

LECTURE 2

Last time, we explained the following principle:

theory of ∞ -groupoids = homotopy theory of topological spaces.

theory of $(\infty, 1)$ -categories = homotopy theory of topological categories.

In this lecture, we give precise meanings to the RHS's.

1. WHAT IS A HOMOTOPY THEORY?

1.1. Recall the homotopy theory of topological spaces is encoded by the *homotopy category* \mathbf{hTop} , which can be defined in the following two equivalent ways:

- As the category obtained from \mathbf{Top} , the category of *all* spaces, by inverting *weak* homotopy equivalence.
- As the category of nice topological spaces and homotopy classes of continuous maps between such spaces.

1.2. This suggests a homotopy theory should be understood as follows:

- In a given category \mathcal{C} , such as \mathbf{Top} , we single out a collection W of morphisms, called **weak equivalences**, and study the category \mathbf{hC} obtained from \mathcal{C} by inverting the morphisms in W .
- We find a full subcategory $\mathcal{C}^\circ \subset \mathcal{C}$ of *nice* objects in \mathcal{C} , such that for $X, Y \in \mathcal{C}^\circ$, the set $\mathbf{Hom}_{\mathbf{hC}}(X, Y)$ can be calculated as the quotient of $\mathbf{Hom}_{\mathcal{C}}(X, Y) / \sim$. Here ' \sim ' stands for the equivalence relation defined by

$$(1.1) \quad \mathbf{Hom}_{\mathcal{C}}(CX, Y) \rightrightarrows \mathbf{Hom}_{\mathcal{C}}(X, Y),$$

where CX is a **cylinder** of X , behaving like $X \times [0, 1]$ when $X \in \mathbf{Top}$.

1.3. However, the passage from \mathbf{Top} to \mathbf{hTop} , or more generally from \mathcal{C} to \mathbf{hC} , loses a significant amount of *homotopy-invariant* information. For example, the notions of homotopy limits/colimits cannot be extracted from the category \mathbf{hTop} .

Definition 1.4. The **homotopy pushout** of a diagram $X \leftarrow Y \rightarrow Z$ in \mathbf{Top} is defined to be

$$X \mathop{\sqcup}_Y^{\mathbf{h}} Z := X \mathop{\sqcup}_{Y \times \{0\}} (Y \times [0, 1]) \mathop{\sqcup}_{Y \times \{1\}} Z.$$

The **homotopy pullback** of a diagram $X \rightarrow Y \leftarrow Z$ in \mathbf{Top} is defined to be

$$X \mathop{\times}_Y^{\mathbf{h}} Z := X \mathop{\times}_{Y^{(0)}} (Y^{[0,1]}) \mathop{\times}_{Y^{(1)}} Z.$$

Exercise 1.5. Check the above definitions are homotopy invariant, i.e., the homotopy types of the results only depend on the images of the diagrams under $\mathbf{h} : \mathbf{Top} \rightarrow \mathbf{hTop}$.

Exercise 1.6. Consider the following diagrams in \mathbf{hTop} . Prove:

- (1) The pushout of $* \leftarrow S^1 \rightarrow D^2$ in \mathbf{hTop} is equivalent to $*$, while the homotopy pushout is S^2 .
- (2) The pushout of $* \leftarrow S^1 \xrightarrow{2} S^1$ in \mathbf{hTop} does not exist, while the homotopy pushout is $\mathbb{R}P^2$.
- (3) The pullback of $* \rightarrow S^1 \leftarrow \mathbb{R}^1$ in \mathbf{hTop} is equivalent to \mathbb{R}^1 , while the homotopy pullback is \mathbb{Z} .
- (4) The pullback of $* \rightarrow \mathbb{C}P^\infty \leftarrow \mathbb{R}P^\infty$ in \mathbf{hTop} does not exist, while the homotopy pullback is S^1 .

Remark 1.7. Note that in (1)/(3), the homotopy pushout/pullback can also be calculated as the homotopy types of the usual pushout/pullback inside \mathbf{Top} . This is related to the facts that $S^1 \rightarrow D^2$ is a nice inclusion while $\mathbb{R}^1 \rightarrow S^1$ is a nice surjection.

2. MODEL CATEGORIES

2.1. In the 1960s, Quillen (see [Qui67]) realized that classical homotopy theory can be carried out in any category equipped with three classes of morphisms: weak equivalences, cofibrations and fibrations, as long as they satisfy a list of axioms motivated by the example of \mathbf{Top} . This motivated the definition of model categories.

Definition 2.2. A **weak factorization system** on a category \mathcal{C} is a pair (L, R) of classes of morphisms such that

- Every morphism can factor as $p \circ i$ such that $i \in L$ and $p \in R$.
- L is precisely the class of morphisms having the **left lifting property** against every morphism in R . In other words, $i \in L$ iff for any $p \in R$ and a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y, \end{array}$$

there exists a morphism $B \rightarrow X$ making the two triangles commute.

- R is precisely the class of morphisms having the **right lifting property** against every morphism in L .

Example 2.3. The category \mathbf{Set} has a weak factorization system $(\text{inj}, \text{surj})$ given by classes of injections and surjections.

Remark 2.4. The following is a typical way to construct weak factorization systems. Let \mathcal{C} be a category admits all small colimits, and I be a set of morphisms such that the sources of these morphisms are small in a suitable sense. Then there is a weak factorization system (L, R) such that

- R is the class of morphisms with right lifting property with respect to I ;
- L is the weakly saturated class¹ generated by I .

This is known as Quillen's small object argument. See *HTT.A.1.2*.

Definition 2.5. A **model structure** on a category \mathcal{C} is a choice of three classes of morphisms W , C and F , respectively called **weak equivalences**, **cofibrations** and **fibrations**, such that

¹Being weakly saturated means closed under pushouts, transfinite compositions and retracts. See *HTT.A.1.2.2*.

- W contains all isomorphisms and is closed under 2-out-of-3².
- $(C, F \cap W)$ and $(C \cap W, F)$ are weak factorization systems on \mathcal{C} .

We say \mathcal{C} is a model category if it is locally small³ and has small limits and colimits⁴, and is equipped with a model structure.

Definition 2.6. In a model category \mathcal{C} , morphisms in $C \cap W$ (resp. $F \cap W$) are called **acyclic cofibrations** (resp. **acyclic fibrations**).

Example 2.7. Quillen's classical model structure on \mathbf{Top} is given by:

- (W) A weak equivalence is a weak homotopy equivalence.
- (C) A cofibration is a retract of a relative cell complex.
- (F) A fibration is a Serre fibration.

Here the weak factorization system $(C, F \cap W)$ is obtained by applying the small object argument to $\{S^{n-1} \rightarrow D^n\}$, while $(C \cap W, F)$ is obtained by applying to $\{D^n \times \{0\} \rightarrow D^n \times [0, 1]\}$.

Example 2.8. Let \mathcal{A} be a nice⁵ abelian category, such as \mathbf{Mod}_R for a ring R . Let $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ be the category of (cochain) complexes in non-positive degrees⁶. Quillen's **projective model structure** on $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ is given by

- (W) A weak equivalence is a quasi-isomorphism, i.e., a map that induces isomorphisms between cohomologies.
- (C) A cofibration is a degreewise monomorphism with degreewise projective cokernel.
- (F) A fibration is a degreewise epimorphism.

Dually, there is an **injective model structure** on $\mathbf{Ch}^{\geq 0}(\mathcal{A})$.

Example 2.9. Let k be a field of characteristic zero. Let $\mathbf{dgcAlg}_k^{\leq 0}$ be the category of differential graded-commutative algebras over k in non-positive degrees. The **projective model structure** on $\mathbf{dgcAlg}_k^{\leq 0}$ is given by

- (W) A weak equivalence is a quasi-isomorphism.
- (F) A fibration is a degreewise epimorphism.
- (C) The class of cofibrations is determined by $W \cap F$.

Remark 2.10. In the above two examples, the story becomes more complicated but interesting when we consider all unbounded complexes.

Definition 2.11. Let \mathcal{C} be a model category, and X, Y be objects in \mathcal{C} .

- (1) A **cylinder object** for X is an object CX together with a factorization of the co-diagonal map $X \sqcup X \rightarrow X$ as

$$X \sqcup X \xrightarrow{i} CX \xrightarrow{p} X$$

such that i is a cofibration and p is a weak equivalence.

²For composable morphisms f, g , if two out of the three morphisms f, g, gf are in W , so is the third.

³This means the $\mathbf{Hom}_{\mathcal{C}}(-, -)$ are sets rather than proper classes.

⁴Quillen's original definition only requires *finite* limits/colimits.

⁵More precisely, \mathcal{A} admits small limits and colimits, and has enough projectives.

⁶We use cohomological convention.

- (2) A **path object** for Y is an object PY together with a factorization of the diagonal map $Y \rightarrow Y \times Y$ as

$$Y \xrightarrow{i} PY \xrightarrow{p} Y \times Y$$

such that i is a weak equivalence and p is a fibration.

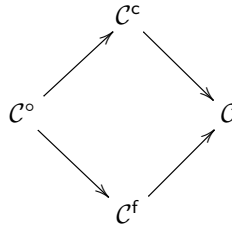
Remark 2.12. Note that the existence of cylinder objects and path objects follows from the factorization axiom in model structures.

Exercise 2.13. Consider the model categories in Example 2.8.

- Find path objects in the projective model category $\text{Ch}^{\leq 0}(\mathcal{A})$.
- Find cylinder objects in the injective model category $\text{Ch}^{\geq 0}(\mathcal{A})$.

Definition 2.14. Let \mathcal{C} be a model category. An object X is **fibrant** (resp. **cofibrant**) if $X \rightarrow *$ (resp. $\emptyset \rightarrow X$) is a fibration (resp. cofibration). Here $*$ and \emptyset stand for the final and initial objects in \mathcal{C} .

An object is **bifibrant** if it is both fibrant and cofibrant. Let



be the full subcategories of fibrant, cofibrant and bifibrant objects.

2.15. As we will soon see, bifibrant objects are the “nice” objects in the sense of Sect. 1.2.

Exercise 2.16. For any object x , there exist weak equivalences $x' \rightarrow x$ and $x \rightarrow x''$ such that x' is cofibrant and x'' is fibrant. Such object x' (resp. x'') is called a **cofibrant replacement** (resp. **fibrant replacement**) of x .

Exercise 2.17. What are cofibrant replacements in the projective model category $\text{Ch}^{\leq 0}(\mathcal{A})$? How about fibrant replacements in the injective model category $\text{Ch}^{\geq 0}(\mathcal{A})$?

2.18. In order to get a feeling about the axioms in the definition of model categories, the readers are encouraged to prove the following result on their own.

Proposition-Definition 2.19. Let \mathcal{C} be a model category, X be a cofibrant object and Y be a fibrant object. For morphisms $f, g : X \rightarrow Y$, the following conditions are equivalent:

- (1) For every/some cylinder object CX for X , there exists a commutative diagram

$$\begin{array}{ccc}
 X \sqcup X & \xrightarrow{(f,g)} & Y \\
 & \searrow & \nearrow \\
 & CX &
 \end{array}$$

(2) For every/some path object PY for Y , there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{(f,g)} & Y \times Y \\ & \searrow \text{dotted} & \nearrow \\ & PY & \end{array}$$

We say f and g are **homotopic**, or $f \sim g$, if they satisfy the above equivalent conditions. The relation \sim is an equivalence relation on the set $\text{Hom}_{\mathcal{C}}(X, Y)$.

Definition 2.20. Let \mathcal{C} be a model category. The **homotopy category** $\text{h}\mathcal{C}$ of \mathcal{C} is defined as follows:

- Objects are the bifibrant objects of \mathcal{C} ;
- Morphisms are homotopy classes of morphisms in \mathcal{C} .

2.21. Quillen proved the homotopy category is canonically equivalent to the localization of \mathcal{C} by W , i.e., the category $\mathcal{C}[W^{-1}]$ obtained from \mathcal{C} by inverting weak equivalences⁷. More precisely:

Theorem 2.22 (Quillen). *The following functors are equivalences*

$$\begin{array}{ccccc} & & \mathcal{C}^f[W^{-1}] & & \\ & \swarrow & & \searrow & \\ \mathcal{C}[W^{-1}] & & & & \mathcal{C}^o[W^{-1}] \longrightarrow \text{h}\mathcal{C} \\ & \swarrow & & \searrow & \\ & & \mathcal{C}^c[W^{-1}] & & \end{array}$$

Exercise 2.23. What does the theorem say when \mathcal{C} is the projective model category $\text{Ch}^{\leq 0}(\mathcal{A})$ or the injective model category $\text{Ch}^{\geq 0}(\mathcal{A})$?

Remark 2.24. Note that $\mathcal{C}[W^{-1}]$ depends only on (\mathcal{C}, W) , while $\text{h}\mathcal{C}$ depends on the model structure of \mathcal{C} , i.e., on the choices of fibrations and cofibrations. One may compare these auxiliary choices with local coordinates on a manifold: both provide powerful tools to do calculations, but more or less obstruct births of an intrinsic theory.

As will be explained in future lectures, this “intrinsic” theory of model categories is exactly the theory of $(\infty, 1)$ -categories.

3. TOPOLOGICAL CATEGORIES

3.1. Now let us try to develop the homotopy theory of topological categories via the formalism of model categories.

⁷This is the category characterized by the following universal property: knowing a functor $\mathcal{C}[W^{-1}] \rightarrow \mathcal{D}$ is equivalent to knowing a functor $\mathcal{C} \rightarrow \mathcal{D}$ that sends morphisms in W to isomorphisms.

3.2. There is only one reasonable choice for weak equivalences, defined as follows.

Definition 3.3. Let \mathcal{C} be a topological category. Its **homotopy category** $\pi_0\mathcal{C}$ is defined by

$$\text{Ob}(\pi_0\mathcal{C}) := \text{Ob}(\mathcal{C}), \quad \text{Hom}_{\pi_0\mathcal{C}}(x, y) := \pi_0\text{Hom}_{\mathcal{C}}(x, y).$$

Definition 3.4. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between topological categories. We say F is a **weak equivalence** if:

- It induces an equivalence $\pi_0 F: \pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$.
- It induces weak equivalences between the Hom-spaces, i.e. the continuous map $\text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$ is a weak equivalence for any $x, y \in \text{Ob}(\mathcal{C})$.

3.5. However, it is not an easy task to define fibrations and cofibrations between topological categories. Even if one finds such a definition, the obtained model structure will not be practical for calculation. Namely, unlike the case of topological spaces, many seemingly innocent topological spaces can fail to bifibrant:

Exercise 3.6. Consider the following commutative diagram of spaces:

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{h'} & S^1 \\ & \searrow h & \nearrow f \\ & & S^1 \times D^2, \end{array}$$

where $h(t_1, t_2) := (t_1 t_2, t_2)$ and $f(t_1, t_2) := t_1$. Let \mathcal{C} be the topological category with three objects x, y, z and

$$\text{Hom}_{\mathcal{C}}(x, y) := S^1, \quad \text{Hom}_{\mathcal{C}}(y, z) := S^1, \quad \text{Hom}_{\mathcal{C}}(x, z) := S^1 \times D^2$$

such that the composition map

$$\text{Hom}_{\mathcal{C}}(x, y) \times \text{Hom}_{\mathcal{C}}(y, z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$$

is given by h . Similarly, let \mathcal{D} be the topological category with three objects x', y', z' and

$$\text{Hom}_{\mathcal{D}}(x', y') := S^1, \quad \text{Hom}_{\mathcal{D}}(y', z') := S^1, \quad \text{Hom}_{\mathcal{D}}(x', z') := S^1$$

such that the composition law is given by h' . Let

$$F: \mathcal{C} \rightarrow \mathcal{D}, \quad x \mapsto x', \quad y \mapsto y', \quad z \mapsto z'$$

be the obvious functor induced by f . Prove:

- (1) The functor F is a weak equivalence.
- (2) There exists no functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G$ is homotopic to $\text{id}_{\mathcal{D}}$ in any reasonable sense.
- (3) Deduce that either \mathcal{D} is not cofibrant or \mathcal{C} is not fibrant (for any model structure on Cat_{Top} such that weak equivalences are given by Definition 3.4).

3.7. Another difficulty in developing the theory of $(\infty, 1)$ -categories via topological categories is to define the $(\infty, 1)$ -category \mathbf{Cat}_∞ of small $(\infty, 1)$ -categories. Namely, if $(\infty, 1)$ -categories are understood *only* via topological categories, we need to define a topological category $\mathbf{Cat}_{\text{Top}}^+$ of small topological categories such that

$$\pi_0 \text{Hom}_{\mathbf{Cat}_{\text{Top}}^+}(\mathcal{C}, \mathcal{D}) \simeq \text{Hom}_{\mathbf{hCat}_{\text{Top}}}(\mathcal{C}, \mathcal{D}).$$

The following exercises suggest this is not an easy task.

Exercise 3.8. For any abstract group G , let $\mathbb{B}G$ be the topological category with a single object $*$ such that $\text{Hom}_{\mathbb{B}G}(*, *) := G$. Use homotopy hypothesis to show

$$\text{Hom}_{\mathbf{Cat}_{\text{Top}}^+}(\mathbb{B}*, \mathbb{B}G)$$

is weakly homotopic to the Eilenberg–MacLane space $K(G, 1)$.

Challenge 3.9. Define a functor $\text{Grp} \rightarrow \text{Top}_*$ such that $G \in \text{Grp}$ is sent to a representative of $K(G, 1)$.

3.10. In future lectures, we will study and compare two more models of $(\infty, 1)$ -categories:

- (1) Joyal’s model structure on the category \mathbf{Set}_Δ of **simplicial sets**, whose bifibrant objects are **quasi-categories**.
- (2) Bergner’s model structure on the category \mathbf{Cat}_Δ of **simplicial categories**, whose bifibrant objects cannot be easily described.

We will show that the *category of small quasi-categories* $\mathbf{QCat} \subset \mathbf{Set}_\Delta$ has a natural simplicial enrichment, i.e. we have a *simplicial category of small quasi-categories*, which serves as a model of \mathbf{Cat}_∞ , the $(\infty, 1)$ -category of $(\infty, 1)$ -categories.

Remark 3.11. In fact, \mathbf{QCat} is Cartesian closed and therefore naturally enriched by itself. This reflects the idea that functors between $(\infty, 1)$ -categories form an $(\infty, 1)$ -category.

APPENDIX A. MORE HOMOTOPY (CO)LIMITS

Exercise A.1. Let \mathcal{C} be a model category. Define the notion of homotopy pushouts in \mathcal{C} . Prove that the homotopy pushout of $X \leftarrow Y \rightarrow Z$ can be calculated as pushout in \mathcal{C} in either of the following cases:

- The morphism $Y \rightarrow Z$ is a cofibration, and X, Y are cofibrant objects.
- The morphism $Y \rightarrow Z$ is a cofibration, and \mathcal{C} is **left proper**.

Exercise A.2. Show that homotopy pushouts/pullbacks are not functorial in \mathbf{hTop} . In other words, we do not have desired functors $\text{Span}(\mathbf{hTop}) \rightarrow \mathbf{hTop}$ or $\text{coSpan}(\mathbf{hTop}) \rightarrow \mathbf{hTop}$.

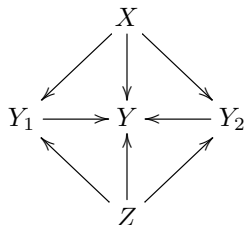
Exercise A.3. Let $X \xrightarrow{p} Y \xleftarrow{q} Z$ be a diagram in \mathbf{Top} and S be a testing object. Compare the data encoded in morphisms

$$S \rightarrow X \times_Y Z, S \rightarrow X \times_{\mathbf{h}Y}^{\mathbf{h}} Z, \mathbf{h}S \rightarrow \mathbf{h}X \times_{\mathbf{h}Y} \mathbf{h}Z.$$

A.4. Convince yourself that:

To capture all homotopy-invariant information, it is not enough to know two maps are homotopic, rather, we need to know how they are homotopic.

Exercise A.5. *Let*



be a commutative diagram in Top. Give a definition of the homotopy limit of this diagram, and study it as in Exercise A.3.

A.6. Convince yourself that:

To capture all homotopy-invariant information, it is not enough to record homotopies between maps, rather, we need to record all the higher homotopies.

A.7. **Suggested readings.** [BK72, Chapter XI-XII] (original), [Rie14, Chapter V-VI] (more recent).

REFERENCES

- [BK72] Aldridge Knight Bousfield and Daniel Marinus Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.
- [Qui67] Daniel G Quillen. Homotopical algebra. *Lecture Notes in Mathematics*, 1967.
- [Rie14] Emily Riehl. *Categorical Homotopy Theory*, volume 24. Cambridge University Press, 2014.