsf{op}} \times \mathsf{D} \\ {\scriptstyle (\mathsf{Id}, G)} \downarrow & & \downarrow {\scriptstyle \mathsf{Hom}_{\mathsf{D}}(-,-)} \\ \mathsf{C}^{\mathsf{op}} \times \mathsf{C} & \xrightarrow[\mathsf{Hom}_{\mathsf{C}}(-,-)]{} & \mathsf{Set}. \end{array}$$

Note however that in the ∞-categorical setting, the construction of the functor $\mathsf{Maps}_{\mathsf{C}}(-,-) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Grpd}_\infty$ is more complicated. Hence we prefer to encode $\rho_{\bullet,\bullet}$ in an alternative way.

Definition 1.5. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be functors between ∞-categories.*

(1) *We say a natural transformation $\mu : \mathsf{Id}_{\mathsf{C}} \to G \circ F$* ***exhibits F as a left adjoint of*** *G if for any objects $x \in \mathsf{C}$ and $y \in \mathsf{D}$, the composition*

$$\mathsf{Maps}_{\mathsf{D}}(F(x), y) \xrightarrow{G(-)} \mathsf{Maps}_{\mathsf{C}}(G \circ F(x), G(y)) \xrightarrow{- \circ \mu(x)} \mathsf{Maps}_{\mathsf{C}}(x, G(y)) \tag{1.1}$$

is invertible.

(2) *We say a natural transformation $\kappa : F \circ G \to \mathsf{Id}_{\mathsf{D}}$* ***exhibits F as a left adjoint of*** *G if for any objects $x \in \mathsf{C}$ and $y \in \mathsf{D}$, the composition*

$$\mathsf{Maps}_{\mathsf{C}}(x, G(y)) \xrightarrow{F(-)} \mathsf{Maps}_{\mathsf{D}}(F(x), F \circ G(y)) \xrightarrow{\kappa(y) \circ -} \mathsf{Maps}_{\mathsf{D}}(F(x), y) \tag{1.2}$$

is invertible.

Definition 1.6. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be functors between ∞-categories. We say a pair of natural transformations $\mu : \mathsf{Id}_{\mathsf{C}} \to G \circ F$ and $\kappa : F \circ G \to \mathsf{Id}_{\mathsf{D}}$ are* ***compatible up to homotopy*** *if they satisfy the following two conditions:*

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(i) *The composition*

$$F \xrightarrow{\mathrm{id}_F \circ \mu} F \circ G \circ F \xrightarrow{\kappa \circ \mathrm{id}_F} F$$

is homotopic to id_F *in* $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$.

(ii) *The composition*

$$G \xrightarrow{\mu \circ \mathrm{id}_G} G \circ F \circ G \xrightarrow{\mathrm{id}_G \circ \kappa} G$$

is homotopic to id_F *in* $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$.

Warning 1.7. *In the above definition, we do* ***not*** *supply* 2*-simplex*

$$\begin{array}{ccc} & F \circ G \circ F & \\ {}^{\mathrm{id} \circ \mu}\nearrow & & \searrow^{\kappa \circ \mathrm{id}} \\ F & = = = = = & F \end{array} \qquad \begin{array}{ccc} & G \circ F \circ G & \\ {}^{\mu \circ \mathrm{id}}\nearrow & & \searrow^{\mathrm{id} \circ \kappa} \\ G & = = = = = & G \end{array}$$

to witness the homotopies.

Definition 1.8. *We say:*

(1) *A natural tranformation* μ *as above is the* ***unit of an adjunction between*** F ***and*** G *if there exists* κ *compatible with it up to homotopy.*

(2) *A natural tranformation* κ *as above is the* ***counit of an adjunction between*** F ***and*** G *if there exists* μ *compatible with it up to homotopy.*

Proposition 1.9 (Ker.02FX)**.** *Let* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *be functors between* ∞*-categories and* $\mu : \mathsf{Id}_{\mathsf{C}} \to G \circ F$ *be a natural transformation. The following are equivalent:*

- μ *is the unit of an adjunction between* F *and* G.
- μ *exhibits* F *as a left adjoint of* G.

Dually, for $\kappa : F \circ G \to \mathsf{Id}_{\mathsf{D}}$*, the following are equivalent:*

- κ *is the counit of an adjunction between* F *and* G.
- κ *exhibits* F *as a left adjoint of* G.

Remark 1.10. *The* $\Rightarrow$ *direction is easy. Suppose* (μ, κ) *is compatible up to homotopy, then we have the following commutative diagram*

$$\begin{array}{ccccc} [Fx, y] & \xrightarrow{G} & [GFx, Gy] & \xrightarrow{-\circ\mu(x)} & [x, Gy] \\ \| & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ [Fx, y] & \xrightarrow{FG} & [FGFx, FGy] & \xrightarrow{-\circ\mu(Fx)} & [Fx, FGy] \\ {\scriptstyle \mathsf{Id}_{\mathsf{D}}}\downarrow & & \downarrow{\scriptstyle \kappa(y)\circ-} & & \downarrow{\scriptstyle \kappa(y)\circ-} \\ [Fx, y] & \xrightarrow[-\circ\kappa(Fx)]{} & [FGFx, y] & \xrightarrow[-\circ\mu(Fx)]{} & [Fx, y]. \end{array}$$

Now Axiom (i) in Definition 1.6 implies the bottom horizontal composition is homotopic to the identity functor. It follows that the composition of the top horizontal and the right vertical functors, i.e., (1.2)$\circ$(1.1) *is homotopic to the identity functor. Dually, Axiom (ii) implies* (1.1)$\circ$(1.2) *is homotopic to the identity functor. Therefore both* (1.1) *and* (1.2) *are invertible.*

The $\Leftarrow$ *direction can be proven if one can assign an inverse of* (1.6) *functorial enough to induce a natural transformation* κ.

Corollary-Definition 1.11. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor between ∞-categories. The following are equivalent:*

- *There exists $G : \mathsf{D} \to \mathsf{C}$ and $\mu : \mathsf{Id}_{\mathsf{C}} \to G \circ F$ such that μ exhibits G as a right adjoint of F.*
- *There exists $G : \mathsf{D} \to \mathsf{C}$ and $\kappa : F \circ G \to \mathsf{Id}_{\mathsf{D}}$ such that κ exhibits G as a right adjoint of F.*
- *There exists $G : \mathsf{D} \to \mathsf{C}$ and a pair (μ, κ) as above which is compatible up to homotopy.*

We say F ***admits a right adjoint****, or is* ***right adjointable****, if it satisfies the above conditions. Dually, we define the notion of* ***left adjointable functors****.*

1.12. In practice, the adjunction datum is more adapted for formal proofs. The following exercise provides an example.

Exercise 1.13. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be an adjunction exhibited either by its unit or counit. Let E be any ∞-category, show that*

(1) *We have an adjunction*

$$F \circ - : \mathsf{Fun}(\mathsf{E}, \mathsf{C}) \rightleftarrows \mathsf{Fun}(\mathsf{E}, \mathsf{D}) : G \circ -$$

exhibited by the same type of datum.

(2) *We have an adjunction*

$$- \circ G : \mathsf{Fun}(\mathsf{D}, \mathsf{E}) \rightleftarrows \mathsf{Fun}(\mathsf{C}, \mathsf{E}) : - \circ F$$

exhibited by the same type of datum.

2. Criteria for adjointability

Proposition 2.1. *We have:*

(1) *If $G : \mathsf{D} \to \mathsf{C}$ admits a left adjoint, then G preserves all small limits that exist in D.*
(2) *If $F : \mathsf{C} \to \mathsf{D}$ admits a right adjoint, then F preserves all small colimits that exist in D.*

Sketch. We may assume both C and D are essentially small. Then

$$\begin{aligned}\mathsf{Maps}_{\mathsf{C}}(x, G(\lim y_i)) \simeq \mathsf{Maps}_{\mathsf{D}}(F(x), \lim y_i) \simeq \lim \mathsf{Maps}_{\mathsf{D}}(F(x), y_i) \simeq \\ \simeq \lim \mathsf{Maps}_{\mathsf{D}}(x, G(y_i)) \simeq \mathsf{Maps}_{\mathsf{D}}(x, \lim G(y_i)).\end{aligned}$$

This implies $G(\lim y_i) \simeq \lim G(y_i)$. □

Exercise 2.2. *Rewrite the above sketch as a rigorous proof.*

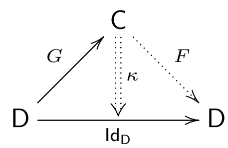
Remark 2.3. *In future lectures, we will see the converse of the proposition is true if C and D are* presentable*.*

Proposition 2.4. *Let $G : \mathsf{D} \to \mathsf{C}$ be a functor between ∞-categories. The following are equivalent:*

(i) *The functor G admits a left adjoint.*
(ii) *For any $x \in \mathsf{C}$, the functor $\mathsf{Maps}_{\mathsf{C}}(x, G(-)) : \mathsf{D} \to \mathsf{Grpd}_\infty$ is corepresentable*[1].
(iii) *For any $x \in \mathsf{C}$, the fiber product $\mathsf{D} \times_{\mathsf{C}} \mathsf{C}_{x/}$ admits an initial object.*

[1] We choose the size bound of objects in Grpd_∞ such that C and D are locally small with respect to it.

Moreover, under these conditions, the pointwise $\mathsf{RKE}_G \mathsf{Id}_\mathsf{D}$ *exists, and*



exhibits F *as the* RKE *iff it exhibits* F *as a left adjoint of* G.

Sketch. (i)⇒(ii): let F be a left adjoint of G exhibited by $\mu : \mathsf{Id}_\mathsf{C} \to G \circ F$. Then $\mu(x) \in \mathsf{Maps}_\mathsf{C}(x, G \circ F(x))$ exhibits $\mathsf{Maps}_\mathsf{C}(x, G(-))$ as corepresented by $F(x)$.

(ii)⇒(iii): let $e \in \mathsf{Maps}_\mathsf{C}(x, G(y))$ exhibit $\mathsf{Maps}_\mathsf{C}(x, G(-))$ as corepresented by y. Then (y, e) is an initial object in $\mathsf{D} \times_\mathsf{C} \mathsf{C}_{x/}$.

(Then one can check $\mathsf{RKE}_G \mathsf{Id}_\mathsf{D}$ is indeed a left adjoint of G.

□

Exercise 2.5. *Fill in the details of the above sketch.*

Exercise 2.6. *State the dual version of the above proposition.*

Warning 2.7. *The existence of the pointwise* $\mathsf{RKE}_G \mathsf{Id}_\mathsf{D}$ *does not imply the existence of a left adjoint of* G.

Exercise 2.8. *Let* $G : \mathsf{D} \to \mathsf{C}$ *be any functor with* $\mathsf{D} := [0]$. *Show that* $\mathsf{RKE}_G \mathsf{Id}_\mathsf{D}$ *always exists, while* G *admits a left adjoint iff* G *is final.*

Exercise 2.9. *Let* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *be functors between* ∞*-categories and consider the induced functors* $\mathsf{h}F : \mathsf{hC} \rightleftarrows \mathsf{hD} : \mathsf{h}G$ *between the homotopy categories. Suppose* G *is left adjointable. Show that* μ *(resp.* κ*) exhibits* F *as a left adjoint of* G *iff* $\mathsf{h}\mu$ *(resp.* $\mathsf{h}\kappa$*) exhibits* $\mathsf{h}F$ *as a left adjoint of* $\mathsf{h}G$.

Exercise 2.10. *Let* $G : \mathsf{D} \to \mathsf{C}$ *be a functor between* ∞*-categories. Show that* $x \in \mathsf{C}$ *satisfies (ii) in Proposition 2.4 iff it satisfies (iii).*

Exercise 2.11. *In above, let* $\mathsf{C}_0 \subset \mathsf{C}$ *be the full sub-*∞*-category consisting of such objects. Challenge: construct a functor* $F_0 : \mathsf{C}_0 \to \mathsf{D}$ *such that* $F_0(x)$ *corepresents* $\mathsf{Maps}_\mathsf{C}(x, G(-))$.

Exercise 2.12. *Let* $G : \mathsf{D} \to \mathsf{C}$ *be a functor between* ∞*-categories and* $\iota : \mathsf{C}_0 \to \mathsf{C}$ *be a fully faithful functor. Find the correct definition for the following notion: a natural transformation* $\mu_0 : \iota \to G \circ F_0$ ***exhibits*** F_0 ***as a partially defined left adjoint of*** G.

3. Uniqueness

3.1. Proposition 2.4 implies for a left adjointable functor G, the data (F, κ) such that κ exhibits F as a left adjoint of G are essentially unique. In fact, we have the following stronger result, which unfortunately is not documented yet[2].

Claim 3.2. *Let* C *and* D *be* ∞*-categories. The following data are classified by equivalent* ∞*-categories:*

(1) *The datum of a right adjointable functor* $F : \mathsf{C} \to \mathsf{D}$.

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[2]Keep an eye on the missing tag in Ker.02F4. See Ker.02D4 for the equivalences between the homotopy categories of (1), (1′), (2) and (2′).

(1′) *The datum of a left adjointable functor* $G : \mathsf{D} \to \mathsf{C}$.
(2) *The datum of* (F, G, μ) *such that* $\mu : \mathsf{Id}_{\mathsf{C}} \to G \circ F$ *is the unit of an adjunction.*
(2′) *The datum of* (F, G, κ) *such that* $\kappa : F \circ G \to \mathsf{Id}_{\mathsf{D}}$ *is the counit of an adjunction.*
(3) *The datum of* $(F, G, \mu, \kappa, \sigma)$ *such that* μ *and* κ *are compatible up to homotopy and* σ *is a 2-simplex*

$$\begin{array}{ccc} & F \circ G \circ F & \\ {\scriptstyle \mathrm{id}\circ\mu}\nearrow & & \searrow{\scriptstyle \kappa\circ\mathrm{id}} \\ F & = \!=\!=\!=\!= & F \end{array}$$

witnessing Axiom (i) in Definition 1.6.
(3′) *The datum of* $(F, G, \mu, \kappa, \tau)$ *such that* μ *and* κ *are compatible up to homotopy and* τ *is a 2-simplex*

$$\begin{array}{ccc} & G \circ F \circ G & \\ {\scriptstyle \mu\circ\mathrm{id}}\nearrow & & \searrow{\scriptstyle \mathrm{id}\circ\kappa} \\ G & = \!=\!=\!=\!= & G \end{array}$$

witnessing Axiom (ii) in Definition 1.6.
...

Exercise 3.3. *Compare the above claim with [Lecture 4, Appendix A].*

Warning 3.4. *Note that the following data are not listed in the above hierarchy:*

- *The datum of* (F, G) *such that there* exists *an adjunction between* F *and* G.
- *The datum of* (F, G, μ, κ) *such that* μ *and* κ *are compatible up to homotopy.*
- *The datum of* $(F, G, \mu, \kappa, \sigma, \tau)$ *such that ....*
- ...

Exercise 3.5. *Compare the above warning with the facts that* $\mathbb{S}^1, \mathbb{S}^2, \mathbb{S}^3 \cdots$ *are not contractible while* $\mathbb{D}^0, \mathbb{D}^1, \mathbb{D}^2 \cdots$ *are contractible.*

Warning 3.6. *The* ∞*-categories indicated in Claim 3.2 are not the naive ones. For example, if a morphism in* (1) *is given by* $F_1 \to F_2$, *then the corresponding morphism in* (1′) *is given by* $G_1 \leftarrow G_2$. *Even worse, the corresponding morphism in* (2) *is given by a commutative diagram*

$$\begin{array}{ccc} \mathsf{Id}_{\mathsf{C}} & \xrightarrow{\mu_1} & G_1 \circ F_1 \\ {\scriptstyle \mu_2}\downarrow & & \downarrow \\ G_2 \circ F_2 & \longrightarrow & G_1 \circ F_2. \end{array}$$

These can be seen from the following construction.

Construction 3.7. *Let* $F_1, F_2 : \mathsf{C} \to \mathsf{D}$ *be functors between* ∞*-categories and* $\alpha : F_1 \to F_2$ *be a natural transformation. Suppose*

- $\mu_1 : \mathsf{Id}_{\mathsf{C}} \to G_1 \circ F_1$ *exhbits* G_1 *as a right adjoint of* F_1.
- $\kappa_2 : F_2 \circ G_2 \to \mathsf{Id}_{\mathsf{D}}$ *exhibits* G_2 *as a right adjoint of* F_2.

Then we have a natural transformation $G_2 \to G_1$ constructed as the following composition:

$$G_2 \xrightarrow{\mu_1 \circ \mathsf{id}} G_1 \circ F_1 \circ G_2 \xrightarrow{\mathsf{id} \circ \alpha \circ \mathsf{id}} G_1 \circ F_2 \circ G_2 \xrightarrow{\mathsf{id} \circ \kappa_2} G_1.$$

3.8. Let C and D be ∞-categories. We denote the full sub-∞-category of $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ consisting of functors that admit right adjoints by[3]

$$\mathsf{LFun}(\mathsf{C}, \mathsf{D}) \subset \mathsf{Fun}(\mathsf{C}, \mathsf{D}).$$

Dually, we denote the full sub-∞-category of $\mathsf{Fun}(\mathsf{D}, \mathsf{C})$ consisting of functors that admit left adjoints by

$$\mathsf{RFun}(\mathsf{D}, \mathsf{C}) \subset \mathsf{Fun}(\mathsf{D}, \mathsf{C}).$$

Then the equivalence between (1) and (1′) in Claim 3.2 can be written as:

Proposition 3.9 (HTT.5.2.6.5)**.** *There is a canonical equivalence*

$$\mathsf{LFun}(\mathsf{C}, \mathsf{D}) \simeq \mathsf{RFun}(\mathsf{D}, \mathsf{C})^{\mathsf{op}}$$

3.10. Let F be a right adjointable functor. By Claim 3.2, we can talk about *the* right adjoint of F as long as we incorporate μ *or* κ as part of the datum in its definition. We denote this right adjoint functor by F^{R}.

Dually, we denote *the* left adjoint of a left adjointable functor G by G^{L}.

4. Fully faithful functors and adjunctions

Exercise 4.1. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be an adjunction[4]. Show that*

1. *F is fully fatihful iff the unit natural transformation $\mu : \mathsf{Id}_{\mathsf{C}} \to G \circ F$ is invertible.*
2. *G is fully fatihful iff the counit natural transformation $\kappa : F \circ G \to \mathsf{Id}_{\mathsf{D}}$ is invertible.*

Deduce that F and G are inverse to each other iff F and G are both fully faithful.

Exercise 4.2. *Show that in above, we actually only need one functor to be fully faithful and the other to be conservative.*

Exercise 4.3. *Let $F : \mathsf{C} \to \mathsf{D}$ be a functor that admits a fully faithful right adjoint. Show that for any E, the functor*

$$- \circ F : \mathsf{Fun}(\mathsf{D}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{E})$$

is fully faithful. Deduce that F exhibits D as $\mathsf{C}[W^{-1}]$ where W is the collection of morphisms $\mu(x) : x \to F^{\mathsf{R}} \circ F(x)$.

Warning 4.4. *Some authors, including Lurie, say a functor $F : \mathsf{C} \to \mathsf{D}$ is a* localization functor *if it admits a fully faithful right adjoint. Note however that for general W, the functor $\mathsf{C} \to \mathsf{C}[W^{-1}]$ may not admit any adjoint.*

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[3]This notation is compatible with [Lecture 11, Definition 5.1] when C in *loc.cit.* is essentially small. This follows from Remark 2.3.

[4]When saying this, we treat μ *or* κ as part of the datum.

5. Beck–Chevalley natural transformations

Construction 5.1. *Let*

$$
\begin{array}{ccc}
\mathsf{C}_1 & \xrightarrow{f} & \mathsf{C}_2 \\
\big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\
\mathsf{D}_1 & \xrightarrow{g} & \mathsf{D}_2,
\end{array}
\tag{5.1}
$$

be a commutative diagram of ∞-categories such that f^{R} and g^{R} exist. We can construct a natural transformation

$$
\begin{array}{ccc}
\mathsf{C}_1 & \xleftarrow{f^{\mathsf{R}}} & \mathsf{C}_2 \\
\big\downarrow{\scriptstyle p} & \searrow\!\!\searrow & \big\downarrow{\scriptstyle q} \\
\mathsf{D}_1 & \xleftarrow[g^{\mathsf{R}}]{} & \mathsf{D}_2,
\end{array}
\tag{5.2}
$$

as the following composition

$$p \circ f^{\mathsf{R}} \xrightarrow{\mu \circ \mathrm{id}} g^{\mathsf{R}} \circ g \circ p \circ f^{\mathsf{R}} \simeq g^{\mathsf{R}} \circ q \circ f \circ f^{\mathsf{R}} \xrightarrow{\mathrm{id} \circ \kappa} g^{\mathsf{R}} \circ q.$$

*We call it the **(right) Beck–Chevalley natural transformation**, or **(right) base-change natural transformation** obtained from* (5.1).

Exercise 5.2. *In above, suppose p^{L} and q^{L} also exist. Construct the left Beck–Chevalley natural transformation*

$$
\begin{array}{ccc}
\mathsf{C}_1 & \xrightarrow{f} & \mathsf{C}_2 \\
{\scriptstyle p^{\mathsf{L}}}\big\uparrow & \nwarrow\!\!\nwarrow & \big\uparrow{\scriptstyle q^{\mathsf{L}}} \\
\mathsf{D}_1 & \xrightarrow{g} & \mathsf{D}_2.
\end{array}
\tag{5.3}
$$

Show that (5.3) *and* (5.2) *correspond to each other as morphisms in*

$$\mathsf{LFun}(\mathsf{D}_1, \mathsf{C}_2) \simeq \mathsf{RFun}(\mathsf{C}_2, \mathsf{D}_1)^{\mathsf{op}}.$$

Remark 5.3. *The higher functorialities of the Beck–Chevalley natural transformations can be described by the theory of $(\infty, 2)$-categories. Unfortunately, I fail to find a satisfying reference.*

Appendix A. Dualities and adjunctions

Exercise A.1. *Define the notion of adjunctions in any $(\infty, 2)$-category such that for the $(\infty, 2)$-category of small ∞-categories, one recovers the notion of adjunctions in this lecture.*

Exercise A.2. *Let $\mathbf{B}$ be an $(\infty, 2)$-category with a single object. Describe the meaning of an adjunction between a morphism f and g in terms of the ∞-category $\mathsf{A} := \mathsf{Maps}_{\mathbf{B}}(*, *)$ and the multiplicative structure on A.*

A.3. **Suggested readings.** Ker.02CA.