LECTURE 20

In this lecture, for a (small) ∞-category D, we explain a canonical equivalence between:

(i) the ∞-category of Cartesian fibrations over D;

(ii) the ∞-category of functors $\mathsf{D}^{\mathsf{op}} \to \mathsf{Cat}_\infty$.

The construction (i)⇒(ii) is called **straightening**, while (ii)⇒(i) is called **unstraightening**. For ordinary categories, the similar equivalence was proved by Grothendieck and known as the **Grothendieck construction**. For ∞-categories, people call it the **Grothendieck–Lurie construction**

1. Transport functors

1.1. In this section, we explain how to construct functors between fibers of a Cartesian fibration.

Let $p : \mathsf{C} \to [1]$ be a Cartesian fibration between ∞-categories. Consider the ∞-category of sections[1] of p:

$$\mathsf{Sect}_{/[1]}(\mathsf{C}) := \mathsf{Fun}([1], \mathsf{C}) \underset{\mathsf{Fun}([1],[1])}{\times} \{\mathsf{Id}\}.$$

Let

$$\mathsf{Sect}^{\mathsf{Cart}}_{/[1]}(\mathsf{C}) \subseteq \mathsf{Sect}_{/[1]}(\mathsf{C})$$

be the full sub-∞-category whose objects are given by p-Cartesian arrows in C.

Proposition 1.2. *Let $p : \mathsf{C} \to [1]$ be a Cartesian fibration between ∞-categories. Then the functor*

$$\mathsf{ev}_1 : \mathsf{Sect}^{\mathsf{Cart}}_{/[1]}(\mathsf{C}) \to \mathsf{C}_1$$

is an equivalence between ∞-categories.

Sketch. The functor is essentially surjective because there are enough p-Cartesian arrows. It remains to show it is fully faithful. For objects $x \xrightarrow{f} y$ and $x' \xrightarrow{f'} y'$ in $\mathsf{Sect}^{\mathsf{Cart}}_{/[1]}(\mathsf{C})$, we have

$$\mathsf{Maps}(f, f') \simeq \mathsf{Maps}_{\mathsf{C}_0}(x, x') \underset{\mathsf{Maps}_{\mathsf{C}}(x,y')}{\times} \mathsf{Maps}_{\mathsf{C}_1}(y, y').$$

Since f' is locally p-Cartesian, the functor $\mathsf{Maps}_{\mathsf{C}_0}(x, x') \to \mathsf{Maps}_{\mathsf{C}}(x, y')$ is an equivalence. It follows that the functor $\mathsf{Maps}(f, f') \to \mathsf{Maps}_{\mathsf{C}_1}(y, y')$ is an equivalence as desired. □

Construction 1.3. *Let $p : \mathsf{C} \to \mathsf{D}$ be a Cartesian fibration between ∞-categories. By the above proposition, for any morphism $f : x \to y$ in D, we have an equivalence*

$$\mathsf{Sect}^{\mathsf{Cart}}_{[1]}(\mathsf{C} \underset{\mathsf{D}}{\times} [1]) \xrightarrow{\simeq} \mathsf{C}_y,$$

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[1] There is no ambiguity for the fiber product below because $\mathsf{Fun}([1], \mathsf{C}) \to \mathsf{Fun}([1], [1])$ is a categorical fibration.

where $[1] \to \mathsf{D}$ *is the functor given by* f. *By choosing an inverse of the above equivalence, we obtain a functor*

$$\mathsf{C}_y \xrightarrow{\simeq} \mathsf{Sect}^{\mathsf{Cart}}_{[1]}(\mathsf{C} \underset{\mathsf{D}}{\times} [1]) \xrightarrow{\mathsf{ev}_0} \mathsf{C}_x.$$

We denote this functor by $f^\dagger : \mathsf{C}_y \to \mathsf{C}_x$, *and call it the* ***contravariant transport functor*** *along* f.

Dually, for a coCartesian fibration $p : \mathsf{C} \to \mathsf{D}$, *we define the* ***covariant transport functor*** $f_\dagger : \mathsf{C}_x \to \mathsf{C}_y$.

1.4. By construction, $f^\dagger$ is well-defined up to a constractible space of choices.

Exercise 1.5. *Let* $p : \mathsf{C} \to \mathsf{D}$ *be a Cartesian fibration between* ∞*-categories, and* $f, f' : x \to y$ *be morphisms in* D *that are homotopic to each other. Show that* $f^\dagger \simeq (f')^\dagger$.

Exercise 1.6. *Let* $p : \mathsf{C} \to \mathsf{D}$ *be a Cartesian fibration between* ∞*-categories, and* $x \xrightarrow{f} y \xrightarrow{g} z$ *be a chain in* D. *Show that* $f^\dagger \circ g^\dagger \simeq (g \circ f)^\dagger$.

Exercise 1.7. *Note that the construction* $f \mapsto f^\dagger$ *makes sense for any* locally *Cartesian fibration* $p : \mathcal{C} \to \mathcal{D}$ *between quasi-categories. Show that in this setting, we have a canonical natural transformation* $f^\dagger \circ g^\dagger \to (g \circ f)^\dagger$.

Exercise 1.8. *Let* $p : \mathsf{C} \to \mathsf{D}$ *be a functor that is both a Cartesian and coCartesian fibration. For any morphism* $f : x \to y$ *in* D, *show that* $f_\dagger$ *is left adjoint to* $f^\dagger$.

1.9. By the above exercises, for any Cartesian fibration $p : \mathsf{C} \to \mathsf{D}$ between ∞-categories, we have a canonial functor

$$\mathsf{D}^{\mathsf{op}} \to \mathsf{hCat}_\infty,\ x \mapsto \mathsf{C}_x,\ (x \xrightarrow{f} y) \mapsto f^\dagger.$$

Similarly, for any coCartesian fibration $p : \mathsf{C} \to \mathsf{D}$, we have a canonical functor $\mathsf{D} \to \mathsf{hCat}_\infty$.

2. Marked simplicial sets

2.1. Let $p : \mathsf{C} \to \mathsf{D}$ be a Cartesian fibration between ∞-categories, we want to lift the above functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{hCat}_\infty$ to a functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{Cat}_\infty$. Moreover, we want such a construction to be functorial in p. To precisely describe our goal, we introduce following ∞-categories.

Construction 2.2. *For* $\mathsf{D} \in \mathsf{Cat}_\infty$, *let*

$$\mathsf{Cart}_{/\mathsf{D}} \subseteq (\mathsf{Cat}_\infty)_{/\mathsf{D}}$$

be the sub-∞*-category such that:*

- (i) *Objects are Cartesian fibrations over* D;
- (ii) *Morphisms are functors defined over* D *that preserves Cartesian arrows.*

Remark 2.3. *More precisely, we realize* $(\mathsf{Cat}_\infty)_{/\mathsf{D}}$ *as a quasi-category and define* $\mathsf{Cart}_{/\mathsf{D}}$ *to be the maximal simplicial subset with the prescribed 1-skeleton.*

Warning 2.4. *The embedding* $\mathsf{Cart}_{/\mathsf{D}} \subseteq (\mathsf{Cat}_\infty)_{/\mathsf{D}}$ *is* not fully faithful. *Instead, it is* 1*-fully faithful, which means it induces fully faithful functors between the mapping* ∞*-groupoids.*

2.5. Our goal is to construct an equivalence

$$\mathsf{Cart}_{/\mathsf{D}} \simeq \mathsf{Fun}(\mathsf{D}^{\mathsf{op}}, \mathsf{Cat}_\infty). \tag{2.1}$$

Recall the only two known constructions of Cat_∞ are based on model categories. Namely, Cat_∞ is equivalent to the ∞-category underlying the model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, which can be constructed in the following equivalent ways:

- Invert categorical equivalences in Set_Δ in an ∞-categorical way.
- Enrich the full subcategory $\mathsf{QCat} \simeq (\mathsf{Set}_\Delta^{\mathsf{Joyal}})^\circ$ over simplicial sets, and take the simplicial nerve functor $\mathfrak{N}_\bullet$.

Hence the only reasonable strategy to construct the equivalence (2.1) is to realize it as a Quillen equivalence between model categories, whose underlying ∞-categories are $\mathsf{Cart}_{/\mathsf{D}}$ and $\mathsf{Fun}(\mathsf{D}^{\mathsf{op}}, \mathsf{Cat}_\infty)$.

2.6. Let us first seek a model for $\mathsf{Cart}_{/\mathsf{D}}$. Realizing D as a quasi-category $\mathcal{D}$, one may want to find a model structure on $(\mathsf{Set}_\Delta)_{/\mathcal{D}}$ that achieves the above goal. However, this hope is *not* reasonable because even if it has the correct bifibrant objects, i.e., Cartesian fibrations over $\mathcal{D}$, morphisms between these objects will be *all* functors defined over $\mathcal{D}$ rather than those perserving Cartesian arrows. This suggests we need to modify $(\mathsf{Set}_\Delta)_{/\mathsf{D}}$ such that morphisms automatically preserve some special arrows. This motivates the following definition.

Definition 2.7. *Let Set_Δ^+ be the ordinary category defined as follows.*

- *An object is a pair (X, E), where X is a simplicial set and $E \subseteq X_1$ is a subset of edges that contains all the degenerate edges.*
- *A morphism $(X, E) \to (X', E')$ is a morphism $X \to X'$ between simplicial sets that sends E into E'.*

We call a pair (X, E) as above a ***marked simplicial set****, and call edges in E the* ***marked edges****.*

Example 2.8. *For any simplicial set S,*

- *let $S^\sharp \in \mathsf{Set}_\Delta^+$ be the object such that all edges in S are marked;*
- *let $S^\flat \in \mathsf{Set}_\Delta^+$ be the object such that only degenerate edges in S are marked.*

Exercise 2.9. *Show that $\mathsf{Set}_\Delta \to \mathsf{Set}_\Delta^+$, $S \mapsto S^\sharp$ is right adjoint to the forgetful functor $\mathsf{oblv} : \mathsf{Set}_\Delta^+ \to \mathsf{Set}_\Delta$. Dually, $S \mapsto S^\flat$ is left adjoint to the forgetful functor.*

Definition 2.10. *Let S be a simplicial set. Define*

$$(\mathsf{Set}_\Delta^+)_{/S} := (\mathsf{Set}_\Delta^+)_{/S^\sharp},$$

and call it the category of ***marked simplicial sets over*** *S.*

Example 2.11. *Let $X \to S$ be a Cartesian fibration between simplicial sets and $E \subseteq X_1$ be the subset of all Cartesian edges. Then $X^\natural := (X, E)$ is an object in $(\mathsf{Set}_\Delta^+)_{/S}$.*

2.12. We are going to construct a model structure on $(\mathsf{Set}_\Delta^+)_{/S}$ such that

- cofibrations are monomorphisms;
- fibrant objects are $X^\natural$ for Cartesian fibrations $X \to S$.

As explained in [Lecture 5, §4], these requirements determine the model structure as long as we know how to calculate the mapping spaces in the desired ∞-category $(\mathsf{Set}_\Delta^+)_{/S}[W^{-1}]$.

Construction 2.13. *For $Y, Z \in \mathsf{Set}_\Delta^+$, consider the simplicial set $\mathsf{Hom}^\sharp(Y,Z)$ defined by the following universal property: for any $K \in \mathsf{Set}_\Delta$,*

$$\mathsf{Hom}_{\mathsf{Set}_\Delta}(K, \mathsf{Hom}^\sharp(Y,Z)) \simeq \mathsf{Hom}_{\mathsf{Set}_\Delta^+}(K^\sharp \times Y, Z).$$

For $Y, Z \in (\mathsf{Set}_\Delta^+)_{/S}$, let

$$\mathsf{Hom}_S^\sharp(Y,Z) \subseteq \mathsf{Hom}^\sharp(Y,Z)$$

be the full simplicial subset containing those vertices given by morphisms $Y \to Z$ defined over S.

Exercise 2.14. *Let $Y \in (\mathsf{Set}_\Delta^+)_{/S}$ and $Z \to S$ be a Cartesian fibration. Show that $\mathsf{Hom}_S^\sharp(Y, Z^\natural)$ is a Kan complex. Convince yourself that this Kan complex models the* space *of functors $Y \to Z$ defined over S that send marked edges in Y into Cartesian arrows in Z.*

Exercise 2.15. *What would happen if we consider $\mathsf{Hom}^\flat(Y, Z^\natural)$ instead?*

Definition 2.16. *A morphism $Y \to Y'$ in $(\mathsf{Set}_\Delta^+)_{/S}$ is called a **Cartesian equivalence** if for any Cartesian fibration $Z \to S$, the functor*

$$\mathsf{Hom}_S^\sharp(Y', Z^\natural) \to \mathsf{Hom}_S^\sharp(Y, Z^\natural)$$

is a weak equivalence between Kan complices.

Remark 2.17. *By HTT.3.1.3.3, we obtain the same notion if using $\mathsf{Hom}^\flat(Y, Z^\natural)$ instead.*

Theorem 2.18 (HTT.3.1.3.7, 3.1.4.1, 3.1.4.4)**.** *Let S be a simplicial set. There exists a left proper combinatorial model structure, called the **Cartesian model structure**, on $(\mathsf{Set}_\Delta^+)_{/S}$ such that*

(C) *cofibrations are monomorphism;*
(W) *weak equivalences are Cartesian equivalences.*

Moreover,

(1) *An object is fibrant iff it is of the form $X^\natural$ for some Cartesian fibration $X \to S$.*
(2) *The model structure is compatible with the $\mathsf{Set}_\Delta^{\mathsf{KQ}}$-enrichment*[2] *given by $\mathsf{Hom}^\sharp(-,-)$.*

2.19. We denote the above model category by $(\mathsf{Set}_\Delta^+)_{/S}^{\mathsf{Cart}}$, and the $\mathsf{Set}_\Delta^{\mathsf{KQ}}$-enriched model category by $(\mathbb{Set}_\Delta^+)_{/S}^{\mathsf{Cart}}$. Dually, we can define the coCartesian model structure on $(\mathsf{Set}_\Delta^+)_{/S}$.

2.20. When $S = \Delta^0$, the Cartesian model structure coincides with the coCartesian one, hence we denote the obtained model category just by Set_Δ^+. As one would expect, the homotopy theory of Set_Δ^+ is also equivalent to $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.

Theorem 2.21 (HTT.3.1.5.1)**.** *When $S = \Delta^0$, the adjunction*

$$(-)^\flat : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Set}_\Delta^+ : \mathsf{oblv}$$

is an Quillen equivalence.

Exercise 2.22. *Let $(X, E) \in \mathsf{Set}_\Delta^+$ be a marked simplicial set. Show that its image in $\mathsf{Set}_\Delta^+[W^{-1}] \simeq \mathsf{Cat}_\infty$ is equivalent to $X[E^{-1}]$.*

[2]See [Lecture 8, §2] for what this means.

Exercise 2.23. *Let $X \to \Delta^1$ be a Cartesian fibration of quasi-categories. Show that*

$$(\mathbb{S}\mathrm{et}_\Delta^+)_{/\Delta^1} \to \mathbb{S}\mathrm{et}_\Delta^+ \to \mathsf{Set}_\Delta^+[W^{-1}] \simeq \mathsf{Cat}_\infty$$

sends $X^\natural$ to an ∞-category equivalent to X_0. Moreover, the image of $X_1^\natural \to X^\natural$ models the contravariant transport functor $X_1 \to X_0$. Here $X_1^\natural$ has all the isomorphisms as marked.

Exercise 2.24. *Let D be an ∞-category realized as a quasi-category $\mathcal{D}$. Show that the underlying ∞-category of*

$$(\mathsf{Set}_\Delta^+)_{/\mathcal{D}}^{\mathsf{Cart}}$$

is equivalent to $\mathsf{Cart}_{/\mathsf{D}}$.

Exercise 2.25. *Let D be an ∞-category realized as a simplicial category $\mathbb{D}$. Consider the projective model category*[3] *of simplicial functors $\mathbb{D} \to \mathbb{S}\mathrm{et}_\Delta^+$:*

$$\mathsf{Fun}(\mathbb{D}, \mathbb{S}\mathrm{et}_\Delta^+)_{\mathsf{proj}}.$$

Show that the underlying ∞-category of it is equivalent to $\mathsf{Fun}(\mathsf{D}, \mathsf{Cat}_\infty)$.

3. Straightening and unstraightening

3.1. Let D be an ∞-category realized both as a quasi-category $\mathcal{D}$ and a simplicial category $\mathbb{D}$. By the above exercises, to construct the desired equivalence, we only need to construct a Quillen equivalence

$$(\mathsf{Set}_\Delta^+)_{/\mathcal{D}}^{\mathsf{Cart}} \rightleftarrows \mathsf{Fun}(\mathbb{D}^{\mathsf{op}}, \mathbb{S}\mathrm{et}_\Delta^+)_{\mathsf{proj}}.$$

By definition, we have a weak equivalence $\phi : \mathfrak{C}(\mathcal{D}) \to \mathbb{D}$ between simplicial categories.

Theorem 3.2 (HTT.3.2.0.1)**.** *Let $\mathcal{D}$ be a simplicial set and $\mathbb{D}$ be a simplicial category. For any functor $\phi : \mathfrak{C}(\mathcal{D}) \to \mathbb{D}$, there is a canonical Quillen adjunction*

$$\mathsf{St}_\phi^+ : (\mathsf{Set}_\Delta^+)_{/\mathcal{D}}^{\mathsf{Cart}} \rightleftarrows \mathsf{Fun}(\mathbb{D}^{\mathsf{op}}, \mathbb{S}\mathrm{et}_\Delta^+)_{\mathsf{proj}} : \mathsf{Un}_\phi^+,$$

which is a Quillen equivalence if ϕ is a weak equivalence.

Corollary 3.3. *Let D be any ∞-category. There is a canonical equivalence*

$$\mathsf{Cart}_{/\mathsf{D}} \simeq \mathsf{Fun}(\mathsf{D}^{\mathsf{op}}, \mathsf{Cat}_\infty).$$

3.4. Let us explain the main ideas in the construction of the functor St_ϕ^+. Let us first treat the case when $\mathbb{D} = \mathfrak{C}(\mathcal{D})$.

For a marked simplicial set $X \in (\mathsf{Set}_\Delta^+)_{/\mathcal{D}}$ over $\mathcal{D}$, we wish to construct a simplicial functor

$$F : \mathfrak{C}(\mathcal{D})^{\mathsf{op}} \to \mathbb{S}\mathrm{et}_\Delta^+$$

such that $F(d)$ is weak equivalent to X_d for any object d in $\mathcal{D}$ (viewed as an object in $\mathfrak{C}(\mathcal{D})$). The naive construction $d \mapsto X_d$ would not work because it is impossible to work out the functorialities. Instead, Exercise 2.23, suggests that

$F(d)$ should encode the information of X over all *possible arrows $d \to d'$.*

[3] See [Lecture 8, §2] for what this means.

For instance, we want any Cartesian arrow in X lying over $d \to d'$ to be a marked edge in $F(d)$.

Suppose we find a way to achieve the above purpose, we would obtain a morphism $F(d') \to F(d)$ which models the contravariant transport functor along $d \to d'$. However, to get the homotopy coherent composition laws, we realize that

> *$F(d)$ should encode the information of X over* all *possible chains $d \to d' \to d'' \to \cdots$*

In other words, we want vertices in $F(d)$ to be chains $x_0 \to x_1 \to \cdots x_n$ with $x_0 \in X_d$, such that refinements of chains can be encoded as edges in $F(d)$. The following construction provides a way to realize this idea.

Construction 3.5. *Let $p : X \to \mathcal{D}$ be any simplicial set over $\mathcal{D}$. Define $\mathbb{D}_X$ to be the following pushout in Cat_Δ:*

$$\begin{array}{ccc}
\mathfrak{C}(X) & \longrightarrow & \mathfrak{C}(X^{\triangleright}) \\
\downarrow & & \vdots \\
\mathfrak{C}(D) & & \vdots \\
\downarrow{\scriptstyle \phi} & & \downarrow \\
\mathbb{D} & \cdots\cdots\to & \mathbb{D}_X.
\end{array}$$

Consider the functor

$$\mathsf{St}_\phi X : \mathbb{D}^{\mathsf{op}} \to \mathbb{Set}_\Delta,\ d \mapsto \mathsf{Hom}_{\mathbb{D}_X}(d, *).$$

We call it the ***straightening*** *of X respect to ϕ.*

Construction 3.6. *Let $p : X \to \mathcal{D}$ be a marked simplicial set over $\mathcal{D}$. For a marked edge $x \to x'$ over $d \to d'$, consider the 2-simplex $\Delta^2 \to X^{\triangleright}$, $0 \mapsto x$, $1 \mapsto x'$, $2 \mapsto *$. The functor*

$$\mathfrak{C}(\Delta^2) \to \mathfrak{C}(X^{\triangleright}) \to \mathfrak{C}(\mathbb{D}_X)$$

induces a morphism between simplicial sets

$$\Delta^1 \simeq \mathsf{Hom}_{\mathfrak{C}(\Delta^2)}(0,2) \to \mathsf{Hom}_{\mathfrak{C}(\mathbb{D}_X)}(d, *) =: \mathsf{St}_\phi X(d).$$

We declare any edge in $\mathsf{St}_\phi X(d)$ obtained as above as special, and define the marked simplcial set

$$\mathsf{St}^+_\phi X(d) := (\mathsf{St}_\phi X(d), E_d)$$

such that $\{E_d\}$ is the smallest collection of sets containing all special edges and depend functorially on d. The obtained functor

$$\mathsf{St}^+_\phi X : \mathbb{D}^{\mathsf{op}} \to \mathbb{Set}^+_\Delta$$

is compatible with simplicial enrichements and depends functorially in X. This gives the desired functor

$$\mathsf{St}^+_\phi : (\mathsf{Set}^+_\Delta)^{\mathsf{Cart}}_{/\mathcal{D}} \to \mathsf{Fun}(\mathbb{D}^{\mathsf{op}}, \mathbb{Set}^+_\Delta)_{\mathsf{proj}}$$

4. Base-change of Cartesian fibrations

Proposition 4.1 (HTT.3.2.1.4)**.** *For $i = 0, 1$, let $\mathcal{D}_i$ be a simplicial set and $\mathbb{D}_i$ be a simplicial category. Suppose we have the following commutative diagram in Cat_Δ:*

$$\begin{tikzcd} \mathfrak{C}(\mathcal{D}_1) \arrow[r, "\phi_1"] \arrow[d, "\mathfrak{C}(F)"] & \mathbb{D}_1 \arrow[d, "\mathbb{F}"] \\ \mathfrak{C}(\mathcal{D}_2) \arrow[r, "\phi_2"] & \mathbb{D}_2. \end{tikzcd}$$

Then we have the following commutative diagram of left Quillen functors

$$\begin{tikzcd} (\mathsf{Set}_\Delta^+)^{\mathsf{Cart}}_{/\mathcal{D}_1} \arrow[r, "\mathsf{St}^+_{\phi_1}"] \arrow[d, "F\circ-"] & \mathsf{Fun}(\mathbb{D}_1^{\mathsf{op}}, \mathbb{Set}_\Delta^+)_{\mathsf{proj}} \arrow[d, "\mathsf{LKE}_{\mathbb{F}^{\mathsf{op}}}"] \\ (\mathsf{Set}_\Delta^+)^{\mathsf{Cart}}_{/\mathcal{D}_2} \arrow[r, "\mathsf{St}^+_{\phi_2}"] & \mathsf{Fun}(\mathbb{D}_2^{\mathsf{op}}, \mathbb{Set}_\Delta^+)_{\mathsf{proj}} \end{tikzcd}$$

and similarly for their right adjoints:

$$\begin{tikzcd} (\mathsf{Set}_\Delta^+)^{\mathsf{Cart}}_{/\mathcal{D}_1} & \mathsf{Fun}(\mathbb{D}_1^{\mathsf{op}}, \mathbb{Set}_\Delta^+)_{\mathsf{proj}} \arrow[l, "\mathsf{Un}^+_{\phi_1}"'] \\ (\mathsf{Set}_\Delta^+)^{\mathsf{Cart}}_{/\mathcal{D}_2} \arrow[u, "-\times_{\mathcal{D}_2}\mathcal{D}_1"] & \mathsf{Fun}(\mathbb{D}_2^{\mathsf{op}}, \mathbb{Set}_\Delta^+)_{\mathsf{proj}}. \arrow[l, "\mathsf{Un}^+_{\phi_2}"'] \arrow[u, "-\circ\mathbb{F}^{\mathsf{op}}"'] \end{tikzcd}$$

Corollary 4.2. *Let $F : \mathsf{D}_1 \to \mathsf{D}_2$ be a functor between ∞-categories. There is a canonical commutative diagram*

$$\begin{tikzcd} \mathsf{Cart}_{/\mathsf{D}_1} & \mathsf{Fun}(\mathsf{D}_1^{\mathsf{op}}, \mathsf{Cat}_\infty) \arrow[l, "\simeq"'] \\ \mathsf{Cart}_{/\mathsf{D}_2} \arrow[u, "-\times_{\mathsf{D}_2}\mathsf{D}_1"] & \mathsf{Fun}(\mathsf{D}_2^{\mathsf{op}}, \mathsf{Cat}_\infty). \arrow[l, "\simeq"'] \arrow[u, "-\circ\mathsf{F}^{\mathsf{op}}"'] \end{tikzcd}$$

5. Applications

5.1. The theory of Cartesian fibrations, equipped with the Grothendieck–Lurie construction, is extremely powerful. Below are some examples.

Example 5.2. *For any ∞-category D, the Cartesian fibration*

$$\mathsf{ev}_0 : \mathsf{Fun}(\Delta^1, \mathsf{D}) \to \mathsf{D}$$

corresponds to a functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{Cat}_\infty$, whose value at each $d \in \mathsf{D}$ is equivalent to $\mathsf{D}_{d/}$. This provides the functoriality for the (co)slice construction.

Example 5.3. *For any functor $F : \mathsf{C} \to \mathsf{D}$ between ∞-categories. One can show*

$$\mathsf{ev}_0 : \mathsf{Fun}(\Delta^1, \mathsf{D}) \underset{\mathsf{Fun}(\{1\},\mathsf{D})}{\times} \mathsf{C} \to \mathsf{D},$$

is a Cartesian fibration, which gives a functor $\mathsf{D}^{\mathsf{op}} \to \mathsf{Cat}_\infty$ whose value at $d \in \mathsf{D}$ is equivalent to $\mathsf{C} \times_{\mathsf{D}} \mathsf{D}_{d/}$. This fibration is used in the proof of Quillen's Theorem A.

Exercise 5.4. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be any categorical fibration between quasi-categories. Show that the morphism*

$$\mathcal{C}^\flat \to (\mathsf{Fun}(\Delta^1, \mathcal{D}) \underset{\mathcal{D}}{\times} \mathcal{C})^\natural$$

induced by $\mathcal{D} \to \mathsf{Fun}(\Delta^1, \mathcal{D})$, $d \mapsto \mathsf{id}_d$, *is a fibrant replacement in* $(\mathsf{Set}^+_\Delta)^{\mathsf{Cart}}_{/\mathcal{D}}$.

Example 5.5. *Let* C *be an* ∞*-category. The left fibration*

$$\mathsf{TwArr}(\mathsf{C}) \to \mathsf{C}^{\mathsf{op}} \times \mathsf{C}$$

corresponds to a functor $\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Cat}_\infty$ *that sends* (x, y) *to an* ∞*-groupoid equivalent to* $\mathsf{Maps}_{\mathsf{C}}(x, y)$.

Exercise 5.6. *Let* $F : \mathsf{D} \to \mathsf{Pr}^{\mathsf{L}}$ *be a functor. The composition*

$$\mathsf{D} \to \mathsf{Pr}^{\mathsf{L}} \to \widehat{\mathsf{Cat}}_\infty$$

corresponds to a (not necessarilly small) coCartesian fibration $\mathsf{E} \to \mathsf{D}$. *Show that this is also a Cartesian fibration, and the corresponding functor* $\mathsf{D}^{\mathsf{op}} \to \widehat{\mathsf{Cat}}_\infty$ *factors through* Pr^{R}.

5.7. By the Grothendieck–Lurie construction, for any simplicial set S, a diagram $u : S^{\mathsf{op}} \to \mathsf{Cat}_\infty$ corresponds to a Cartesian fibration $X \to S$ between simplicial sets as follows:

- Use the equivalence $\mathsf{Cat}_\infty \simeq \mathfrak{N}((\mathbb{S}\mathrm{et}^+_\Delta)^\circ)$ to translate u to a functor
$$\mathfrak{C}(S^{\mathsf{op}}) \to (\mathbb{S}\mathrm{et}^+_\Delta)^\circ$$
- Find a fibrant replacement and apply the unstraightening functor to obtain a fibrant object in $\mathsf{Cart}_{/S}$.
- Let $X \to S$ be the Cartesian fibration corresponding to this fibrant object.

Proposition 5.8 (HTT.3.3.3)**.** *Let* $u : S^{\mathsf{op}} \to \mathsf{Cat}_\infty$ *be a diagram of* ∞*-categories corresponding to a Cartesian fibration* $p : X \to S$. *Then the limit of* u *can be identified with the quasi-category of Cartesian sections of* p*:*

$$\lim u \simeq \mathsf{Sect}^{\mathsf{Cart}}_{/S}(X)$$

Proposition 5.9 (HTT.3.3.4)**.** *Let* $u : S \to \mathsf{Cat}_\infty$ *be a diagram of* ∞*-categories corresponding to a coCartesian fibration* $p : X \to S$. *Then the colimit of* u *can be modelled by the marked simplicial set* $X^\natural$, *where marked edges are* p*-coCartesian ones:*

$$\mathsf{colim}\, u \simeq X^\natural.$$

Appendix A. Dual and conjugate fibrations

A.1. Let $F : S \to \mathsf{Cat}_\infty$ be a diagram of ∞-categories. We can construct the following fibrations:

(i) The Cartesian fibration over S^{op} corresponding to F;
(ii) The coCartesian fibration over S corresponding to F;
(iii) The Cartesian fibration over S^{op} corresponding to $\mathsf{op} \circ F$;
(iv) The coCartesian fibration over S corresponding to $\mathsf{op} \circ F$.

Exercise A.2. *Describe how to construct one of the above fibrations* directly *from another one.*

A.3. **Suggested readings.** Ker.04C0.