LECTURE 16

In this lecture, we introduce t-structures on stable ∞-categories.

1. Definitions

1.1. Let C be a stable ∞-category. A t-structure on C is just a t-structure on the triangulated category hC.

Definition 1.2. *Let C be a stable ∞-category. A **t-structure** on C is defined to be a pair of full sub-∞-categories $(\mathsf{C}^{\le 0}, \mathsf{C}^{\ge 0})$ of C such that when we write*

$$\mathsf{C}^{\le n} := \mathsf{C}^{\le 0}[-n],\ \mathsf{C}^{\ge n} := \mathsf{C}^{\ge n}[-n]$$

we have:

(0) *Both $\mathsf{C}^{\le 0}$ and $\mathsf{C}^{\ge 0}$ are stable under isomorphisms in C.*
(1) *For $X \in \mathsf{C}^{\le 0}$ and $Y \in \mathsf{C}^{\ge 1}$, we have $\mathsf{Maps}_{\mathsf{C}}(X,Y) \simeq \{0\}$.*
(2) *We have inclusions $\mathsf{C}^{\le -1} \subseteq \mathsf{C}^{\le 0}$ and $\mathsf{C}^{\ge 1} \subseteq \mathsf{C}^{\ge 0}$.*
(3) *For any $X \in \mathsf{C}$, there exists a fiber-cofiber sequence $X' \to X \to X''$ such that $X' \in \mathsf{C}^{\le 0}$ and $X'' \in \mathsf{C}^{\ge 1}$.*

Warning 1.3. *We use the **cohomological convention**. To compare with notations in the homological convention, let*

$$\mathsf{C}_{\le n} := \mathsf{C}^{\ge -n},\ \mathsf{C}_{\ge n} := \mathsf{C}^{\le -n}$$

Exercise 1.4. *The assignment*

$$(\mathsf{C}^{\le 0}, \mathsf{C}^{\ge 0}) \mapsto (\mathsf{hC}^{\le 0}, \mathsf{hC}^{\ge 0})$$

gives a bijection between t-structures on C with t-structures on the triangulated category hC.

Exercise 1.5. *The assignment*

$$(\mathsf{C}^{\le 0}, \mathsf{C}^{\ge 0}) \mapsto ((\mathsf{C}^{\ge 0})^{\mathsf{op}}, (\mathsf{C}^{\le 0})^{\mathsf{op}})$$

gives a bijection between t-structures on C with t-structures on C^{op}.

Exercise 1.6. *Show that for $m < n$, $\mathsf{C}^{\le m} \cap \mathsf{C}^{\ge n} \simeq \{0\}$.*

Lemma 1.7. *Let $n \ge 0$. For any $X \in \mathsf{C}^{\le 0}$ and $Y \in \mathsf{C}^{\ge -n}$, the mapping space $\mathsf{Maps}_{\mathsf{C}}(X,Y)$ is a homotopy n-type.*

Sketch. First note that the connected components of $\mathsf{Maps}_{\mathsf{C}}(X,Y)$ are weakly homotopy equivalent to each other because it is the loop space of $\mathsf{Maps}_{\mathsf{C}}(X,\Sigma Y)$. Hence we only need to show $\pi_m \mathsf{Maps}_{\mathsf{C}}(X,Y) \simeq 0$ for $m > n$ and the base point $0 \in \mathsf{Maps}_{\mathsf{C}}(X,Y)$. This follows from observation that $\Omega^{n+1}\mathsf{Maps}_{\mathsf{C}}(X,Y) \simeq \mathsf{Maps}_{\mathsf{C}}(X,\Omega^{n+1}Y)$ is weakly contractible because $\Omega^{n+1}Y \in \mathsf{C}^{\ge 1}$. □

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Definition 1.8. *Let C be a stable ∞-category equipped with a t-structure. The **heart** of this t-structure is defined to be*

$$\mathsf{C}^{\heartsuit} := \mathsf{C}^{\leq 0} \cap \mathsf{C}^{\geq 0}.$$

Theorem 1.9. *The ∞-category $\mathsf{C}^{\heartsuit}$ is an ordinary abelian category. Moreover, $0 \to X'' \to X \to X' \to 0$ is a short exact sequence in $\mathsf{C}^{\heartsuit}$ iff $X'' \to X \to X'$ is a fiber-cofiber sequence of C contained in $\mathsf{C}^{\heartsuit}$.*

Sketch. By Lemma 1.7, $\mathsf{C}^{\heartsuit}$ is an ordinary category. It follows it can be identified with $\mathsf{hC}^{\leq 0} \cap \mathsf{hC}^{\geq 0}$, which is the heart of the triangulated category hC. Now the claims follow from the corresponding well-known claims for triangulated categories. □

Warning 1.10. *A t-structure is not determined by its heart. Most information in a stable ∞-category cannot be recovered from the heart. For instance, for $X, Y \in \mathsf{C}^{\heartsuit}$, the groups $\mathsf{Ext}^{\bullet}_{\mathsf{C}}(X,Y)$ and $\mathsf{Ext}^{\bullet}_{\mathsf{C}^{\heartsuit}}(X,Y)$ are not isomorphic in general.*

Definition 1.11. *Let C and D be stable ∞-categories equipped with t-structures. For an exact functor $F : \mathsf{C} \to \mathsf{D}$, we say*

- *the functor F is **left t-exact** if it sends $\mathsf{C}^{\geq 0}$ into $\mathsf{D}^{\geq 0}$*
- *the functor F is **right t-exact** if it sends $\mathsf{C}^{\leq 0}$ into $\mathsf{D}^{\leq 0}$*

Exercise 1.12. *Let $F : \mathsf{C} \to \mathsf{D}$ be a left t-exact functor. Show that the composition*

$$\mathsf{H}^0 F : \mathsf{C}^{\heartsuit} \to \mathsf{C}^{\geq 0} \xrightarrow{F} \mathsf{D}^{\geq 0} \xrightarrow{\tau^{\leq 0}} \mathsf{D}^{\heartsuit}$$

is left exact.

Warning 1.13. *As in Warning 1.10, even a t-exact functor F cannot be recovered from $\mathsf{H}^0 F$.*

Remark 1.14. *Next time, we will define various versions of* derived ∞-categories *of an abelian categories A, which are equipped with t-structures whose hearts can be identified with A. By definition, all information about these derived ∞-categories can be recovered from A.*

Under certain assumptions, a left/right t-exact functor F out of these derived ∞-categories can be identified with the right/left derived functor of $\mathsf{H}^0 F$.

2. Cohomologies

Proposition-Definition 2.1. *Let C be a stable ∞-category equipped with a t-structure. For any n, the inclusion functor $\mathsf{C}^{\leq n} \to \mathsf{C}$ admits a right adjoint $\tau^{\leq n}$, and the inclusion functor $\mathsf{C}^{\geq n} \to \mathsf{C}$ admits a left adjoint $\tau^{\geq n}$. These functors are called the **truncation functors** for the t-structure.*

Remark 2.2. *One can memorize the above handedness as $\mathsf{C}^{\leq 0} \rightleftarrows \mathsf{C} \rightleftarrows \mathsf{C}^{\geq 1}$.*

Exercise 2.3. *Prove the above proposition by verifying the following claim. Let $X \in \mathsf{C}$ and $X' \to X \to X''$ be any fiber-cofiber sequence with $X' \in \mathsf{C}^{\leq 0}$ and $X'' \in \mathsf{C}^{\geq 1}$, then*

(1) *For any $Y \in \mathsf{C}^{\leq 0}$, the morphism $X' \to X$ induces equivalences*

$$\mathsf{Maps}_{\mathsf{C}}(Y, X') \xrightarrow{\simeq} \mathsf{Maps}_{\mathsf{C}}(Y, X).$$

(2) *For any $Z \in \mathsf{C}^{\geq 1}$, the morphism $X \to X''$ induces equivalences*

$$\mathsf{Maps}_{\mathsf{C}}(X, Z) \xrightarrow{\simeq} \mathsf{Maps}_{\mathsf{C}}(X'', Z).$$

Deduce that $X' \simeq \tau^{\leq 0} X$ *and* $X'' \simeq \tau^{\geq 1} X$.

Exercise 2.4. *We have Cartesian squares*

$$\begin{array}{ccc} \mathsf{C}^{\leq 0} & \xrightarrow{\subseteq} & \mathsf{C} \\ \downarrow & & \downarrow{\scriptstyle \tau^{\geq 1}} \\ \{0\} & \xrightarrow{\subseteq} & \mathsf{C}^{\geq 1} \end{array} \qquad \begin{array}{ccc} \mathsf{C}^{\geq 0} & \xrightarrow{\subseteq} & \mathsf{C} \\ \downarrow & & \downarrow{\scriptstyle \tau^{\leq -1}} \\ \{0\} & \xrightarrow{\subseteq} & \mathsf{C}^{\leq -1} \end{array}$$

In particular, $\mathsf{C}^{\leq 0}$ *and* $\mathsf{C}^{\geq 0}$ *determine each other.*

Exercise 2.5. *The full sub-*∞*-category* $\mathsf{C}^{\leq 0} \subseteq \mathsf{C}$ *is stable under colimits, while* $\mathsf{C}^{\leq 0} \subseteq \mathsf{C}$ *is stable under limits.*

Exercise 2.6. *A limit in* $\mathsf{C}^{\leq 0}$ *is isomorphic to the* $\tau^{\leq 0}$*-truncation of the corresponding limit in* C*. Dually, a colimit in* $\mathsf{C}^{\geq 0}$ *is isomorphic to the* $\tau^{\geq 0}$*-truncation of the corresponding colimit in* C*.*

2.7. The following exercises say that for a fiber-cofiber sequence $X' \to X \to X''$, *amplitude estimations* of two terms give an estimation for the third one.

Exercise 2.8. *Let* $X' \to X \to X''$ *be a fiber-cofiber sequence.*

(1) *If* $X', X'' \in \mathsf{C}^{\leq 0}$*, then* $X \in \mathsf{C}^{\leq 0}$.
(2) *If* $X \in \mathsf{C}^{\leq 0}$ *and* $X'' \in \mathsf{C}^{\leq -1}$*, then* $X' \in \mathsf{C}^{\leq 0}$.
(3) *If* $X \in \mathsf{C}^{\leq 0}$ *and* $X' \in \mathsf{C}^{\leq 1}$*, then* $X'' \in \mathsf{C}^{\leq 0}$.

In particular, the full sub-∞*-categories* $\mathsf{C}^{\leq 0} \subseteq \mathsf{C} \supseteq \mathsf{C}^{\geq 0}$ *are stable under extensions.*

Exercise 2.9. *Give examples to show the above estimations are optimal.*

Exercise 2.10. *The truncation functors* $\tau_{\leq \bullet}$ *and* $\tau_{\geq \bullet}$ *commute with each other.*

Remark 2.11. *The precise meaning of the above exercise consists of the following. For* m *and* n*, we have*

- $\tau_{\geq m} \circ \tau_{\geq n} \simeq \tau_{\geq \max\{m,n\}}$, $\tau_{\leq m} \circ \tau_{\leq n} \simeq \tau_{\leq \max\{m,n\}}$.
- *The commutative square*

$$\begin{array}{ccc} \mathsf{C}^{\leq m} \cap \mathsf{C}^{\geq n} & \xrightarrow[\subseteq]{} & \mathsf{C}^{\leq m} \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\ \mathsf{C}^{\geq n} & \xrightarrow[\subseteq]{} & \mathsf{C} \end{array}$$

is left adjointable along the horizontal direction, and the induced commutative square

$$\begin{array}{ccc} \mathsf{C}^{\leq m} \cap \mathsf{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathsf{C}^{\leq m} \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\ \mathsf{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathsf{C} \end{array}$$

is right adjointable along the vertical direction, i.e., induces a commutative square

$$\begin{array}{ccc} \mathsf{C}^{\leq m} \cap \mathsf{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathsf{C}^{\leq m} \\ {\scriptstyle \tau^{\leq m}}\uparrow & & \uparrow{\scriptstyle \tau^{\leq m}} \\ \mathsf{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathsf{C} \end{array}$$

Definition 2.12. *Let C be a stable ∞-category equipped with a t-structure. Consider the functor*

$$\mathsf{H}^n : \mathsf{C} \xrightarrow{\tau^{\le n}\circ\tau^{\ge n}} \mathsf{C}^{\le n} \cap \mathsf{C}^{\ge n} \xrightarrow{[-n]} \mathsf{C}^{\heartsuit}.$$

*For $X \in \mathsf{C}$, we call $\mathsf{H}^n(X) \in \mathsf{C}^{\heartsuit}$ the **n-th cohomology of** X.*

Warning 2.13. *It may happen that $\mathsf{H}^n(X) \simeq 0$ while X is not isomorphic to 0. For instance, $\mathsf{C}^{\le 0} := \mathsf{C}$ and $\mathsf{C}^{\ge 0} := \{0\}$ give a t-structure with $\mathsf{C}^{\heartsuit} \simeq \{0\}$. There are lots of interesting examples in*

- non-regular *algebraic geometry*
- infinite type *algebraic geometry*
- infinite dimensional *representation theory*
- *...*

2.14. The following result follows from the corresponding well-known result for triangulated categories.

Proposition 2.15. *Let C be a stable ∞-category equipped with a t-structure. For any fiber-cofiber sequence $X' \to X \to X''$, we have a long exact sequence in $\mathsf{C}^{\heartsuit}$*

$$\cdots \to \mathsf{H}^n(X') \to \mathsf{H}^n(X) \to \mathsf{H}^n(X'') \xrightarrow{\delta} \mathsf{H}^{n+1}(X') \to \cdots,$$

where the connecting morphism δ is induced by the morphism $X'' \to X'[1]$.

3. Bounded, separated and complete

Definition 3.1. *Let C be a stable ∞-category equipped with a t-structure. For an object $X \in \mathsf{C}$,*

- *we say X is **connective** if $X \in \mathsf{C}^{\le 0}$;*
- *we say X is **coconnective** if $X \in \mathsf{C}^{\ge 0}$;*
- *we say X is **n-connective** if $X \in \mathsf{C}^{\le -n}$;*
- *we say X is **n-coconnective** if $X \in \mathsf{C}^{\ge -n}$;*
- *we say X is **eventually connective**, or **right bounded** if*
$$X \in \mathsf{C}^{-} := \bigcup \mathsf{C}^{\le n}$$
- *we say X is **eventually coconnective**, or **left bounded** if*
$$X \in \mathsf{C}^{+} := \bigcup \mathsf{C}^{\ge n}$$
- *we say X is **bounded** if X is both left bounded and right bounded:*
$$X \in \mathsf{C}^{\mathsf{b}} := \mathsf{C}^{+} \bigcap \mathsf{C}^{-}.$$

Warning 3.2. *As in Warning 2.13, the above properties cannot be tested via the cohomologies.*

Exercise 3.3. *Show that if X is left bounded, then $X \in \mathsf{C}^{\ge 0}$ iff $\mathsf{H}^i(X) \simeq 0$ for $i < 0$.*

Definition 3.4. *Let C be a stable ∞-category equipped with a t-structure.*

- *We say C is **right bounded** if $\mathsf{C} \simeq \mathsf{C}^{-}$.*
- *We say C is **left bounded** if $\mathsf{C} \simeq \mathsf{C}^{+}$.*
- *We say C is **bounded** if $\mathsf{C} \simeq \mathsf{C}^{\mathsf{b}}$.*
- *We say C is **right separated** if*
$$\mathsf{C}^{\ge \infty} := \bigcap_n \mathsf{C}^{\ge n} \simeq \{0\}.$$

- *We say* C *is* ***left separated*** *if*

$$\mathsf{C}^{\leq -\infty} := \bigcap_n \mathsf{C}^{\leq n} \simeq \{0\}.$$

Definition 3.5. *Let* C *be a stable* ∞*-category equipped with a t-structure. The* ***left completion*** $\widehat{\mathsf{C}}$ *of* C *is defined to be the limit of the following diagram*

$$\cdots \xrightarrow{\tau} \mathsf{C}^{\geq -2} \xrightarrow{\tau} \mathsf{C}^{\geq -1} \xrightarrow{\tau} \mathsf{C}^{\geq 0} \xrightarrow{\tau} \cdots.$$

We say C *is* ***left complete*** *if the functor* $\mathsf{C} \to \widehat{\mathsf{C}}$ *is an equivalence.*

Remark 3.6. *Dually, the* ***right completion*** *is defined to be the limit of*

$$\cdots \xrightarrow{\tau} \mathsf{C}^{\leq 2} \xrightarrow{\tau} \mathsf{C}^{\leq 1} \xrightarrow{\tau} \mathsf{C}^{\leq 0} \xrightarrow{\tau} \cdots.$$

Most unbounded t-structures appearing naturally are right complete, hence people do not give a notation to the right completion. Note however that the right completion can be identified with $(\widehat{\mathsf{C}^{\mathsf{op}}})^{\mathsf{op}}$.

Exercise 3.7. *Show that the left completion* $\widehat{\mathsf{C}}$ *is stable and the functor* $\mathsf{C} \to \widehat{\mathsf{C}}$ *is exact.*

Exercise 3.8. *Show that the left completion* $\widehat{\mathsf{C}}$ *admits an essential unique t-structure such that* $\mathsf{C}^{\geq 0} \xrightarrow{\simeq} (\widehat{\mathsf{C}})^{\geq 0}$.

Exercise 3.9. *Show that* C *is left separated iff the functor* $\mathsf{C} \to \widehat{\mathsf{C}}$ *detects zero objects. In particular,* C *is left separated if it is left complete.*

Warning 3.10. *A left separated t-structure may not be left complete. For example, any left bounded t-structure is left separated, but is almost never left complete.*

Exercise 3.11. *Suppose* C *admits countable products and* $\mathsf{C}^{\leq 0} \subseteq \mathsf{C}$ *is stable under countable products*[1]*. Then* C *is left separated iff it is left complete.*

Exercise 3.12. *Show that* C *is left complete iff it is* ***Postnikov complete****. The latter means*

- *Every object* $X \in \mathsf{C}$ *is the limit of its* ***Postnikov tower****:*

$$X \simeq \lim [\cdots \to \tau^{\geq n-1} X \to \tau^{\geq n} X \to \tau^{\geq n+1} X \to \cdots]$$

- *Any* ***Postnikov tower*** *in* C *converges. In other words, any collection* $X_n \in \mathsf{C}^{\geq n}$ *equipped with isomorphisms* $\tau^{\geq n} X_{n-1} \xrightarrow{\simeq} X_n$ *is the Postnikov tower of*

$$X := \lim [\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots] \in \mathsf{C}$$

4. t-structures on presentable stable ∞-categories

Proposition-Definition 4.1 (HA.1.4.4.13)**.** *Let* C *be a presentable stable* ∞*-category equipped with a t-structure. The following conditions are equivalent:*

- *The* ∞*-category* $\mathsf{C}^{\leq 0}$ *is presentable.*
- *The* ∞*-category* $\mathsf{C}^{\leq 0}$ *is accessible.*
- *The* ∞*-category* $\mathsf{C}^{\geq 0}$ *is presentable.*
- *The* ∞*-category* $\mathsf{C}^{\geq 0}$ *is accessible.*
- *The composition* $\mathsf{C} \xrightarrow{\tau_{\leq 0}} \mathsf{C}^{\leq 0} \to \mathsf{C}$ *is accessible.*

[1] These conditions are often referred as: taking countable products is right t-exact.

- *The composition* $\mathsf{C} \xrightarrow{\tau_{\geq 0}} \mathsf{C}^{\geq 0} \to \mathsf{C}$ *is accessible.*

We say a t-structure on C *is* ***accessible*** *if it satisfies the above conditions.*

Proposition 4.2 (HA.1.4.4.11)**.** *Let* C *be a presentable stable* ∞*-category. A full sub-*∞*-category* $\mathsf{C}' \subseteq \mathsf{C}$ *determines an accessible t-structure with* $\mathsf{C}^{\leq 0} \simeq \mathsf{C}'$ *iff*

- C' *is accessible*
- $\mathsf{C}' \subseteq \mathsf{C}$ *is closed under colimits and extensions.*

Remark 4.3. *In practice, one can apply the above proposition to the smallest full sub-*∞*-category* C' *generated by a* small *collection of objects under small colimits and extensions. Such* C' *is always accessible.*

Definition 4.4. *Let* C *be a presentable stable* ∞*-category. We say a t-structure on* C *is* ***compatible with filtered colimits*** *if taking filtered colimits is t-exact.*

Exercise 4.5. *A t-structure on* C *is compatible with filtered colimits iff taking small coproducts is t-exact.*

Definition 4.6. *Let* C *be a presentable stable* ∞*-category. We say a t-structure on* C *is* ***compactly generated*** *if* $\mathsf{C}^{\leq 0}$ *is compactly generated.*

Exercise 4.7. *Let* C *be a presentable stable* ∞*-category equipped with a compactly generated t-structure.*

(1) *Show that the t-structure is right complete and compatible with filtered colimits.*
(2) *Show that* C *is compactly generated, and compact objects in* C *are exactly given by* $X[n]$ *such that* X *is compact object in* $\mathsf{C}^{\leq 0}$.

Exercise 4.8. *Let* C_0 *be a small stable* ∞*-category equipped with a bounded t-structure. Show that* $\mathsf{C} := \mathsf{Ind}(\mathsf{C}_0)$ *has a compactly generated t-structure with* $\mathsf{C}^{\leq 0} := \mathsf{Ind}(\mathsf{C}_0^{\leq 0})$.

Warning 4.9. *Compact objects in* C *may not be stable under truncations. Hence not all compactly generated t-structures come from ind-completion. There are lots of interesting examples in the settings listed in Warning 2.13.*

Remark 4.10. *In practice, most t-structures on presentable stable* ∞*-categories are right complete and compatible with filtered colimits, or even compactly generated.*

5. t-STRUCTURE ON Sptr

Proposition-Construction 5.1 (HA.1.4.3.4)**.** *Let* C *be a pointed presentable* ∞*-category. We have an accessible t-structure on* $\mathsf{Sptr}(\mathsf{C})$ *given by the following.*

- *Let* $\mathsf{Sptr}(\mathsf{C})^{\geq 1}$ *be the full sub-*∞*-category of objects* X *such that* $\Omega^\infty(X) \simeq 0$.
- *Let* $\mathsf{Sptr}(\mathsf{C})^{\leq 0}$ *be the full sub-*∞*-category generated by the essential image of the functor* Σ^∞ *under extensions and small colimits.*

Exercise 5.2. *Show that* $\mathsf{Sptr}(\mathsf{Sptr}(\mathsf{C})^{\leq 0}) \to \mathsf{Sptr}(\mathsf{C})$ *is a t-exact equivalence.*

Proposition 5.3 (HA.1.4.3.6)**.** *Let* Sptr *be equipped with the above t-structure. Then*

(1) *This t-structure is compactly generated*[2] *and left complete.*

[2] Compact generation is not proved in HA, but it follows from the fact that Spc_* is compactly generated.

(2) *The Eilenberg–Maclane spectrum functor* $A \mapsto \mathbb{H}A$ *gives an equivalence* $\mathsf{Ab} \xrightarrow{\simeq} \mathsf{Sptr}^{\heartsuit}$.

Exercise 5.4. *Show that the cohomology functor* $\mathsf{H}^i : \mathsf{Sptr} \to \mathsf{Sptr}^{\heartsuit}$ *can be identified with* π_{-i} *via the equivalence* $\mathsf{Ab} \xrightarrow{\simeq} \mathsf{Sptr}^{\heartsuit}$.

Warning 5.5. *Let* A_1 *and* A_2 *be abelian groups. In general, the extension groups* $\mathsf{Ext}^i_{\mathsf{Sptr}}(\mathbb{H}A_1, \mathbb{H}A_2)$ *and* $\mathsf{Ext}^i(A_1, A_2)$ *are not isomorphic. For instance, the graded ring* $\mathsf{Ext}^{\bullet}_{\mathsf{Sptr}}(\mathbb{H}\mathbb{F}_q, \mathbb{H}\mathbb{F}_q)$ *is the* ***mod*** p ***Steenrod algebra****.*

5.6. Next time, we will construct a natural homomorphism

$$\mathsf{Ext}^i(A_1, A_2) \to \mathsf{Ext}^i_{\mathsf{Sptr}}(\mathbb{H}A_1, \mathbb{H}A_2)$$

which should be viewed as derived direct images along $\mathsf{Spec}\,\mathbb{Z} \to \mathsf{Spec}\,\mathbb{S}$.

Appendix A. Prestable ∞-categories

A.1. Let C be a stable ∞-category equipped with a t-structure. Sometimes it is more convenient to study $\mathsf{C}^{\leq 0}$ rather then C. It is possible to find several axioms that characterize ∞-categories of the form $\mathsf{C}^{\leq 0}$. Such ∞-categories are called **prestable ∞-categories**.

A.2. **Suggested readings.** SAG.C.