LECTURE 17

In this lecture, we introduce two examples of stable ∞-categories:

- the naive ∞-category $\mathsf{K}(\mathsf{A})$ of cochain complexes in an *additive category* A
- the derived ∞-category $\mathsf{D}(\mathsf{A})$ of an *abelian* category A

1. $\mathsf{K}(\mathsf{A})$ and $\mathsf{D}(\mathsf{A})$ via localization

Definition 1.1. *Let A be an additive category and $\mathsf{Ch}(\mathsf{A})$ be the* ordinary category *of cochain complexes in A. Let $f, g : M^\bullet \to N^\bullet$ be morphisms in $\mathsf{Ch}(\mathsf{A})$. A* ***cochain homotopy*** *from f to g is a collection of maps $h^n : M^n \to N^{n-1}$ such that*

$$f^n - g^n = d \circ h^n + h^{n+1} \circ d$$

for any n. We say f ***and*** *g* ***are homotopic****, or $f \sim g$, if there exists a cochain homotopy between them.*

Remark 1.2. *Being homotopic is an equivalence relation on $\mathsf{Hom}_{\mathsf{Ch}(\mathsf{A})}(M^\bullet, N^\bullet)$.*

Example 1.3. *Let $F, G : X \to Y$ be continuous maps between topological spaces. Then any homotopy from F to G induces a cochain homotopy from F^* to G^*, where $F^* : \mathsf{C}^\bullet(Y) \to \mathsf{C}^\bullet(X)$ is the cochain homomorphism between the singular cochain complexes.*

Exercise 1.4. *Let A be an abelian category (so that we can define cohomologies). If $f \sim g$, then $\mathsf{H}^n(f) = \mathsf{H}^n(g)$ as morphisms $\mathsf{H}^n(M^\bullet) \to \mathsf{H}^n(N^\bullet)$.*

Definition 1.5. *Let A be an additive category and $f : M^\bullet \to N^\bullet$ be a morphism in $\mathsf{Ch}(\mathsf{A})$. We say f is a* ***cochain homotopy equivalence*** *if there exists $g : N^\bullet \to M^\bullet$ such that $f \circ g \sim \mathsf{id}_{N^\bullet}$ and $g \circ f \simeq \mathsf{id}_{M^\bullet}$. Let S be the collection of cochain homotopy equivalences in $\mathsf{Ch}(\mathsf{A})$.*

Definition 1.6. *Let A be an abelian category and $f : M^\bullet \to N^\bullet$ be a morphism in $\mathsf{Ch}(\mathsf{A})$. We say f is a* ***quasi-isomorphism*** *if it induces isomorphisms between the cohomologies. Let W be the collection of quasi-isomorphisms in $\mathsf{Ch}(\mathsf{A})$.*

Exercise 1.7. *Let A be an abelian category. Show that a cochain homotopy equivalence in $\mathsf{Ch}(\mathsf{A})$ is a quasi-isomorphism.*

Exercise 1.8. *Let A be an abelian category and $0 \to L \to M \to N \to 0$ be a short exact sequence. Show that*

$$\begin{array}{ccccccc} [\cdots & \longrightarrow & L & \longrightarrow & M & \longrightarrow & \cdots] \\ & & \downarrow & & \downarrow & & \\ [\cdots & \longrightarrow & 0 & \longrightarrow & N & \longrightarrow & \cdots] \end{array}$$

is always a quasi-isomorphism, but is a cochain homotopy equivalence iff the given short exact sequence has a splitting.

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Definition 1.9. *Let* A *be an additive category. The* ***naive*** ∞***-category of cochain complexes*** *in* A *is defined to be the* ∞*-categorical localization*[1]

$$\mathsf{K}(\mathsf{A}) \coloneqq \mathsf{Ch}(\mathsf{A})[S^{-1}].$$

Definition 1.10. *Let* A *be an abelian category. The* ***unbounded derived*** ∞***-category*** *of* A *is defined to be the* ∞*-categorical localization*

$$\mathsf{D}(\mathsf{A}) \coloneqq \mathsf{Ch}(\mathsf{A})[W^{-1}].$$

Exercise 1.11. *Write* $\mathsf{D}(A)$ *as a localization of* $\mathsf{K}(\mathsf{A})$.

Warning 1.12. *In the non-*∞*-categorical literatures,* $\mathsf{K}(\mathsf{A})$ *and* $\mathsf{D}(\mathsf{A})$ *often means the 1-categorical localizations, which are the homotopy categories of our* ∞*-categories.*

1.13. It is well-known (see [Sta24, Tag 05QI]) that $\mathsf{hK}(\mathsf{A})$ has a structure of triangulated categories such that for any $f : M^\bullet \to N^\bullet$ in $\mathsf{Ch}(\mathsf{A})$, there is a distinguished triangle

$$M^\bullet \xrightarrow{f} N^\bullet \to \mathsf{cone}(f) \to M^\bullet[1]$$

for the corresponding objects in $\mathsf{hK}(A)$. Moreover, when A is abelian, this triangulated is compatible with the collection of quasi-isomorphisms and thereby induces a triangulated structure on the 1-categorical localization $\mathsf{hD}(\mathsf{A})$.

Warning 1.14. *The category* $\mathsf{Ch}(\mathsf{A})$ *is almost never triangulated because any abelian triangulated category is semisimple.*

1.15. The following result says the above triangulated categories come from stable ∞-categories.

Theorem 1.16 (HA.1.3.2.10)**.** *Let* A *be an additive category. The* ∞*-category* $\mathsf{K}(\mathsf{A})$ *is stable and the triangulated structure on* $\mathsf{hK}(\mathsf{A})$ *coincides with the well-known one*[2].

Theorem 1.17 (HA.1.3.5.9, 1.3.5.21)**.** *Let* A *be a* Grothendieck[3] *abelian category. The* ∞*-category* $\mathsf{D}(\mathsf{A})$ *is presentable and stable and the triangulated structure on* $\mathsf{hD}(\mathsf{A})$ *coincides with the well-known one.*

Warning 1.18. *Let* A *be a Grothendieck abelian category. The* ∞*-category* $\mathsf{K}(\mathsf{A})$ *is almost never presentable. See this MathOverflow question.*

Remark 1.19. *Note that the* ∞*-category* $\mathsf{D}(\mathsf{A})$ *defined above makes sense for any abelian category* A*. One can prove* $\mathsf{D}(\mathsf{A})$ *is always stable (for example using [NS18, Theorem I.3.3]). However,* $\mathsf{D}(\mathsf{A})$ *may be ill-behaved in general. For example, when* A *is small, it is better to consider* $\mathsf{D}^{\mathsf{b}}(\mathsf{A})$.

2. $\mathsf{K}(\mathsf{A})$ and $\mathsf{D}(\mathsf{A})$ via differential graded nerve

2.1. Definition 1.9 and Definition 1.10 are concise but inconvenient for calculations. In this section, we provide a more explicit construction of these ∞-categories.

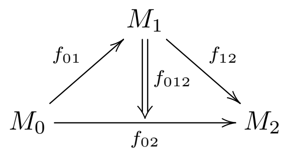
[1] According to our convention, we use the same symbol to denote an ordinary category C and the corresponding ∞-*category*, realized as the *quasi-category* $\mathsf{N}_\bullet(\mathsf{C})$. Hence $\mathsf{K}(\mathsf{A})$ can be represented as the *quasi-categorical localization* $\mathsf{N}_\bullet(\mathsf{Ch}(\mathsf{A}))[S^{-1}]$, which is the notation used in Lurie's books.

[2] This claim follows from the proof of HA.1.3.2.10.

[3] This means A is presentable and taking filtered colimits is exact.

2.2. The alternative construction for $\mathsf{K}(\mathsf{A})$ will be a quasi-category such that

- an object is a cochain complex M_0
- a morphism is a cochain homomorphism $M_0 \to M_1$
- a 2-simplex is given by



  where h_{012} stands for a homotopy from $f_{12} \circ f_{01}$ to f_{02},
- $\cdots$

To give a description for higher simplices, it is convenient to introduce some terminologies.

Construction 2.3. *Let* $\mathsf{Ch}(\mathsf{Ab})$ *be the ordinary category of cochain complexes. For* $M, N \in \mathsf{Ch}(\mathsf{Ab})$*, the* ***graded tensor product*** $M \otimes N$ *is defined as follows.*

- *For each integer* k*,*
$$(M \otimes N)^k := \bigoplus_{i+j=k} M^i \otimes N^j.$$
- *For each integer* k*, the differential* $(M \otimes N)^k \to (M \otimes N)^{k+1}$ *is given by the* ***graded Lubniz rule****, which sends a pure tensor* $m \otimes n \in M^i \otimes N^j$ *to*
$$d(m \otimes n) := d(m) \otimes n + (-1)^j m \otimes d(n).$$

There is a natural monoidal structure *on* $\mathsf{Ch}(\mathsf{Ab})$ *with multiplication given by the graded tensor products.*

Example 2.4. *Let* X *and* Y *be topological spaces. Then we have* $\mathsf{C}^\bullet(X \times Y) \simeq \mathsf{C}^\bullet(X) \otimes \mathsf{C}^\bullet(Y)$*.*

Exercise 2.5. *Let* $M \in \mathsf{Ch}(\mathsf{Ab})$*. Show that a 0-cocycle in* M *is the same as a morphism* $\mathbb{1} \to M$*, where* $\mathbb{1}$ *is the monoidal unit.*

Definition 2.6. *A* ***differential graded category****, or a* ***dg-category****, is a* $\mathsf{Ch}(\mathsf{Ab})$*-enriched category.*

2.7. By definition, a dg-category $\mathbb{C}$ consists of the following datum:

- A cochain complex $\mathsf{Hom}_{\mathbb{C}}(x,y) \in \mathsf{Ch}(\mathsf{Ab})$ for objects $x, y \in \mathbb{C}$ called the **mapping complex**
- A composition law given by cochain homomorphisms
$$- \circ - : \mathsf{Hom}_{\mathbb{C}}(x,y) \otimes \mathsf{Hom}_{\mathbb{C}}(y,z) \to \mathsf{Hom}_{\mathbb{C}}(x,z)$$
  which is associative in the obvious sense
- A *0-cocycle* $\mathsf{id}_x \in \mathsf{Hom}_{\mathbb{C}}(x,x)^0$ such that $f \circ \mathsf{id}_x = f$ and $\mathsf{id}_x \circ g = g$ for any $f \in \mathsf{Hom}_{\mathbb{C}}(x,y)^m$ and $g \in \mathsf{Hom}_{\mathbb{C}}(y,x)^n$.

Exercise 2.8. *Show that any* $\mathsf{id}_x \in \mathsf{Hom}_{\mathbb{C}}(x,x)^0$ *satisfying the above condition is automatically a cocycle.*

Definition 2.9. *Let* $\mathbb{C}$ *be a dg-category. The* ***underlying category*** C *of* $\mathbb{C}$ *is defined such that the* Hom*-sets are given by*
$$\mathsf{Hom}_{\mathsf{C}}(x,y) := \mathsf{Hom}_{\mathsf{Ch}(\mathsf{A})}(\mathbb{1}, \mathsf{Hom}_{\mathbb{C}}(x,y)).$$

2.10. We will identify $\mathsf{Hom}_{\mathsf{C}}(x,y)$ with the abelian group of 0-cocycles in $\mathsf{Hom}_{\mathbb{C}}(x,y)$.

Definition 2.11. *Let $\mathbb{C}$ be a dg-category. The **homotopy category** $\mathsf{h}\mathbb{C}$ of $\mathbb{C}$ is defined such that the* Hom*-sets are given by*

$$\mathsf{Hom}_{\mathsf{h}\mathbb{C}}(x,y) \coloneqq \mathsf{H}^0(\mathsf{Hom}_{\mathbb{C}}(x,y)).$$

Example 2.12. *For any additive category A, the ordinary category $\mathsf{Ch}(\mathsf{A})$ has a natural enrichment $\mathbb{Ch}(\mathsf{A})$ over $\mathsf{Ch}(\mathsf{Ab})$ such that*

$$\mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^k \coloneqq \prod_i \mathsf{Hom}_{\mathsf{A}}(M^i, N^{i+k})$$

with the differential given by the graded Lubniz rule

$$(df)(x) = d(f(x)) - (-1)^k f(dx)$$

for $f \in \mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^k$ and $x \in M^i$.

Exercise 2.13. *Show that the underlying category of $\mathbb{Ch}(\mathsf{A})$ is indeed $\mathsf{Ch}(\mathsf{A})$. What is its homotopy category?*

Exercise 2.14. *Let $f,g \in \mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^0$ be 0-cocycles, viewed as morphisms in the ordinary category $\mathsf{Ch}(\mathsf{A})$. Show that a homotopy from f to g is the same as an element $h \in \mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^{-1}$ such that $dh = f - g$.*

Proposition-Construction 2.15 (HA.1.3.1.10)**.** *Let $\mathbb{C}$ be a dg-category. We have a quasi-category $\mathsf{N}_{\mathsf{dg}}(\mathbb{C})$, called the **dg-nerve** of $\mathbb{C}$, defined as follows.*

- *A 0-simplex is an object in $\mathbb{C}$*
- *A 1-simplex consists of its boundary $\{X_0, X_1\}$ together with a 0-cocycle in $f_{01} \in \mathsf{Hom}_{\mathbb{C}}(X_0,X_1)^0$*
- *A 2-simplex consists of its boundary $\{f_{01}, f_{02}, f_{12}\}$ together with an element $g_{012} \in \mathsf{Hom}_{\mathbb{C}}(X_0,X_2)^{-1}$ such that*

$$df_{012} = -(f_{02} - f_{12} \circ f_{01}).$$

- *A 3-simplex consists of its boundary $\{f_{012}, f_{013}, f_{023}, f_{123}\}$ together with an element $f_{0123} \in \mathsf{Hom}_{\mathbb{C}}(X_0,X_3)^{-2}$ such that*

$$df_{0123} = -(f_{013} - f_{23} \circ f_{012}) + (f_{023} - f_{123} \circ f_{01}).$$

- *Higher simplices are given similarly (see HA.1.3.1.6).*

Proposition 2.16. *Let $\mathbb{C}$ be a dg-category and X, Y be objects in $\mathbb{C}$. Then there are canonical isomorphisms*

$$\pi_i(\mathsf{Maps}_{\mathsf{N}_{\mathsf{dg}}(\mathbb{C})}(X,Y)) \simeq \mathsf{H}^{-i}(\mathsf{Hom}_{\mathbb{C}}(X,Y))$$

Remark 2.17. *We will prove the above proposition next time. In fact, the space $\mathsf{Maps}_{\mathsf{N}_{\mathsf{dg}}(\mathbb{C})}(X,Y)$ can be obtained from the truncation $\tau^{\le 0}\mathsf{Hom}_{\mathbb{C}}(X,Y)$ in a canonical way, known as the* Dold–Kan correspondence.

Exercise 2.18. *Construct a monomorphism $\mathsf{N}(\mathsf{C}) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{C})$ that is bijective on n-simplices with $n \le 1$, where $\mathsf{N}(\mathsf{C})$ is the usual nerve of the underlying ordinary category C of $\mathbb{C}$.*

Exercise 2.19. *Construct an equivalence $\mathsf{h}\mathsf{N}_{\mathsf{dg}}(\mathbb{C}) \simeq \mathsf{h}\mathbb{C}$.*

Proposition 2.20 (HA.1.3.4.5)**.** *Let* A *be an additive category. Then the functor* $\mathsf{N}(\mathsf{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$ *exhibits* $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$ *as the localization of* $\mathsf{N}(\mathsf{Ch}(\mathsf{A}))$ *for the collection of cochain homotopy equivalences. In other words,* $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$ *represents the* ∞*-category* $\mathsf{K}(\mathsf{A})$*.*

Exercise 2.21. *Let* A *be an abelian category and* $0 \to L \to M \to N \to 0$ *be a short exact sequence. Show that it represents a zero object in* $\mathsf{K}(\mathsf{A})$ *iff it admits a splitting.*

2.22. Our next goal is to realize $\mathsf{D}(\mathsf{A})$ as the dg-nerve of a dg-category when A is a Grothendieck abelian category. Recall such A has enough injective objects.

Theorem 2.23. *Let* A *be a Grothendieck abelian category. Then* $\mathsf{Ch}(\mathsf{A})$ *admits a left proper combinatorial model structure determined by the following.*

- (W) *weak equivalences are quasi-isomorphisms*
- (C) *cofibrations are degreewise monomoprhisms*

Exercise 2.24. *If* $\mathsf{M} \in \mathsf{Ch}(\mathsf{A})$ *is fibrant in the above model structure, then each* $\mathsf{M}^n \in \mathsf{A}$ *is injective. Conversely, if* M *is bounded below and each* M^n *is injective, then* M *is fibrant.*

Proposition 2.25 (HA.1.3.5.13)**.** *Let* $\mathsf{Ch}(\mathsf{A})^\circ \subseteq \mathsf{Ch}(\mathsf{A})$ *be the full subcategory of bifibrant objects. Let* $\mathbb{Ch}(\mathsf{A})^\circ \subseteq \mathbb{Ch}(\mathsf{A})$ *be the corresponding dg-category. Then the embedding*

$$\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$$

admits a left adjoint

$$\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$$

such that the composition

$$\mathsf{N}(\mathsf{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$$

exhibits $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$ *as the localization of* $\mathsf{N}(\mathsf{Ch}(\mathsf{A}))$ *for the collection of quasi-isomorphisms. In other words,* $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$ *represents the* ∞*-category* $\mathsf{D}(\mathsf{A})$*.*

2.26. The above proposition implies for any Grothendieck abelian category A, we have an adjunction of exact functors

$$\mathsf{K}(\mathsf{A}) \rightleftarrows \mathsf{D}(\mathsf{A}) \tag{2.1}$$

such that the right adjoint functor is fully faithful, and the left adjoint is given by taking fibrant replacements.

Exercise 2.27. *Find the kernel of the left adjoint functor* $\mathsf{K}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$*.*

3. t-structures

Proposition 3.1 (HA.1.3.5.19, 1.3.5.21)**.** *Let* A *be a Grothendieck abelian category.*

- *Let* $\mathsf{D}(\mathsf{A})^{\le 0}$ *be the full sub-*∞*-category consisting of objects represented by cochain complexes* M *with* $H^n(M) \simeq 0$ *for* $n > 0$*.*
- *Let* $\mathsf{D}(\mathsf{A})^{\ge 0}$ *be the full sub-*∞*-category consisting of objects represented by cochain complexes* M *with* $H^n(M) \simeq 0$ *for* $n < 0$*.*

Then $(\mathsf{D}(\mathsf{A})^{\le 0}, \mathsf{D}(\mathsf{A})^{\ge 0})$ *determines a t-structure on* $\mathsf{D}(\mathsf{A})$*. Moreover,*

- *The functor* $\mathsf{H}^0 : \mathsf{Ch}(\mathsf{A}) \to \mathsf{A}$ *induces an equivalence* $\mathsf{D}(\mathsf{A})^\heartsuit \to \mathsf{A}$*.*
- *This t-structure is accessible, left separated, right complete and compatible with filtered colimits.*

Warning 3.2. *The t-structure on* $\mathsf{D}(\mathsf{A})$ *is generally not left complete. See [Nee11] for a counterexample. The left completion is often denoted by* $\widehat{\mathsf{D}}(\mathsf{A})$.

Example 3.3. *Let* $\mathsf{A} = \mathsf{Mod}_R^\heartsuit$ *be the abelian category of* R*-modules for an associative ring* R*. In future lectures, we will construct*

- *a symmetric monoidal structure on* Sptr *given by* smash products*;*
- *an* $\mathbb{E}_1$-algebra *structure on the* Eilenburg–Maclane spectrum $\mathbb{H}R \in \mathsf{Sptr}$*;*
- *a stable* ∞*-category* $\mathsf{Mod}_{\mathbb{H}R}$ *of* $\mathbb{H}R$*-modules in* Sptr*, equipped with a t-structure induced from that of* Sptr*;*
- *a t-exact equivalence*

$$\mathsf{Mod}_{\mathbb{H}R} \simeq \mathsf{D}(\mathsf{Mod}_R^\heartsuit) =: \mathsf{D}(R).$$

Moreover, when R *is* commutative*, we will equip* $\mathsf{D}(R)$ *with a symmetric monoidal structure given by* relative tensor product *over* $\mathbb{H}R$.

Exercise 3.4. *Let* A *be a Grothendieck abelian category. Show that the left bounded part of* $\mathsf{D}(\mathsf{A})$ *can be identified with*

$$\mathsf{D}^+(\mathsf{A}) \simeq \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}^+(\mathsf{A}_{\mathsf{inj}}))$$

where $\mathbb{Ch}^+(\mathsf{A}_{\mathsf{inj}})$ *is the dg-category of left bounded complexes of injective objects in* A*. Note that the proof should* not *work for* $\mathsf{D}^-(\mathsf{A})$.

3.5. Motivated by the above exercise, we make the following definition.

Definition 3.6. *Let* A *be an abelian category with enough injectives. The* ***left bounded derived-∞-category*** *of* A *is defined to be*

$$\mathsf{D}^+(\mathsf{A}) := \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}^+(\mathsf{A}_{\mathsf{inj}})).$$

Dually, let A *be an abelian category with enough projectives. The* ***right bounded derived-∞-category*** *of* A *is defined to be*

$$\mathsf{D}^-(\mathsf{A}) := \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}^-(\mathsf{A}_{\mathsf{proj}})) \simeq \mathsf{D}^+(\mathsf{A}^{\mathsf{op}})^{\mathsf{op}}.$$

Exercise 3.7. *Show that* $\mathsf{D}^-(\mathsf{A})$ *is a stable sub-*∞*-category of* $\mathsf{K}(\mathsf{A}_{\mathsf{proj}})$.

Warning 3.8. *The* ∞*-categories* $\mathsf{D}^+(\mathsf{A})$ *and* $\mathsf{D}^-(\mathsf{A})$ *are not presentable because countable coproducts may not exist.*

3.9. The following results characterize $\mathsf{D}^-(\mathsf{A})$ via universal properties.

Proposition 3.10 (HA.1.3.2.19, 1.3.3.16)**.** *Let* A *be an abelian category with enough projectives. The obvious choice defines a left complete t-structure on* $\mathsf{D}^-(\mathsf{A})$ *whose heart is canonically equivalent to* A.

Theorem 3.11 (HA.1.3.3.2. 1.3.3.6)**.** *Let* A *be an abelian category with enough projectives. For any stable* ∞*-category* C *equipped with a left complete t-structure, the following data are equivalent:*

(i) *A right t-exact functor* $F : \mathsf{D}^-(\mathsf{A}) \to \mathsf{C}$ *that sends* $\mathsf{A}_{\mathsf{proj}}$ *into* $\mathsf{C}^\heartsuit$
(ii) *A right exact functor* $f : \mathsf{A} \to \mathsf{C}^\heartsuit$

where $f := \tau^{\geq 0} \circ F|_{\mathsf{A}}$*. Moreover, the following conditions are equivalent*

- F *is t-exact.*
- F *preserves the hearts.*
- f *is exact.*

Definition 3.12. *Let* A *be an abelian category with enough projectives and* C *be a stable* ∞*-category equipped with a left complete t-structure. For a right exact functor* $f : \mathsf{A} \to \mathsf{C}^{\heartsuit}$*, the corresponding functor* F *is called the* ***left derived functor*** *of* f *and denote as*

$$\mathbb{L}f : \mathsf{D}^{-}(\mathsf{A}) \to \mathsf{C}.$$

Exercise 3.13. *Show that* $\mathbb{L}f(M)$ *can be calculated via a projective resolution of the complex* M*.*

Exercise 3.14. *Consider the t-exact functor* $\mathsf{D}^{-}(\mathsf{Ab}) \to \mathsf{Sptr}$ *corresponding to the equivalence* $\mathsf{Ab} \simeq \mathsf{Sptr}^{\heartsuit}$*,* $A \mapsto \mathbb{H}A$*. Taking* right completion*, we obtain a functor*

$$\pi_* : \mathsf{D}(\mathsf{Ab}) \to \mathsf{Sptr}.$$

Show that:

(1) *The functor can also be obtained via the universal property of* $\mathsf{D}^{+}(\mathsf{Ab})$ *described by the dual of Theorem 3.11.*
(2) *Make a guess for the composition* $\Omega^{\infty} \circ \pi_*$*.*

3.15. Let A be a Grothendieck abelian category with enough *projective* objects. The following result says the two definitions of $\mathsf{D}^{-}(\mathsf{A})$ coincide.

Proposition 3.16 (HA.1.3.5.24)**.** *Let* A *be a Grothendieck abelian category with enough projective objects. Then the composition*

$$\mathsf{D}^{-}(\mathsf{A}) \xrightarrow{\subseteq} \mathsf{K}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$$

is fully faithful with essential image given by the right bounded part of $\mathsf{D}(\mathsf{A})$*. In particular,* $\mathsf{D}(\mathsf{A})$ *is left complete.*

3.17. In fact, there is also a canonical t-structure on $\mathsf{K}(\mathsf{A})$ when A is a Grothendieck abelian category. However, one needs to be careful about the coconnective part because there are nonzero objects in $\mathsf{K}(\mathsf{A})$ with zero cohomologies.

Proposition 3.18 (HA.1.3.5.18)**.** *Let* A *be a Grothendieck abelian category.*

- *Let* $\mathsf{K}(\mathsf{A})^{\leq 0}$ *be the full sub-*∞*-category consisting of objects represented by cochain complexes* M *with* $H^n(M) \simeq 0$ *for* $n > 0$*.*
- *Let* $\mathsf{K}(\mathsf{A})^{\geq 0}$ *be the full sub-*∞*-category consisting of objects represented by cochain complexes* M *with* $M^n \simeq 0$ *for* $n < 0$ *such that* M^n *is* injective *for any* n*.*

Then $(\mathsf{K}(\mathsf{A})^{\leq 0}, \mathsf{K}(\mathsf{A})^{\geq 0})$ *determines a t-structure on* $\mathsf{K}(\mathsf{A})$*.*

Exercise 3.19. *Identify* $\mathsf{K}(\mathsf{A})^{\geq 0}$ *with the essential image of the fully faithful functor*

$$\mathsf{D}(\mathsf{A})^{\geq 0} \to \mathsf{D}(\mathsf{A}) \to \mathsf{K}(\mathsf{A})$$

where the first functor is the right adjoint in (2.1)*.*

Exercise 3.20. *Show that both functors in* $\mathsf{K}(\mathsf{A}) \rightleftarrows \mathsf{D}(\mathsf{A})$ *are t-exact, and they induce equivalences between the hearts.*

Exercise 3.21. *Show that the t-structure on* $\mathsf{K}(\mathsf{A})$ *is right complete and compatible with filtered colimits, but is not left separated.*

Appendix A. $\breve{\mathsf{D}}(\mathsf{A})$

Construction A.1. *Let* A *be a Grothendieck abelian category. Define the* ***unseparated derived*** ∞***-category*** *of* A *to be*

$$\breve{\mathsf{D}}(\mathsf{A}) \coloneqq \mathsf{N}^{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}_{\mathsf{inj}})) \simeq \mathsf{K}(\mathsf{A}_{\mathsf{inj}}).$$

There is a t-structure on $\breve{\mathsf{D}}(\mathsf{A})$ *defined similarly as that on* $\mathsf{K}(\mathsf{A})$ *such that*

- *The heart is canonically identified with* A*;*
- *This t-structure is accessible, right complete and compatible with filtered colimits;*
- *This t-structure is (left)* ***anti-complete****.*

Remark A.2. *Roughly speaking, being anti-complete means the t-structure is "orthogonal" to complete ones. In fact, there is an essentially unique colimit-preserving t-exact functor*

$$\breve{\mathsf{D}}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$$

that restricts to the identity functor on the hearts. Moreover, this functor exhibits $\breve{\mathsf{D}}(\mathsf{A})$ *as the (left)* ***anti-completion*** *of* $\mathsf{D}(\mathsf{A})$.

Example A.3. *For Noetherian commutative ring* R*,*

$$\breve{\mathsf{D}}(\mathsf{Mod}_R^{\heartsuit}) \simeq \mathsf{Ind}(\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}_{R,\mathsf{fg}}^{\heartsuit})),$$

where $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}_{R,\mathsf{fg}}^{\heartsuit})$ *is the full sub-*∞*-category of* $\mathsf{D}(\mathsf{Mod}_R^{\heartsuit})$ *consisting of complexes with bounded finite generated cohomologies.*

A.4. **Suggested readings.** SAG.C.

Appendix B. dg-category vs. $\mathbb{H}Z$-linear stable ∞-categories

B.1. The homotopy category $\mathsf{h}\mathbb{C}$ of any dg-category $\mathbb{C}$ has a canonical *candidate* for triangulated structures. If this candidate is indeed a triangulated structure, we say $\mathbb{C}$ is **pretriangulated**. The following notions are essentially equivalent:

- pretriangulated dg-categories over a commutative ring R;
- $\mathbb{H}R$-linear stable ∞-categories.

B.2. **Suggested readings.** [Coh13].

References

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