

LECTURE 15

In this lecture, we define and study *stable* ∞ -categories, which are exactly ∞ -categories of the form $\mathbf{Sptr}(\mathbf{C})$.

1. STABILITY

Definition 1.1. Let \mathbf{C} be a pointed ∞ -category. A **triangle** in \mathbf{C} is a diagram $\Delta^1 \times \Delta^1 \rightarrow \mathbf{C}$ depicted as

$$(1.1) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

where $0 \in \mathbf{C}$ is the zero object. We say such a triangle is a **fiber sequence** if it is a pullback square, and a **cofiber sequence** if it is a pushout square.

For a morphism $f : X \rightarrow Y$, a **cofiber of f** is a fiber sequence (1.1). Dually, for a morphism $g : Y \rightarrow Z$, a **fiber of g** is a cofiber sequence (1.1).

1.2. We often abuse notation and write a triangle as $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Warning 1.3. The datum of a triangle (1.1) is not determined by the chain $X \xrightarrow{f} Y \xrightarrow{g} Z$, even up to homotopy. Indeed, knowing such a triangle is equivalent to knowing a null-homotopy of $g \circ f$, which is not unique even up to homotopy.

1.4. Note however that a fiber sequence (1.1) is essentially uniquely determined by the morphism g . Dually, a cofiber sequence (1.1) is essentially uniquely determined by the morphism f . Hence we can use the notations

$$\mathbf{Fib}(g), \mathbf{Cofib}(f) \in \mathbf{C}$$

as long as we incorporate (1.1) as data in their definitions.

Definition 1.5. An ∞ -category \mathbf{C} is **stable** if it satisfies the following conditions:

- it is pointed
- any morphism in \mathbf{C} admits a fiber and a cofiber
- a triangle in \mathbf{C} is a fiber sequence iff it is a cofiber sequence.

1.6. For stable ∞ -categories, we can use the words **fiber-cofiber sequences**.

Exercise 1.7. Find all ordinary categories that are stable when viewed as ∞ -categories.

Proposition 1.8. Let \mathbf{C} be a pointed ∞ -category admitting both finite limits and colimits. Then the following conditions are equivalent.

- (i) The functor $\Sigma : \mathbf{C} \rightarrow \mathbf{C}$ is fully faithful.
- (ii) Any cofiber sequence in \mathbf{C} of the form $X \rightarrow 0 \rightarrow Z$ is also a fiber sequence.

- (iii) Any cofiber sequence in \mathcal{C} is a fiber sequence.
- (iv) Any pushout square in \mathcal{C} is a pullback square.

Corollary 1.9. *Let \mathcal{C} be a pointed ∞ -category admitting both finite limits and colimits. Then the following conditions are equivalent.*

- (i) The functors $\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega$ are equivalences.
- (ii) A triangle in \mathcal{C} of the form $X \rightarrow 0 \rightarrow Z$ is a cofiber sequence iff it is a fiber sequence.
- (iii) A triangle in \mathcal{C} is a cofiber sequence iff it is a fiber sequence.
- (iv) A square in \mathcal{C} is a pushout iff it is a pullback.

Proof of Proposition 1.8. The implications (ii) \Leftarrow (iii) \Leftarrow (iv) are obvious. It remains to show (i) \Rightarrow (iv). Suppose Σ is fully faithful. Since Ω is a right adjoint of Σ , we have $\text{Id}_{\mathcal{C}} \simeq \Omega \circ \Sigma$. For a pushout square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z, \end{array}$$

we need to show $W \rightarrow X \times_Z Y$ is invertible. Consider the following commutative diagram

$$\begin{array}{ccccc} W & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Z & \dashrightarrow & \Sigma W \\ \downarrow & & & & \downarrow \\ 0 & \longrightarrow & & & \Sigma W. \end{array}$$

Here both the inner and outer squares are pushout squares, hence there exists an essentially unique dotted arrow $Z \rightarrow \Sigma W$ making the above diagram commute. By functoriality of pullbacks, we obtain a morphism $X \times_Z Y \rightarrow 0 \times_{\Sigma W} 0 \simeq \Omega \Sigma W$ fitting into the following commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & X \times_Z Y \\ \downarrow \simeq & \swarrow \dashrightarrow & \downarrow \simeq \\ \Omega \Sigma W & \longrightarrow & \Omega \Sigma X \times_{\Omega \Sigma Z} \Omega \Sigma Y, \end{array}$$

where the vertical morphisms are isomorphisms because of $\text{Id}_{\mathcal{C}} \simeq \Omega \circ \Sigma$. By the 2-out-of-6 property of isomorphisms, we obtain $X \xrightarrow{\simeq} X \times_Z Y$ as desired. \square

Proposition 1.10. *Let \mathcal{C} be a pointed ∞ -category. The following conditions are equivalent.*

- (a) The ∞ -category \mathcal{C} is stable.
- (b) The ∞ -category \mathcal{C}^{op} is stable.
- (c) The ∞ -category \mathcal{C} admits finite colimits and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence.
- (d) The ∞ -category \mathcal{C} admits finite limits and $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence.
- (e) The ∞ -category \mathcal{C} admits finite colimits and limits, and a square in \mathcal{C} is a pushout square iff it is a pullback square.

(f) The ∞ -category \mathbf{C} admits finite limits and $\Omega^\infty : \mathbf{Sptr}(\mathbf{C}) \rightarrow \mathbf{C}$ is an equivalence.

Proof. The equivalence (a) \Leftrightarrow (b) is obvious. The equivalence (d) \Leftrightarrow (f) was proved last time. It remains to show (c) \Leftrightarrow (a) \Leftrightarrow (e) because (d) \Leftrightarrow (b) would follow by passing to the opposite ∞ -category.

Suppose \mathbf{C} admits finite colimits and limits, then (c) \Leftrightarrow (a) \Leftrightarrow (e) follow from (i) \Leftrightarrow (iii) \Leftrightarrow (iv) in Proposition 1.8 (and its dual version). Hence it remains to show (a) or (c) implies \mathbf{C} admits finite colimits and limits.

For (a), we only need to show a stable ∞ -category admits coequalizers. This follows from Exercise 2.13 below.

For (c), we can use $\iota : \mathbf{C} \rightarrow \mathbf{Ind}(\mathbf{C})$ to embed \mathbf{C} into a presentable ∞ -category. Note that $\mathbf{Ind}(\mathbf{C})$ is pointed because ι preserves and detects both finite colimits and limits. Moreover, $\Sigma_{\mathbf{Ind}(\mathbf{C})}$ can be identified with $\mathbf{Ind}(\Sigma_{\mathbf{C}}) : \mathbf{Ind}(\mathbf{C}) \rightarrow \mathbf{Ind}(\mathbf{C})$ and thereby is also an equivalence. It follows from the previous discussion that $\mathbf{Ind}(\mathbf{C})$ satisfies all the properties in the proposition. In particular, any pushout square in $\mathbf{Ind}(\mathbf{C})$ is a pullback square. Since ι preserves and detects both finite colimits and limits, we see the same holds for \mathbf{C} . In particular, it admits pullbacks and therefore all finite limits as desired. \square

Corollary 1.11. *Let \mathbf{C} be a pointed ∞ -category that admits finite limits. The ∞ -category $\mathbf{Sptr}(\mathbf{C})$ is stable.*

Exercise 1.12. *Let \mathbf{C} be a stable ∞ -category. Then $f : X \rightarrow Y$ is an isomorphism iff $\mathbf{Fib}(f)$ is a zero object iff $\mathbf{Cofib}(f)$ is a zero object.*

Exercise 1.13. *Let $f : X \rightarrow Y$ be a morphism in a stable ∞ -category \mathbf{C} . Show that $\Sigma \mathbf{Fib}(f) \simeq \mathbf{Cofib}(f)$.*

2. HOMOTOPY CATEGORY OF STABLE ∞ -CATEGORY

2.1. In this section, let \mathbf{C} be a stable ∞ -category.

Exercise 2.2. *Show that the canonical morphism $X \sqcup Y \rightarrow X \times Y$ is invertible.*

Hint:

$$\begin{array}{ccc} X \sqcup Y & \longrightarrow & X \sqcup 0 \\ \downarrow & & \downarrow \\ 0 \sqcup Y & \longrightarrow & 0 \sqcup 0 \end{array}$$

is a pushout square.

2.3. Since there is a canonical equivalence between $X \sqcup Y$ and $X \times Y$, we use $X \oplus Y$ to denote both of them.

Exercise 2.4. *Let $f, g : X \rightarrow Y$ be two morphisms. Show that the composition*

$$X \rightarrow X \oplus X \xrightarrow{(f,g)} Y \oplus Y \rightarrow Y$$

gives a well-defined binary operator on $\pi_0 \mathbf{Maps}_{\mathbf{C}}(X, Y)$. We denote the above composition by $f + g$.

Exercise 2.5. *Let $f, g : X \rightarrow Y$ be two morphisms. Show that the above binary operator coincides with the addition operator on the abelian group*

$$\pi_0 \mathbf{Maps}_{\mathbf{C}}(X, Y) \simeq \pi_0 \mathbf{Maps}_{\mathbf{C}}(X, \Omega^2 \Sigma^2 Y) \simeq \pi_0 \Omega^2 \mathbf{Maps}_{\mathbf{C}}(X, \Sigma^2 Y) \simeq \pi_2 \mathbf{Maps}_{\mathbf{C}}(X, \Sigma^2 Y).$$

Exercise 2.6. Let $\sigma : \Delta^1 \times \Delta^1 \rightarrow \mathbf{C}$ be a diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma Y \end{array}$$

and σ' be its transpose. By the universal property of pushouts, σ induces a morphism $\eta_\sigma : X \rightarrow \Omega \Sigma Y \simeq Y$, which is an well-defined element in $\pi_0 \text{Hom}_{\mathbf{C}}(X, Y)$. Show that $\eta_\sigma + \eta_{\sigma'} = 0$.

Corollary 2.7. The homotopy category \mathbf{hC} is an additive category.

2.8. From now on, we write $X[n] := \Sigma^n X$, where for $n < 0$ we take $\Sigma^n := \Omega^{-n}$. Note that these objects are well-defined up to homotopy.

Definition 2.9. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a chain in \mathbf{hC} . We say it is a **distinguished triangle in \mathbf{hC}** if there exists a diagram $\Delta^1 \times \Delta^2 \rightarrow \mathbf{C}$ depicted as

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0' & \longrightarrow & Z & \longrightarrow & W \end{array}$$

such that

- the objects 0 and $0'$ are zero objects
- the morphisms \tilde{f} and \tilde{g} lift f and g respectively
- the outer square is a fiber-cofiber sequence
- the composition $Z \rightarrow W \xrightarrow{\cong} X[1]$ (which is well-defined up to homotopy) lifts h .

Theorem 2.10 (HA.1.1.2.14). The above choice of the translation functor and the distinguished triangles makes \mathbf{hC} a triangulated category.

Exercise 2.11. What would happen if we use diagrams of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & W \end{array}$$

to define distinguished triangles in \mathbf{hC} ?

Exercise 2.12. For $X \in \mathbf{C}$, construct a fiber-cofiber sequence

$$X \xrightarrow{(\text{id}, -\text{id})} X \oplus X \xrightarrow{(\text{id}, \text{id})} X.$$

Exercise 2.13. Show that the coequalizer of $f, g : X \rightrightarrows Y$ is canonically equivalent to $\text{Cofib}(f - g)$. *Hint:*

$$\begin{array}{ccccc} X & \xrightarrow{(\text{id}, -\text{id})} & X \oplus X & \xrightarrow{(f, g)} & Y \\ \downarrow & & \downarrow (\text{id}, \text{id}) & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & \bullet \end{array}$$

3. MAPPING SPECTRA

Construction 3.1. Let \mathcal{C} be a stable ∞ -category. For $X, Y \in \mathcal{C}$, define

$$\underline{\text{Hom}}(X, Y)_n := \text{Maps}(X, Y[n]).$$

Note that for any $n \geq 0$, we have an isomorphism

$$\alpha_n : \text{Maps}(X, Y[n]) \simeq \text{Maps}(X, \Omega Y[n+1]) \simeq \Omega \text{Maps}(X, Y[n+1]).$$

Let $\underline{\text{Hom}}(X, Y) \in \text{Sptr}$ be the spectrum given by the spaces $\{\underline{\text{Hom}}(X, Y)_n\}$ and the isomorphisms α_n . We call it the **mapping spectrum** between X and Y .

Remark 3.2. The above spectrum $\underline{\text{Hom}}(X, Y)$ is well-defined up to homotopy. In future lectures, we will equip Sptr with a symmetric monoidal structure such that any stable ∞ -category \mathcal{C} is canonically enriched over Sptr .

Definition 3.3. Let \mathcal{C} be a stable ∞ -category. For $X, Y \in \mathcal{C}$, define

$$\text{Ext}^n(X, Y) := \pi_0 \underline{\text{Hom}}(X, Y)_n \simeq \pi_0 \text{Maps}(X, Y[n])$$

and call it the n -th extension group between X and Y .

3.4. Note that the extension groups only depend on the images of X and Y in the triangulated category hC . It is well-known that for any distinguished triangle $Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_0[1]$ in hC , we have a long exact sequence

$$\cdots \rightarrow \text{Ext}^n(X, Y_0) \rightarrow \text{Ext}^n(X, Y_1) \rightarrow \text{Ext}^n(X, Y_2) \rightarrow \text{Ext}^{n+1}(X, Y_0) \rightarrow \cdots$$

4. EXACT FUNCTORS

4.1. Proposition 1.10 and Exercise 2.13 imply the following result.

Proposition-Definition 4.2. Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be a functor between stable ∞ -category. The following conditions are equivalent.

- The functor F preserves zero objects and fiber-cofiber sequence.
- The functor F is left exact, i.e., preserves finite limits.
- The functor F is right exact, i.e., preserves finite colimits.

We say F is **exact** if it satisfies the above conditions.

Exercise 4.3. Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor between stable ∞ -category. Show that $\text{h}F : \text{hC} \rightarrow \text{hC}'$ has a natural structure of an exact functor¹ between triangulated categories.

Definition 4.4. Let $\text{Cat}_\infty^{\text{ex}} \subseteq \text{Cat}_\infty$ be the sub- ∞ -category of small stable ∞ -categories and exact functors between them.

¹Also known as a triangulated functor. Warning: being a triangulated functor is a structure rather than property.

Exercise 4.5. Let \mathcal{C} be a stable ∞ -category. Show that Σ and Ω are exact. What are the triangulated functors induced by them?

Exercise 4.6. Let $F : \mathcal{C} \rightarrow \mathcal{C}'$ be an exact functor between stable ∞ -categories. Show that F is conservative iff F detects zero objects.

5. CLOSURE PROPERTIES

Exercise 5.1. Let \mathcal{C} be a stable ∞ -category and K be a simplicial set. Show that $\text{Fun}(K, \mathcal{C})$ is stable.

Exercise 5.2. Let \mathcal{C} and \mathcal{D} be stable ∞ -categories. Show that $\text{Fun}_{\text{ex}}(\mathcal{C}, \mathcal{D})$ is stable.

Exercise 5.3. Let \mathcal{C} and \mathcal{D} be presentable ∞ -categories such that \mathcal{D} is stable. Show that $\text{LFun}(\mathcal{C}, \mathcal{D})$ is stable.

Exercise 5.4. Let \mathcal{C} be a small stable ∞ -category, then $\text{Ind}(\mathcal{C})$ is stable.

Exercise 5.5. Let \mathcal{C} be a stable ∞ -category, then \mathcal{C}^{cpt} is stable. In particular, Sptr^{fin} is stable.

Exercise 5.6. Let \mathcal{C} be a stable ∞ -category, show that the idempotent completion of \mathcal{C} is also stable.

Exercise 5.7. Let \mathcal{C} be a stable ∞ -category, is $\text{PShv}(\mathcal{C})$ stable?

Theorem 5.8. The ∞ -category $\text{Cat}_{\infty}^{\text{ex}}$ admits small limits and the inclusion $\text{Cat}_{\infty}^{\text{ex}} \rightarrow \text{Cat}_{\infty}$ preserves and detects small limits.

Sketch. We only need to show for any small diagram $K \rightarrow \text{Cat}_{\infty}$, $i \mapsto \mathcal{C}_i$ such that each \mathcal{C}_i is stable and each connecting functor $\mathcal{C}_i \rightarrow \mathcal{C}_j$ is exact, we have

- the limit ∞ -category $\mathcal{C} := \lim_i \mathcal{C}_i$ is stable
- the evaluating functors $\mathcal{C} \rightarrow \mathcal{C}_i$ are exact.

Both claims can be checked using the explicit description of objects and mapping spaces in \mathcal{C} . \square

5.9. Similarly, one can prove the following result.

Theorem 5.10. The ∞ -category $\text{Cat}_{\infty}^{\text{ex}}$ admits small filtered colimits and the inclusion $\text{Cat}_{\infty}^{\text{ex}} \rightarrow \text{Cat}_{\infty}$ preserves and detects small filtered colimits.

6. A UNIVERSAL PROPERTY OF Sptr

6.1. The following result implies Sptr is the stable ∞ -category freely generated by one object under small colimits.

Exercise 6.2. Let \mathcal{D} be a presentable stable ∞ -category. Show that evaluating at $\mathbb{S} \in \text{Sptr}$ induces an equivalence

$$\text{LFun}(\text{Sptr}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

Hint: show $\text{RFun}(\mathcal{D}, \text{Sptr}) \xrightarrow{\Omega^{\infty} \circ -} \text{RFun}(\mathcal{D}, \text{Spc})$ is an equivalence.

Exercise 6.3. Let \mathcal{D} be a presentable stable ∞ -category. Show that evaluating at $\mathbb{S} \in \text{Sptr}^{\text{fin}}$ induces an equivalence

$$\text{Fun}_{\text{ex}}(\text{Sptr}^{\text{fin}}, \mathcal{D}) \xrightarrow{\simeq} \mathcal{D}.$$

APPENDIX A. TRIANGULATED CATEGORIES WITHOUT MODELS

- A.1. There are triangulated categories that are not the homotopy category of any stable ∞ -category.
- A.2. There are exact functors between homotopy categories of stable ∞ -categories that do not come from exact functors between the stable ∞ -categories.
- A.3. **Suggested readings.** [MSS07].

REFERENCES

- [MSS07] Fernando Muro, Stefan Schwede, and Neil Strickland. Triangulated categories without models. *Inventiones mathematicae*, 170(2):231–241, 2007.