LECTURE 18

In this lecture, we discuss the Dold–Kan correspondence and its generalizations. We use *homological convention* because such is standard in the literature about the DK correspondences.

1. classical Dold–Kan correspondence

1.1. Let A be an abelian category. The classical Dold–Kan correspondence is an equivalence between:

- The ordinary category $\mathsf{Ch}(\mathsf{A})_{\geq 0}$ of nonnegatively graded chain complexes in A;
- The ordinary category A_Δ of simplicial objects in A.

This equivalence can be constructed explicitly in both directions.

Construction 1.2. *Let $A_\bullet \in \mathsf{A}_\Delta := \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{A})$ be a simplicial object in an additive category A. For any $i \in [n]$, consider the face morphism $d_i : A_n \to A_{n-1}$ corresponding to the unique injection $\delta_n^i : [n-1] \to [n]$ whose image does not contain i. Define*

$$\mathsf{C}_*(A) := [\cdots \to A_2 \to A_1 \to A_0 \to 0 \to \cdots]$$

where

- *the object A_n is put in the homological degree n (= cohomological degree $-n$);*
- *the differential $A_n \to A_{n-1}$ is the alternating sum $\sum_{i=0}^n (-1)^i d_i$.*

We call it the ***unnormalized chain complex*** *associated to $A_\bullet$.*

Exercise 1.3. *Show that $\mathsf{C}_*(A)$ is indeed a complex.*

1.4. By the above exercise, we have $\mathsf{C}_*(A) \in \mathsf{Ch}(\mathsf{A})_{\geq 0}$. However, the obtained functor $\mathsf{A}_\Delta \to \mathsf{Ch}(\mathsf{A})_{\geq 0}$ is not an equivalence.

Exercise 1.5. *For $\mathsf{A} := \mathsf{Ab}$, consider the constant simplicial abelian group $\underline{\mathbb{Z}}$. Show that*

$$\mathsf{C}_*(\underline{\mathbb{Z}}) \simeq [\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots].$$

Deduce that the functor C_* is not even fully faithful.

1.6. In the above example, note however that

- the complex $[\cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots]$ has the correct set of endomorphisms;
- the above complex is quasi-isomorphic to $\mathsf{C}_*(\underline{\mathbb{Z}})$;
- n-simplices in $\underline{\mathbb{Z}}$ are degenerate for $n > 1$;

This suggests C_* fails to be an equivalence because of the degenerate simplices.

There are two ways to get rid of the degenerate simplices.

Date: Nov. 22, 2024.

Construction 1.7. *Let $A_\bullet \in \mathsf{A}_\Delta$ be a simplicial object in an abelian category A. Define*

$$\mathsf{D}_n(A) := \mathsf{im}(\bigoplus_{i=0}^{n-1} A_{n-1} \to A_n),$$

where for $0 \le i \le n-1$, the morphism $A_{n-1} \to A_n$ is the i-th degeneracy morphism.

Exercise 1.8. *Show that $\mathsf{D}_*(A)$ is a subcomplex of $\mathsf{C}_*(A)$. In particular, we can form the quotient complex $\mathsf{C}_*(A)/\mathsf{D}_*(A)$.*

Construction 1.9. *Let $A_\bullet \in \mathsf{A}_\Delta$ be a simplicial object in an abelian category A. Define*

$$\mathsf{N}_*(A) := [\cdots \to \mathsf{N}_2(A) \to \mathsf{N}_1(A) \to \mathsf{N}_0(A) \to 0 \to \cdots]$$

where

- *the object $\mathsf{N}_n(A)$ is the joint kernel of d_i for $1 \le i \le n$;*
- *the differential is given by d_0.*

*We call it the **normalized chain complex**, or the **Moore complex**, associated to $A_\bullet$.*

Exercise 1.10. *Show that d_0 indeed sends $\mathsf{N}_n(A)$ into $\mathsf{N}_{n-1}(A)$, and $\mathsf{N}_*(A)$ is subcomplex of $\mathsf{C}_*(A)$.*

Exercise 1.11. *Show that $\mathsf{C}_*(A) \simeq \mathsf{N}_*(A) \bigoplus \mathsf{D}_*(A)$.*

1.12. In particular, the composition

$$\mathsf{N}_*(A) \to \mathsf{C}_*(A) \to \mathsf{C}_*(A)/\mathsf{D}_*(A)$$

is an isomorphism in $\mathsf{Ch}(\mathsf{A})^{\le 0}$. The following result was proved by Eilenberg–MacLane.

Proposition 1.13 (HA.1.2.3.17)**.** *The chain morphism $\mathsf{N}_*(A) \to \mathsf{C}_*(A)$ is a quasi-isomoprhism.*

Exercise 1.14. *In fact, $\mathsf{N}_*(A) \to \mathsf{C}_*(A)$ is even a* chain homotopy equivalence. *See [GJ09, Theorem III.2.4].*

1.15. Now let us construct the desired inverse of the functor $\mathsf{N}_* : \mathsf{A}_\Delta \to \mathsf{Ch}(\mathsf{A})_{\ge 0}$.

Construction 1.16. *Let A be an additive cateogry and $A_* \in \mathsf{Ch}(\mathsf{A})_{\ge 0}$. Define $\mathsf{DK}_\bullet(A) \in \mathsf{A}_\Delta$ as follows.*

- *For each $n \ge 0$, the object of n-simplices is*

$$\mathsf{DK}_n(A) := \bigoplus_{\alpha:[n]\twoheadrightarrow[k]} A_k,$$

  where the direct sum is indexed by surjections $\alpha : [n] \to [k]$.
- *For $[n'] \to [n]$ in Δ, the connecting morphism $\mathsf{DK}_n(A) \to \mathsf{DK}_{n'}(A)$ is given by the map*

$$\bigoplus_{\alpha:[n]\twoheadrightarrow[k]} A_k \to \bigoplus_{\alpha:[n']\twoheadrightarrow[k']} A_{k'}$$

  where

– *each commutative square*

$$\begin{array}{ccc} [n'] & \longrightarrow & [n] \\ \downarrow & & \downarrow \\ [k] & = & [k] \end{array}$$

contributes to $A_k \xrightarrow{\mathsf{id}} A_k$*;*

– *each commutative square*

$$\begin{array}{ccc} [n'] & \longrightarrow & [n] \\ \downarrow & & \downarrow \\ [k-1] & \xrightarrow{\delta_k^0} & [k] \end{array}$$

contributes to $A_k \xrightarrow{d} A_{k-1}$.

One can check $\mathsf{DK}_\bullet(A)$ *is indeed a simplicial object in* A.

Exercise 1.17. *For* $\mathsf{A} := \mathsf{Ab}$*, show that* $\mathsf{DK}_\bullet(\mathbb{Z}[n]) \simeq \mathbb{Z}\Delta^n/\mathbb{Z}\partial\Delta^n$*, where for a simplicial set* S*,* $\mathbb{Z}S$ *is the free simplicial abelian group.*

1.18. The following result is known as the classical **Dold–Kan correspondence**.

Theorem 1.19 (HA.1.2.3.7, 1.2.3.12)**.** *Let* A *be an additive cateogry.*

(1) *The functor* $\mathsf{DK} : \mathsf{Ch}(\mathsf{A})_{\geq 0} \to \mathsf{A}_\Delta$ *is fully faihtful.*
(2) *If* A *is idempotent complete, then* DK *is an equivalence. In particular,* DK *is an equivalence when* A *is abelian.*
(3) *If* A *is abelian, then* $\mathsf{N} : \mathsf{A}_\Delta \to \mathsf{Ch}(\mathsf{A})_{\geq 0}$ *is an inverse of* DK.

Exercise 1.20. *When* $\mathsf{A} = \mathsf{Ab}$*, show that the functor*

$$\mathsf{Ch}(\mathsf{Ab})_{\geq 0} \to \mathsf{Ab}_\Delta,\ A \mapsto \underline{\mathsf{Hom}}(\mathsf{N}(\mathbb{Z}\Delta^\bullet), A)$$

is right adjoint to N*. Deduce that this functor is equivalent to* DK*. Can you prove this equivalence directly?*

2. Model-categorical Dold–Kan correspondence

2.1. Recall that both $\mathsf{Ch}(\mathsf{Ab})_{\geq 0}$ and Ab_Δ have classical model structures, and the Dold–Kan equivalence is compatible with these model structures in the strongest sense. Results in this section can be found in [Qui67, §II.4].

Proposition 2.2. *There is a model structure on* $\mathsf{Ch}(\mathsf{Ab})_{\geq 0}$*, known as the* ***projective model structure****, such that:*

(W) *weak equivalences are quasi-isomorphisms;*
(C) *cofibrations are degreewise injections with projective kernels;*
(F) *fibrations are degreewise surjections.*

Proposition 2.3. *There is a model structure on* Ab_Δ*, known as the* ***classical model structure****, such that:*

(W) *weak equivalences are weak homotopy equivalences between the underlying simplicial sets;*
(F) *fibrations are Kan fibrations between the underlying simplicial sets.*

Theorem 2.4 (Quillen)**.** *For* $\mathsf{A} := \mathsf{Ab}$*, both of the equivalences* DK *and* N *preserve and detect the collections (W), (C) and (F).*

Exercise 2.5. *Show that any object in* Ab_Δ *is fibrant, i.e., its underlying simplicial set is a Kan complex. Can you prove this directly?*

Exercise 2.6. *Show that homology groups* $\mathsf{H}_*(-)$ *on the* $\mathsf{Ch}(\mathsf{Ab})_{\geq 0}$ *side corresponds to simplicial homotopy groups* $\pi_*(-)$ *on the* Ab_Δ *side.*

3. Connective chain complices and spaces

3.1. By Quillen's theorem, the functor $\mathsf{Ch}(\mathsf{Ab})_{\geq 0} \to \mathsf{Ab}_\Delta \to (\mathsf{Set}_\Delta)_*$ preseves weak equivalences. Hence we obtain a functor between their ∞-categorical localizaitons

$$\mathsf{Ch}(\mathsf{Ab})_{\geq 0}[W^{-1}] \to (\mathsf{Set}_\Delta)_*[W^{-1}].$$

Exercise 3.2. *We have* $\mathsf{Ch}(\mathsf{Ab})_{\geq 0}[W^{-1}] \simeq \mathsf{D}(\mathsf{Ab})_{\geq 0}$.

Exercise 3.3. *We have* $(\mathsf{Set}_\Delta)_*[W^{-1}] \simeq \mathsf{Spc}_*$.

3.4. We abuse notation and denote the obtained functor by

$$\mathsf{DK} : \mathsf{D}(\mathsf{Ab})_{\geq 0} \to \mathsf{Spc}_*.$$

Warning 3.5. *The functor* DK *is not an equivalence because the above construction uses the forgetful functor* $\mathsf{Ab}_\Delta \to (\mathsf{Set}_\Delta)_*$.

Exercise 3.6. *The functor* DK *admits a left adjoint. Hint: construct a Quillen adjunction* $(\mathsf{Set}_\Delta)_* \rightleftarrows \mathsf{Ab}_\Delta$.

3.7. In particular, we can take spectrum objects and obtain a functor $\mathsf{Sptr}(\mathsf{D}(\mathsf{Ab})_{\geq 0}) \to \mathsf{Sptr}(\mathsf{Spc}_*)$.

Exercise 3.8. *We have* $\mathsf{Sptr}(\mathsf{D}(\mathsf{Ab})_{\geq 0}) \simeq \mathsf{D}(\mathsf{Ab})$.

3.9. We abuse notation and denote the obtained functor by

$$\mathsf{DK} : \mathsf{D}(\mathsf{Ab}) \to \mathsf{Sptr}.$$

Exercise 3.10. *Identify the above functor with the functor in [Lecture 17, Exercise 3.14].*

4. Dold–Kan correspondence for stable ∞-categories

4.1. Lurie's Dold–Kan correspondence for a stable ∞-category C establishes a canonical equivalence between

- The stable ∞-category $\mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathsf{C})$ of $\mathbb{Z}_{\geq 0}$-filtered objects in C;
- The stable ∞-category $\mathsf{C}_\Delta := \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{C})$ of simplicial objects in C.

Theorem 4.2 (HA.1.2.4.1)**.** *Let* C *be a stable ∞-category. Then there is a canonical equivalence*

$$\mathsf{Fun}(\mathbb{Z}_{\geq 0}, \mathsf{C}) \simeq \mathsf{C}_\Delta$$

Remark 4.3. *In this remark, we explain the content of the above theorem as well as its relation with the classical Dold–Kan correspondence.*

First, for any filtered object $D(0) \to D(1) \to D(2) \to \cdots$ *in* C*, consider the* shifted graded objects

$$C_k := \mathsf{Cofib}(D(k-1) \to D(k))[-k]$$

where we take $D(-1) := 0$ *for* $k = 0$*. For each* k*, the morphisms* $D(k-1) \to D(k) \to D(k+1)$ *induce a fiber-cofiber sequence*

$$C_k[k] \to \mathsf{Cofib}(D(k-1) \to D(k+1)) \to C_{k+1}[k+1],$$

which corresponds to a morphism $C_{k+1} \to C_k$*.*

One can check (Exercise!) that in the homotopy cateogry hC*, the composition* $C_{k+1} \to C_k \to C_{k-1}$ *is zero, hence we obtain a chain complex*

$$\cdots \to C_2 \to C_1 \to C_0 \to 0 \to \cdots$$

in the additive category hC*. Applying the classical functor* DK*, we obtain a simplicial object in* hC*.*

On the other hand, for any simplicial object $A_\bullet$ *in* C*, the functor* $\mathsf{C} \to \mathsf{hC}$ *induces a simplicial object in* hC*. Now the precise statement of Theorem 4.2 says if* $D(-)$ *corresponds to* $A_\bullet$ *under the claimed equivalence, then the obtained simplicial objects in* hC *are canonically identified.*

Remark 4.4. *In fact, the* n*-th term* $D(n)$ *can be calculated as the colimit of the* k-skeleton *of the simplicial object* $A_\bullet$*. However, since* C *is not an ordinary category, one needs to be careful about the meaning of the* k*-skeleton of a simplicial object. Nevertheless, this suggests that the colimit of the filtered object should be isomorphic to the colimit of the entire simplicial object* $A_\bullet$*. In other words,*

$$\underset{n \in \mathbb{Z}_{\geq 0}}{\mathsf{colim}}\, D(n) \simeq \underset{\Delta^{\mathrm{op}}}{\mathsf{colim}}\, A_\bullet.$$

This is indeed correct, see HA.1.2.4 for more details.

Remark 4.5. *Suppose* C *is equipped with a t-structure. For any* $\mathbb{Z}$*-filtered object* $D(-)$*, one can construct a spectral sequence in the abelian category* $\mathsf{C}^\heartsuit$ *such that the* E_1*-page is given by*

$$E_1^{p,q} = \mathsf{H}_{p+q}\mathsf{Cofib}(D(p) \to D(p+1)).$$

Under certain conditions, this spectral sequence converges to

$$\mathsf{H}_n(\underset{m}{\mathsf{colim}}\, D(m)).$$

Note that this provides a strategy to calculate homologies of the geometric realizations of simplicial objects in C*. See HA.1.2.3 for more details.*

Appendix A. Moore complexes and simplicial homotopy groups

A.1. For a *pointed* simplicial set X, one can construct a simplicial group $\mathsf{G}(X)$, known as the *Kan loop group construction*, which models the loop construction for topological spaces. We have

$$\pi_n(X) \simeq \mathsf{H}_{n-1}(\mathsf{NG}(X))$$

which calculates the simplicial homotopy groups of X via the homology groups of the Moore complex of the Kan loop $\mathsf{G}(X)$. Note that the construction $\mathsf{N}_*(-)$ makes sense for *nonabelian* simplicial groups.

A.2. **Suggested readings.** [Kan58] (original), [GJ09, §V.7] (more recent).

Appendix B. Monoidal Dold–Kan correspondence

B.1. Consider the symmetric monoidal structure on Ab given by tensor products of abelian groups. It induces the following structures:

- A symmetric monoidal structure on $\mathsf{Ch}(\mathsf{Ab})_{\leq 0}$ given by graded tensor products.
- A symmetric monoidal structure on Ab_Δ given by pointwise tensor products.

The Dold–Kan correspondence does *not* commute with these structures strictly. Below is a list of what is known for the functor $\mathsf{N} : \mathsf{Ab}_\Delta \to \mathsf{Ch}(\mathsf{A})_{\geq 0}$.

(i) The functor N has a symmetric lax monoidal structure: for any $A, B \in \mathsf{Ab}_\Delta$, there is a morphism, called the Eilenberg-Zilber map,

$$\mathsf{EZ}_{A,B} : \mathsf{N}(A) \otimes \mathsf{N}(B) \to \mathsf{N}(A \otimes B)$$

compatible with the commutativity constraints.

(ii) The functor N has a oplax monoidal structure: for any $A, B \in \mathsf{Ab}_\Delta$, there is a morphism, called the Alexander-Whitney map

$$\mathsf{AW}_{A,B} : \mathsf{N}(A \otimes B) \to \mathsf{N}(A) \otimes \mathsf{N}(B)$$

that is *not* compatible with the commutativity constraints.

(iii) The composition $\mathsf{AW}_{A,B} \circ \mathsf{EZ}_{A,B}$ is equal to the identity morphism, while $\mathsf{EZ}_{A,B} \circ \mathsf{AW}_{A,B}$ is only a chain homotopy equivalence.

B.2. Note that (ii) implies the inverse functor DK has a natural lax monoidal structure. It follows that

$$\mathsf{Ch}(\mathsf{Ab}) \xrightarrow{\tau_{\geq 0}} \mathsf{Ch}(\mathsf{Ab})_{\geq 0} \to \mathsf{Ab}_\Delta \to \mathsf{Kan}$$

also has a natural lax monoidal structure. In particular, any dg-category $\mathbb{C}$ is also enriched over Kan. Let $\mathbb{C}'$ be the corresponding Kan-enriched category. We have a canonical equivalence (HA.1.3.1.17)

$$\mathfrak{N}(\mathbb{C}') \simeq \mathsf{N}_{\mathsf{dg}}(\mathbb{C}).$$

Exercise B.3. *Using the above equivalence to deduce [Lecture 17, Proposition 2.16].*

B.4. The failure of the Dold–Kan correspondence for being a *monoidal adjunction* makes it non-trivial to compare the theories of

- connective dg-algebras
- simplicial associative algebras.

Although it is still possible to find a Quillen equivalence between the model categories of the above objects, the story becomes much subtler when we consider connective *commutative* dg-algebras and simplicial commutative algebras. In fact, in positive characteristic, these two theories are not equivalent, and there is no reasonable model structure on the former. Moreover, the theory of simplicial commutative algebras of $\mathbb{F}_p$ is *not* the same as that of $\mathbb{E}_\infty$-$\mathbb{H}\mathbb{F}_p$-spectra. This leads to the bipartition between *derived* algebraic geometry and *spectral* algebraic geometry. For more information, see this mathoverflow question (in particular, see the answers by Lurie and May).

B.5. **Suggested readings.** references in this nLab page.

References

[GJ09] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[Kan58] Daniel M Kan. A combinatorial definition of homotopy groups. *Annals of Mathematics*, 67(2):282–312, 1958.

[Qui67] Daniel G Quillen. Homotopical algebra. *Lecture Notes in Mathematics*, 1967.