LECTURE 13

In this lecture, we introduce presentable ∞-categories.

1. Ind-completion

1.1. Let C be an essentailly small ∞-category and

$$\iota : \mathsf{C} \to \mathsf{PShv}(\mathsf{C}) := \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$$

be the Yoneda embedding. Recall we can view $\mathsf{PShv}(\mathsf{C})$ as obtained from C by *freely* adding small colimits. In this section, we consider the ∞-category obtained from C by freely adding small *filtered* colimits.

Definition 1.2. *Let C be a small ∞-category. Define $\mathsf{Ind}(\mathsf{C}) \subset \mathsf{PShv}(\mathsf{C})$ to be the full sub-∞-category generated by $\iota(\mathsf{C}) \subset \mathsf{PShv}(\mathsf{C})$ under small filtered colimits. We call it the* ***ind-completion*** *of C. Objects in $\mathsf{Ind}(\mathsf{C})$ are called* ***ind-objects in*** *C.*

We abuse notation and call the obtained fully faithful functor

$$\iota : \mathsf{C} \to \mathsf{Ind}(\mathsf{C})$$

the Yoneda embedding.

1.3. By definition, any object in $\mathsf{Ind}(\mathsf{C})$ can be non-uniquely written as

$$\operatorname*{colim}_{i \in I} \iota(x_i),$$

where $I \to \mathsf{C}$, $i \mapsto x_i$ is a small filterd diagram in C. We often use the notation

$$\text{“}\operatorname*{colim}_{i \in I} x_i\text{”} := \operatorname*{colim}_{i \in I} \iota(x_i)$$

to indicate such as object.

Exercise 1.4. *Show that*

$$\mathsf{Maps}_{\mathsf{Ind}(\mathsf{C})}(\text{“}\operatorname*{colim}_{i \in I} x_i\text{”}, \text{“}\operatorname*{colim}_{j \in J} y_j\text{”}) \simeq \lim_{i \in I} \operatorname*{colim}_{j \in J} \mathsf{Maps}_{\mathsf{C}}(x_i, y_j).$$

1.5. Recall any object $\mathcal{F} \in \mathsf{PShv}(\mathsf{C})$ can be written as the colimit of an explicit diagram in $\iota(\mathsf{C})$:

$$\mathcal{F} \simeq \operatorname*{colim}_{x \in \mathsf{C}_{/\mathcal{F}}} \iota(x),$$

where

$$\mathsf{C}_{/\mathcal{F}} := \mathsf{C} \underset{\mathsf{PShv}(\mathsf{C})}{\times} \mathsf{PShv}(\mathsf{C})_{/\mathcal{F}}$$

is the ∞-category classifying $x \in \mathsf{C}$ equipped with a morphism $\iota(x) \to \mathcal{F}$.

Proposition 1.6 (HTT.5.3.5.4)**.** *Let C be an essentially small ∞-category and $\mathcal{F} \in \mathsf{PShv}(\mathsf{C})$. The following conditions are equivalent:*

(i) *The object $\mathcal{F}$ is contained in $\mathsf{Ind}(\mathsf{C})$.*
(ii) *The ∞-category $\mathsf{C}_{/\mathcal{F}}$ is filtered.*

If moreover C^{op} admits finite limits, then the above conditions are equivalent to

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(iii) *The functor* $\mathcal{F} : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ *preserves finite limits.*

Proposition 1.7. *Let* C *be an essentially small* ∞*-category. Then the Yoneda embedding* $\iota : \mathsf{C} \to \mathsf{Ind}(\mathsf{C})$ *preserves and detects finite colimits.*

sketch. Let $I \to \mathsf{C}, i \mapsto x_i$ be a finite diagram that admits a colimit. Then for any object

$$\text{“}\operatorname*{colim}_{j\in J} y_j\text{”} \in \mathsf{Ind}(\mathsf{C}),$$

we have

$$\begin{aligned}
&\mathsf{Maps}_{\mathsf{Ind}(\mathsf{C})}(\iota(\operatorname*{colim}_{i\in I} x_i), \text{“}\operatorname*{colim}_{j\in J} y_j\text{”})\\
\simeq\ &\operatorname*{colim}_{j\in J} \mathsf{Maps}_{\mathsf{C}}(\operatorname*{colim}_{i\in I} x_i, y_j)\\
\simeq\ &\operatorname*{colim}_{j\in J} \lim_{i\in I} \mathsf{Maps}_{\mathsf{C}}(x_i, y_j)\\
\simeq\ &\lim_{i\in I} \operatorname*{colim}_{j\in J} \mathsf{Maps}_{\mathsf{C}}(x_i, y_j)\\
\simeq\ &\lim_{i\in I} \mathsf{Maps}_{\mathsf{Ind}(\mathsf{C})}(\iota(x_i), \text{“}\operatorname*{colim}_{j\in J} y_j\text{”})\\
\simeq\ &\mathsf{Maps}_{\mathsf{Ind}(\mathsf{C})}(\operatorname*{colim}_{i\in I} \iota(x_i), \text{“}\operatorname*{colim}_{j\in J} y_j\text{”}).
\end{aligned}$$

Here the first and fourth equivalences are due to Exercise 1.4. The second and fifth equivalences are due to the fact that representable functors preserve limits. The third equivalence is due to the fact that filtered colimits commute with finite limits. □

Exercise 1.8. *Show that* $\iota : \mathsf{C} \to \mathsf{Ind}(\mathsf{C})$ *preserves and detects all limits.*

1.9. Recall the ∞-category $\mathsf{PShv}(\mathsf{C})$ is a cocompletion of C, which says it is characterized by the following universal property:

$$\mathsf{LFun}(\mathsf{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\simeq} \mathsf{Fun}(\mathsf{C}, \mathsf{D}),$$

where the LHS is the ∞-category of functors $\mathsf{PShv}(\mathsf{C}) \to \mathsf{D}$ that perserve small colimits. The ∞-category $\mathsf{Ind}(\mathsf{C})$ can be characterized by a similar universal property with small colimits replaced by small *filtered* colimits.

Definition 1.10. *Let* D *be an* ∞*-category admitting small filtered colimits. We say a functor* $F : \mathsf{D} \to \mathsf{D}'$ *is* ***continuous*** *if it preserves small filtered colimits. Let*

$$\mathsf{Fun}(\mathsf{D}, \mathsf{D}')_{\mathsf{cont}} \subset \mathsf{Fun}(\mathsf{D}, \mathsf{D}')$$

be the full sub-∞*-category of continuous functors.*

Proposition 1.11 (HTT.5.3.5.10)**.** *Let* C *be a small* ∞*-category and* D *be an* ∞*-category admitting small filtered colimits. Then the Yoneda embedding* $\iota : \mathsf{C} \to \mathsf{Ind}(\mathsf{C})$ *induces an equivalence*

$$\mathsf{Fun}(\mathsf{Ind}(\mathsf{C}), \mathsf{D})_{\mathsf{cont}} \xrightarrow{-\circ\iota} \mathsf{Fun}(\mathsf{C}, \mathsf{D})$$

with an inverse given by

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{Ind}(\mathsf{C}), \mathsf{D})_{\mathsf{cont}},\ F \mapsto \mathsf{LKE}_\iota F.$$

2. Compact objects

Definition 2.1. *Let* D *be an* ∞*-category admitting small filtered colimits. We say an object* $x \in \mathsf{D}$ *is* ***compact*** *if the functor*[1] $\mathsf{Maps}_{\mathsf{D}}(x,-)$ *is continuous, i.e., preserves small filtered colimits.*

Let $\mathsf{D}^{\mathsf{cpt}} \subset \mathsf{D}$ *be the full sub-*∞*-category of compact objects in* D*.*

Proposition 2.2 (HTT.5.4.2.4)**.** *Let* C *be a small* ∞*-category. Then* $\mathsf{Ind}(\mathsf{C})^{\mathsf{cpt}}$ *is the smallest full sub-*∞*-category of* $\mathsf{Ind}(\mathsf{C})$ *such that*

(i) *it contains* $\iota(\mathsf{C})$*;*
(ii) *it is* idempotent complete[2].

Exercise 2.3. *Prove by yourself that* $\mathsf{Ind}(\mathsf{C})^{\mathsf{cpt}}$ *satisfies these two properties.*

Exercise 2.4. *Show that* $\mathsf{Ind}(\mathsf{C})$ *is idempotent complete. Deduce that (ii) can be replaced by*

(ii') *the embedding* $\mathsf{C}' \subset \mathsf{Ind}(\mathsf{C})$ *is* closed under retracts[3].

Definition 2.5. *Let* D *be an* ∞*-category. We say* D *is* ***compactly generated*** *if*

(a) *there exists an essentially small* ∞*-category* C *such that* $\mathsf{D} \simeq \mathsf{Ind}(\mathsf{C})$
(b) *the* ∞*-category* D *admits all small colimits.*

Proposition 2.6 (HTT.5.5.1.1)**.** *Let* D *be an* ∞*-category. The following are equivalent:*

- *The* ∞*-category* D *is compactly generated;*
- *The* ∞*-category* $\mathsf{D}^{\mathsf{cpt}}$ *admits finite colimits and generates* D *under small filtered colimits;*
- *There exists a small* ∞*-category* C *admitting finite colimits such that* $\mathsf{D} \simeq \mathsf{Ind}(\mathsf{C})$*.*

Exercise 2.7. *Show that if* D *is compactly generated, then filtered colimits commute with finite limits in* D*.*

Exercise 2.8. *Let* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *be an adjunction such that* C *is compactly generated. Show that* G *is continuous iff* F *preserves compact objects.*

3. Presentable ∞-categories

3.1. Recall an ∞-category J is filtered if any finite diagram $K \to \mathsf{J}$ can be extended to $K^{\triangleright} \to \mathsf{J}$. All the previous discussions about filtered colimits remain correct if we replace "*any finite diagram*" in above by "*any* κ*-small diagram*", where κ is a regular cardinal[4].

For example, we can define κ*-filtered diagrams*, κ*-ind-completions* $\mathsf{Ind}_\kappa(-)$, κ*-compact objects*, κ*-compact generation...* The proofs of the corresponding results are essentially the same as the case when κ is the countable cardinal ω.

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[1]Since we do not assume D is locally small, the target of the functor $\mathsf{Maps}_{\mathsf{D}}(x,-)$ should be $\widehat{\mathsf{Grpd}}_\infty$, which contains not necessarily small ∞-groupoids.

[2]This means $\mathsf{Ind}(\mathsf{C})^{\mathsf{cpt}}$ admits Idem-indexed colimits

[3]This means if $z \in \mathsf{C}'$ and $y \in \mathsf{Ind}(\mathsf{C})$ such that id_y factors as $y \to z \to y$, then $y \in \mathsf{C}'$.

[4]Informally speaking, *cardinals* measure sizes of sets. For a given cardinal κ, a set S is called κ*-small* if its size is strictly smaller than κ. A cardinal κ is *regular* if any κ-small union of κ-small sets is still κ-small.

Exercise 3.2. *Let $\kappa < \kappa'$ be regular cardinals. Show that*

$$\begin{aligned} \kappa\text{-small} &\Rightarrow \kappa'\text{-small} \\ \kappa\text{-filtered} &\Leftarrow \kappa'\text{-filtered} \\ \kappa\text{-continous} &\Leftarrow \kappa'\text{-continuous} \\ \kappa\text{-compact} &\Rightarrow \kappa'\text{-compact} \end{aligned}$$

Definition 3.3. *Let κ be a regular cardinal. We say an ∞-category D is **κ-accessible** if there exists an essentially small ∞-category C such that $\mathsf{D} \simeq \mathsf{Ind}_\kappa(\mathsf{C})$. We say D is **accessible** if it is κ-accessible for some κ.*

Definition 3.4. *Suppose D is accessible. We say a functor $\mathsf{D} \to \mathsf{D}'$ is **accessible** if it is κ-continuous for some regular cardinal κ.*

Lemma 3.5 (HTT.5.4.2.8)**.** *Suppose D is κ-accessible, then it is κ'-accessible for $\kappa' >> \kappa$*[5]*.*

Definition 3.6. *We say an ∞-category D is **presentable** if it is accessible and admits small colimits.*

3.7. By definition, D is presentable iff it is κ-compactly generated for some regular cardinal κ.

Theorem 3.8 (HTT.A.3.7.6)**.** *An ∞-category D is presentable iff there exists a combinatorial simplicial model category $\mathbb{A}$ such that $\mathsf{D} \simeq \mathfrak{N}_\bullet(\mathbb{A}^\circ)$.*

Corollary 3.9. *The ∞-categories Grpd_∞ and Cat_∞ are presentable.*

Theorem 3.10 (HTT.5.5.2.4)**.** *A presentable ∞-category also admits small limits.*

Warning 3.11. *If D is presentable, then D^{op} is almost never accessible. Challenge: show $\mathsf{Grpd}_\infty^{\mathsf{op}}$ is not accessible.*

Theorem 3.12 (HTT.5.5.1.1)**.** *An ∞-category D is presentable iff there exists an essentially small ∞-category C and an adjunction*

$$F : \mathsf{PShv}(\mathsf{C}) \rightleftarrows \mathsf{D} : G$$

such that G is fully faithful and $G \circ F$ is accessible.

Remark 3.13. *The above theorem is a powerful tool to prove results about presentable ∞-categories. Namely, we can first prove in the case when $\mathsf{D} = \mathsf{PShv}(\mathsf{C})$, then formally deduce the general cases.*

Proposition 3.14 (HTT.5.5.3.6)**.** *Let C be presentable and K be a small simplicial set. Then $\mathsf{Fun}(K, \mathsf{C})$ is presentable.*

Proposition 3.15 (HTT.5.5.3.10, 5.5.3.11)**.** *Let C be presentable and $u : K \to \mathsf{C}$ be a small diagram. Then $\mathsf{C}_{/u}$ and $\mathsf{C}_{u/}$ are presentable.*

[5] See HTT.A.2.6.3 for what this means.

4. Representability theorem and adjoint functor theorem

Exercise 4.1. *Presentable ∞-categories are locally small, i.e., have essentially small* $\mathsf{Maps}(-,-)$.

Theorem 4.2 (HTT.5.5.2.2, 5.5.2.7)**.** *Let* C *be a presentable ∞-category.*

(1) *A functor* $\mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ *is representable iff it preserves small limits.*
(2) *A functor* $\mathsf{C} \to \mathsf{Grpd}_\infty$ *is corepresentable iff it is accessible and preserves small limits.*

Remark 4.3. *By Warning 3.11, it does not make sense to ask* $\mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ *to be accessible.*

Corollary 4.4. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a functor between presentable ∞-categories.*

(1) *The functor F is right adjointable iff it preserves small colimits.*
(2) *The functor F is left adjointable iff it is accessible and preserves small limits.*

Sketch. We prove (1) and leave (2) as an exercise. The functor F is right adjointable iff $\mathsf{Maps}(F(-), y)$ is representable for any y. By Theorem 4.2, this is equivalent to $\mathsf{Maps}(F(-), y)$ preserving small limits for any y. Since representable functors preserve and detect all limits, this is equivalent to $F(-)$ preserving small colimits. □

5. Left and right functors between presentable ∞-categories

Definition 5.1. *Let*

$$\mathsf{Pr}^{\mathsf{L}},\ \mathsf{Pr}^{\mathsf{R}}$$

be the ∞-categories of presentable ∞-categories such that morphisms are given by functors that are left adjoints (resp. right adjoints). In other words,

$$\mathsf{Maps}_{\mathsf{Pr}^{\mathsf{L}}}(-,-) := \mathsf{LFun}(-,-)^{\simeq},\ \mathsf{Maps}_{\mathsf{Pr}^{\mathsf{R}}}(-,-) := \mathsf{RFun}(-,-)^{\simeq}$$

Proposition 5.2 (HTT.5.5.3.8)**.** *Let* C *and* D *be presentable ∞-categories. Then* $\mathsf{LFun}(\mathsf{C}, \mathsf{D})$ *is presentable.*

Warning 5.3. *The ∞-category* $\mathsf{RFun}(\mathsf{C}, \mathsf{D})$ *is almost never presentable because*

$$\mathsf{RFun}(\mathsf{C}, \mathsf{D}) \simeq \mathsf{LFun}(\mathsf{D}, \mathsf{C})^{\mathsf{op}}.$$

Exercise 5.4. *What are* $\mathsf{LFun}(\mathsf{Grpd}_\infty, \mathsf{D})$ *and* $\mathsf{RFun}(\mathsf{Grpd}_\infty^{\mathsf{op}}, \mathsf{D})$*? How about* $\mathsf{RFun}(\mathsf{D}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$*?*

Exercise 5.5. *Challenge: for any presentable ∞-categories* C *and* D*, construct a canonical equivalence*

$$\mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}) \simeq \mathsf{RFun}(\mathsf{D}^{\mathsf{op}}, \mathsf{C}).$$

Theorem 5.6 (HTT.5.5.3.4)**.** *There is a canonical equivalnce*

$$\mathsf{Pr}^{\mathsf{L}} \simeq (\mathsf{Pr}^{\mathsf{R}})^{\mathsf{op}}$$

such that

- *it sends a presentable ∞-category* D*, viewed as an object in* Pr^{L} *to (an object equivalent to)* D*, viewed as an object in* Pr^{R}
- *it sends a functor* $F \in \mathsf{LFun}(\mathsf{C}, \mathsf{D})$ *to a right adjoint* F^{R} *of it.*

Remark 5.7. *In particular, given a diagram* $u : K \to \mathsf{Pr}^{\mathsf{L}}$*, we can pass to right adjoint functors and obtain a diagram* $K^{\mathsf{op}} \to \mathsf{Pr}^{\mathsf{R}}$*.*

Theorem 5.8 (HTT.5.5.3.13, 5.5.3.18)**.** *The* ∞*-categories* Pr^{L} *and* Pr^{R} *admits small colimits and limits. And the forgetful functors*[6]

$$\mathsf{Pr}^{\mathsf{L}} \to \widehat{\mathsf{Cat}}_\infty,\ \mathsf{Pr}^{\mathsf{R}} \to \widehat{\mathsf{Cat}}_\infty$$

preserve small limits.

Remark 5.9. *Recall limits in* $\widehat{\mathsf{Cat}}_\infty$ *are calculable ([Lecture 9, Remark 2.9]). It follows that small limits in* Pr^{L} *and* Pr^{R} *are calculable. Using Theorem 5.6, small colimits in* Pr^{L} *and* Pr^{R} *can be calculated by passing to adjoint functors.*

Appendix A. Compact spaces

Exercise A.1. *Show that* Grpd_∞ *is compactly generated and any essentially finite* ∞*-groupoid is a compact object.*

Exercise A.2. *Show that compact objects in* Grpd_∞ *might not be essentially finite.*

A.3. **Suggested readings.** [Wal65] and [Wal66].

Appendix B. Adjoining colimits

B.1. More generally, for any ∞-category C, it is possible to freely adjoin certain types of colimits while keeping a certain collection of colimits that exist in C.

Exercise B.2. *Define the universal property that characterizes the above construction.*

Example B.3. *For an essentially small* ∞*-category* C*, one can freely adjoin small sifted colimits to obtain the so-called* ***nonabelian derived*** ∞***-category of*** C*.*

B.4. **Suggested readings.** HTT.5.3.6, 5.5.8 and 5.5.9.

References

[Wal65] Charles Terence Clegg Wall. Finiteness conditions for cw-complexes. *Annals of Mathematics*, 81(1):56–69, 1965.

[Wal66] C Terence C Wall. Finiteness conditions for cw complexes. ii. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 129–139, 1966.

[6]Note that presentable ∞-categories are in general not essentially small. Hence we need to use $\widehat{\mathsf{Cat}}_\infty$ rather that Cat_∞