LECTURE 21

In this lecture, we introduce ∞-operads and monoidal ∞-categories.

1. Symmetric monoidal ∞-category

1.1. Recall a **monoidal category** is an (ordinary) category C equipped with the following structure:

- A binary functor $-\otimes- : \mathsf{C}\times\mathsf{C}\to\mathsf{C}$, called the **monoidal product**;
- An object $\mathbb{1}\in\mathsf{C}$, called the **monoidal unit**;
- An invertible natural transformation of the form $X\otimes(Y\otimes Z)\simeq(X\otimes Y)\otimes Z$, called the **asssociator**;
- Invertible natural transformations of the form $\mathbb{1}\otimes X\simeq X$, $X\otimes\mathbb{1}\simeq X$, called the **left and right unitors**

such that certain diagrams, including the pentagon diagram, commute.

Informally, we can say a monoidal category is a category equpped with a multiplication which is unital and associative *up to coherent homotopy*. Here the coherence data are finite because ordinary categories form a 2-category. In fact, the definition of monoidal categories only invokes n-ary operators $\mathsf{C}^{\times n}\to\mathsf{C}$ for $0\le n\le 4$.

1.2. Also recall a **symmetric monoidal category** is a monoidal category C equipped with:

- An invertible natural transformation of the form $X\otimes Y\simeq Y\otimes X$, called the **swap natural transformation**

such that certain diagrams, including the inverse law, commute.

1.3. We would like to generalize the above to a notion of *(symmetric) monoidal ∞-categories*. Now the coherence data should invoke n-ary operators for all $n\ge 0$.

Instead of writing down such coherence data, we would like to encode them as a functor $F : J\to\mathsf{Cat}_\infty$, a.k.a. a homotopy coherent diagram of ∞-categories, where J is a clever-designed simplicial set such that

- For $n\ge 0$, there is a vertex $\langle n\rangle$ which is sent by F to the n-th power $\mathsf{C}^{\times n}$;
- There is an edge $\langle 2\rangle\to\langle 1\rangle$ encoding the monoidal product;
- There is an edge $\langle 0\rangle\to\langle 1\rangle$ encoding the monoidal unit;
- The coherence data are encoded by cells in J.

Note that the first point is actually a *structure rather than property*: we need to provide projections $p_k : F(\langle n\rangle)\to F(\langle 1\rangle)$ for each $1\le k\le n$ that exhibit $F(\langle n\rangle)$ as the n-th power of $F(\langle 1\rangle)$. Hence we should have:

- There is an edge $\langle n\rangle\to\langle 1\rangle$ encoding the projection functor $\mathsf{pr}_i : \mathsf{C}^{\times n}\to\mathsf{C}$ for each $i\in\{1,\cdots,n\}$.

Also, in the *symmetric* monoidal setting, to encode the swap natural transformations, we need to be able to swap the factors in $\mathsf{C}^{\times n}$. Hence we should have:

- For $n\ge 0$, the symmetric group Σ_n should act on $\langle n\rangle$.

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In fact, the above two requirements can be combined together as:

- For each *injective* map $\phi : \{1, \cdots, m\} \to \{1, \cdots, n\}$, there is an edge from $\langle n \rangle \to \langle m \rangle$ encoding the projection functor $\mathsf{pr}_\phi : \mathsf{C}^{\times n} \to \mathsf{C}^{\times m}$.

These edges in J will be called the **inert morphisms**. In contrast, edges encoding data such as the product and the unit are called the **active morphisms**.

Our intuition says:

- In the symmetric monoidal setting, there is a unique **active** morphism $\langle n \rangle \to \langle 1 \rangle$.
- In the monoidal setting, active morphisms $\langle n \rangle \to \langle 1 \rangle$ should be acted freely and transitively by Σ_n.

Exercise 1.4. *Show that injective maps $\phi : \{1, \cdots, m\} \to \{1, \cdots, n\}$ are in natural bijection with maps between* marked sets

$$\alpha : \{*, 1, 2, \cdots, n\} \to \{*, 1, 2, \cdots, m\}$$

such that $\alpha^{-1}(\{j\})$ is a singleton for any $1 \le j \le m$.

1.5. The above discussion motivates the following definition by Segal.

Definition 1.6. *Let Fin_* be the* minimal model *for the ordinary category of marked finite sets. For each $n \ge 0$, let $\langle n \rangle \in \mathsf{Fin}_*$ be the object*

$$\langle n \rangle := \{*\} \sqcup \langle n \rangle^\circ := \{*, 1, 2, \cdots, n\}.$$

Let $\alpha : \langle n \rangle \to \langle m \rangle$ be a morphism in Fin_*, we say

(1) *The morphism α is* ***inert*** *if $\alpha^{-1}(\{j\})$ is a singleton for any* non-marked *element $j \in \langle m \rangle^\circ$.*
(2) *The morphism α is* ***active*** *if $\alpha^{-1}(\{*\}) = \{*\}$.*

We also write

$$\mathsf{Comm}^\otimes := \mathsf{Fin}_*.$$

Exercise 1.7. *A morphism in $\mathsf{Comm}^\otimes$ is an isomorphism iff it is both inert and active.*

Exercise 1.8. *Any morphism $\langle n \rangle \to \langle m \rangle$ can be written as $\langle n \rangle \xrightarrow{\beta} \langle l \rangle \xrightarrow{\gamma} \langle m \rangle$ with β inert and γ active. Moreover, such expression is unique up to unique isomorphism.*

Definition 1.9. *A* ***symmetric monoidal ∞-category*** *is a functor*

$$F : \mathsf{Comm}^\otimes \to \mathsf{Cat}_\infty$$

that sends the inert morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle$, $i \in \langle n \rangle^\circ$ to functors

$$F(\rho^i) : F(\langle n \rangle) \to F(\langle 1 \rangle)$$

that exhibit $F(\langle n \rangle)$ as $F(\langle 1 \rangle)^{\times n}$. We call $F(\langle 1 \rangle)$ its ***underlying ∞-category****.*

Exercise 1.10. *If $F(\langle 1 \rangle)$ is ordinary, then F determines a symmetric monoidal category.*

1.11. The non-symmetric case can be encoded by the following variant of $\mathsf{Comm}^{\otimes}$.

Definition 1.12. *Let* $\mathsf{Assoc}^{\otimes}$ *be the ordinary category defined as follows:*

- *An object in* $\mathsf{Assoc}^{\otimes}$ *is a marked finite set* $\langle n\rangle$ *with* $n\geq 0$.
- *A morphism from* $\langle n\rangle$ *to* $\langle m\rangle$ *is a pair* $(\alpha,\leq_j)$ *where*
  - α *is a map* $\langle n\rangle\to\langle m\rangle$ *between marked sets*
  - *For each* $j\in\langle m\rangle^{\circ}$, $\leq_j$ *is a linear ordering on* $\alpha^{-1}(j)$.

For a morphism $(\alpha,\leq_j)$, *we say it is inert (resp. active) if* α *is so.*

Definition 1.13. *A* ***monoidal*** ∞***-category*** *is a functor*

$$F:\mathsf{Assoc}^{\otimes}\to\mathsf{Cat}_{\infty}$$

that sends the inert morphisms $\rho^i:\langle n\rangle\to\langle 1\rangle$, $i\in\langle n\rangle^{\circ}$ *to functors*

$$F(\rho^i):F(\langle n\rangle)\to F(\langle 1\rangle)$$

that exhibit $F(\langle n\rangle)$ *as* $F(\langle 1\rangle)^{\times n}$. *We call* $F(\langle 1\rangle)$ *its* ***underlying*** ∞***-category***.

Exercise 1.14. *Show that Exercise 1.7-1.10 remain valid for* $\mathsf{Assoc}^{\otimes}$ *instead of* Fin_*.

Exercise 1.15. *Show that for both* $\mathsf{Comm}^{\otimes}$ *and* $\mathsf{Assoc}^{\otimes}$, *the group of automorphisms on* $\langle n\rangle$ *can be identified with* Σ_n.

1.16. There is a functor $\mathsf{Assoc}^{\otimes}\to\mathsf{Comm}^{\otimes}$ that sends a symmetric monoidal ∞-category to a monoidal one via restriction.

Warning 1.17. *Although* $\mathsf{Assoc}^{\otimes}\to\mathsf{Comm}^{\otimes}$ *behaves like a quotient functor (it is surjective on both objects and morphisms), one should* not *think a symmetric monoidal* ∞*-category as a monoidal one satisfying certain* properties. *Rather, even for a functor* $\mathsf{Assoc}^{\otimes}\to\mathsf{D}$ *into a* $(2,1)$*-category* D, *such as that of ordinary categories, a* weak *factorization through* $\mathsf{Comm}^{\otimes}$ *is a* structure rather than property.

Remark 1.18. *In fact, we can encode non-symmetric monoidal* ∞*-categories in a more efficient way, where the role of* $\mathsf{Assoc}^{\otimes}$ *is replaced by* Δ^{op}. *This is known as the* simplicial model for monoids. *Roughly speaking, this amounts to cancel out the* Σ_n*-action* $\mathsf{Hom}(\langle n\rangle,\langle 1\rangle)$ *from the data encoded by a functor* $\mathsf{Assoc}^{\otimes}\to\mathsf{Cat}_{\infty}$.

Comparing with $\mathsf{Assoc}^{\otimes}$, *this simplicial model has several advantages. For instance, one can define* $\mathbb{A}_n$*-algebras using the truncation* $(\Delta_{\leq n})^{\mathsf{op}}$. *However, it is hard to* directly *compare monoidal structures encoded by the simplicial model with the symmetric monoidal ones because the* Σ_n*-action is hidden.*

See HA.4.1 and 4.2.2 for more details.

2. ∞-OPERADS

Exercise 2.1. *Show that for both* $\mathsf{Comm}^{\otimes}$ *and* $\mathsf{Assoc}^{\otimes}$, *the following data are equivalent:*

- *An* active *morphism* $\alpha:\langle n\rangle\to\langle m\rangle$;
- *A decomposition* $\langle n\rangle^{\circ}\simeq\bigsqcup_{j\in\langle m\rangle^{\circ}}\langle n_j\rangle^{\circ}$ *and* active *morphisms* $\alpha_j:\langle n_j\rangle\to\langle 1\rangle$ *labelled by* $j\in\langle m\rangle^{\circ}$.

Hint: for each α, *consider the factorization of* $\rho_j\circ\alpha$ *in Exercise 1.8.*

2.2. In fact, the categories $\mathsf{Comm}^{\otimes}$ and $\mathsf{Assoc}^{\otimes}$ can be recovered from the following data:

(i) For each $n \geq 0$, the set $\mathsf{O}(n)$ of active morphisms $\langle n\rangle \to \langle 1\rangle$;
(ii) The action of Σ_n on $\mathsf{O}(n)$;
(iii) For $m \geq 0$ and n_j labelled by $j \in \langle m\rangle^{\circ}$, a map

$$\mathsf{O}(m) \times (\prod_j \mathsf{O}(n_j)) \to \mathsf{O}(n_1 + \cdots + n_m)$$

given by the above exercise.

In the classical literatures, a collection of such data satisfying certain compatibilities is called a *(symmetric) operad.* Here $\mathsf{O}(n)$ is the set of *n-ary operators* and (iii) are called *composition maps.*

2.3. One can also consider *operads with enriched structures.* For instance, we have the notions of *simplicial operads, topological operads, k-linear operads, dg-operads*, using enrichment over the *ordinary* symmetric monoidal categories Set_{Δ}, Top, $\mathsf{Mod}_k^{\heartsuit}$, $\mathsf{Ch}(\mathsf{Ab})$...

We would like to have a notion of operads *weakly* enriched over the ∞-category Spc. In other words, each $\mathsf{O}(n)$ should be a space, and the composition maps are associative up to coherent homotopies.

Following the general philosephy in this course, it is better to *define* an ∞-operad as an ∞-category satisfying certain conditions, such as $\mathsf{Comm}^{\otimes}$ and $\mathsf{Assoc}^{\otimes}$, rather than listing all these coherence data.

2.4. Before giving the definition of ∞-operads, let us conduct one more generalization.

Let A be a symmetric monoidal ∞-category and M be an ∞-category. We want to define an A-action on M via a functor $F : \mathsf{CMod}^{\otimes} \to \mathsf{Cat}_{\infty}$. The index category $\mathsf{CMod}^{\otimes}$ should contain $\mathsf{Comm}^{\otimes}$ as a full subcategory, and the remaining part should encode the A-module structure on M. Hence we should at least have:

- objects $(\mathfrak{a}, \cdots, \mathfrak{a})$ which comes from $\langle n\rangle \in \mathsf{Assoc}^{\otimes}$ and should be sent to $\mathsf{A}^{\times n}$ by F;
- objects $(\mathfrak{a}, \cdots, \mathfrak{a}, \mathfrak{m})$ which should be sent to $\mathsf{A}^{\times n} \times \mathsf{M}$ by F;
- a unique active morphism $(\mathfrak{a}, \cdots, \mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ encoding the action functor $\mathsf{A}^{\times n} \times \mathsf{M} \to \mathsf{M}$.

Note that we should also have objects of the form $(\mathfrak{m}, \mathfrak{a}, \cdots, \mathfrak{a})$ and more generally $(\mathfrak{a}, \cdots, \mathfrak{a}, \mathfrak{m}, \mathfrak{a}, \cdots, \mathfrak{a})$. For technical reasons, it is more convenient to even allow sequences with multiple $\mathfrak{m}$-terms. Of course, we will not allow any *active* morphism from $(\mathfrak{m}, \mathfrak{m})$ to $\mathfrak{m}$.

Definition 2.5. *Let $\mathsf{CMod}^{\otimes}$ be the category defined as folllows:*

- *An object is a pair $(\langle n\rangle, c)$, where $\langle n\rangle \in \mathsf{Fin}_*$ and $c : \langle n\rangle^{\circ} \to \{\mathfrak{a}, \mathfrak{m}\}$ is a map.*
- *A morphism from $(\langle n\rangle, c)$ to $(\langle m\rangle, d)$ is a morphism $\alpha : \langle n\rangle \to \langle m\rangle$ in $\mathsf{Comm}^{\otimes}$ such that*
  - *If $c(i) = \mathfrak{m}$, then either $\alpha(i) = *$ or $d(\alpha(i)) = \mathfrak{m}$.*
  - *If $d(j) = \mathfrak{m}$, then there is a **unique** $i \in \alpha^{-1}(\{j\})$ with $c(i) = \mathfrak{m}$.*

For such a morphism, we say it is inert (resp. active) if the underlying morphism in $\mathsf{Comm}^{\otimes}$ is so.

2.6. Similarly, we can define a category $\mathsf{LMod}^{\otimes}$ encoding monoids and their *left* modules.

Definition 2.7. *Let $\mathsf{LMod}^{\otimes}$ be the category defined as folllows:*

- *An object is a pair $(\langle n\rangle, c)$, where $\langle n\rangle \in \mathsf{Fin}_*$ and $c : \langle n\rangle^{\circ} \to \{\mathfrak{a}, \mathfrak{m}\}$ is a map.*
- *A morphism from $(\langle n\rangle, c)$ to $(\langle m\rangle, d)$ is a morphism $(\alpha, \leq_j) : \langle n\rangle \to \langle m\rangle$ in $\mathsf{Assoc}^{\otimes}$ such that*
  - *If $c(i) = \mathfrak{m}$, then either $\alpha(i) = *$ or $d(\alpha(i)) = \mathfrak{m}$.*
  - *If $d(j) = \mathfrak{m}$, then there is a **unique** $i \in \alpha^{-1}(\{j\})$ with $c(i) = \mathfrak{m}$, which is also the maximal element for the ordering $\leq_j$.*

For such a morphism, we say it is inert (resp. active) if the underlying morphism in $\mathsf{Assoc}^{\otimes}$ is so.

Dually, we define $\mathsf{RMod}^{\otimes}$ by replacing maximal *by* minimal.

2.8. Categories such as $\mathsf{CMod}^{\otimes}$, $\mathsf{LMod}^{\otimes}$ or $\mathsf{RMod}^{\otimes}$ are *colored operads*, and the symbols $\mathfrak{a}, \mathfrak{m}$ are the *colors*.

Exercise 2.9. *Construct an ordinary colored operad $\mathsf{BMod}^{\otimes}$ with three colors $(\mathfrak{a}_l, \mathfrak{m}, \mathfrak{a}_r)$ encoding two monoids and a bimodule of them.*

Exercise 2.10. *Let $\mathsf{O}^{\otimes}$ be either $\mathsf{Comm}^{\otimes}$, $\mathsf{Assoc}^{\otimes}$, $\mathsf{CMod}^{\otimes}$, $\mathsf{LMod}^{\otimes}$, $\mathsf{RMod}^{\otimes}$ or $\mathsf{BMod}^{\otimes}$. Consider the natural functor $p : \mathsf{O}^{\otimes} \to \mathsf{Fin}_*$. Show that*

(1) *For any inert morphism in Fin_*, there are enough p-coCartesian liftings of it. Moreover, a morphism in $\mathsf{O}^{\otimes}$ is inert iff it is a p-coCartesian over an inert morphism in Fin_*.*
(2) *Let $C \in \mathsf{O}^{\otimes}_{\langle n\rangle}$ be an object lying over $\langle n\rangle$, and $C \to C_i$ be p-coCartesian morphisms lying over $\rho^i : \langle n\rangle \to \langle 1\rangle$, $i \in \langle n\rangle^{\circ}$. Then these morphisms exhibit C as a p-*limit[1]*. In other words, for any testing object $D \in \mathsf{O}^{\otimes}$, the following square is Cartesian*

$$\begin{array}{ccc} \mathsf{Maps}_{\mathsf{O}^{\otimes}}(D, C) & \longrightarrow & \prod_i \mathsf{Maps}_{\mathsf{O}^{\otimes}}(D, C_i) \\ \downarrow & & \downarrow \\ \mathsf{Maps}_{\mathsf{Fin}_*}(p(D), \langle n\rangle) & \longrightarrow & \prod_i \mathsf{Maps}_{\mathsf{Fin}_*}(p(D), \langle 1\rangle). \end{array}$$

(3) *The covaraint transport functors $\rho^i_{!}$ exhibit $\mathsf{O}^{\otimes}_{\langle n\rangle}$ as $(\mathsf{O}^{\otimes}_{\langle 1\rangle})^{\times n}$.*

Definition 2.11. *An ∞-**operad**[2] is an ∞-category $\mathsf{O}^{\otimes}$ over Fin_* satisfying the conclusions in the above exercise[3]. We call*

$$\mathsf{O} := \mathsf{O}^{\otimes}_{\langle 1\rangle}$$

*the ∞-**category of colors** in $\mathsf{O}^{\otimes}$.*

Definition 2.12. *Let $p : \mathsf{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad and $f : C \to C'$ be a morphism in $\mathsf{O}^{\otimes}$, we say*

- *The morphism f is **inert** if it is p-coCartesian over an inert morphism in Fin_*.*
- *The morphism f is **active** if $p(f)$ is active.*

[1]See HTT.4.3.2 for a discussion about relative limits in general.

[2]A better terminology might be *colored* ∞-operad.

[3]More precisely, we should say a functor $p : \mathsf{O}^{\otimes} \to \mathsf{Fin}_*$ exhibits $\mathsf{O}^{\otimes}$ as an essential ∞-operad, if any/all realization $\mathcal{O}^{\otimes} \to \mathsf{N}(\mathsf{Fin}_*)$ of p as an inner fibration satisfies the above conditions.

2.13. We often denote an object $C \in \mathsf{O}^{\otimes}_{\langle n \rangle}$ by $(C_1, \cdots, C_n)$[4] with $C_i \in \mathsf{O}$ and treat the inert morphisms $C \to C_i$ as implicit.

Exercise 2.14. *A morphism in $\mathsf{O}^{\otimes}$ is an isomorphism iff it is both inert and active.*

Exercise 2.15. *Show that any morphism in $\mathsf{O}^{\otimes}$ can be essentially uniquely written as a composition of an inert morphism followed by an active one.*

2.16. Let $\mathsf{O}^{\otimes}$ be an ∞-operad. We can define an O-monoidal ∞-category as a functor $\mathsf{O}^{\otimes} \to \mathsf{Cat}_{\infty}$ satisfying certain conditions. In practice, it is better to describe such conditions via the corresponding coCartesian fibration over $\mathsf{O}^{\otimes}$.

Definition 2.17. *Let $\mathsf{O}^{\otimes}$ be an ∞-operad. An **O-monoidal ∞-category** is an (essential) coCartesian fibration $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ such that the composition $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes} \to \mathsf{Fin}_*$ exhibits $\mathsf{C}^{\otimes}$ as an ∞-operad. We call $\mathsf{C} := \mathsf{C}^{\otimes}_{\langle 1 \rangle}$ the **underlying ∞-category** of $\mathsf{C}^{\otimes}$.*

Definition 2.18. *Let $\mathsf{C}^{\otimes}$ and $\mathsf{C}'^{\otimes}$ be O-monoidal ∞-categories. An **O-monoidal functor** is a functor $\mathsf{C}^{\otimes} \to \mathsf{C}'^{\otimes}$ defined over $\mathsf{O}^{\otimes}$ that preserves all coCartesian arrows. Let*

$$\mathsf{Fun}^{\otimes}_{\mathsf{O}}(\mathsf{C}, \mathsf{C}') \subseteq \mathsf{Fun}_{\mathsf{O}^{\otimes}}(\mathsf{C}^{\otimes}, \mathsf{C}'^{\otimes})$$

be the full sub-∞-category of O-monoidal functors.

Let

$$\mathsf{Mon}_{\mathsf{O}}(\mathsf{Cat}_{\infty}) \subseteq (\mathsf{Cat}_{\infty})_{/\mathsf{O}^{\otimes}}$$

be the 1-full sub-∞-category of small O-monoidal ∞-categories and O-monoidal functors between them.

Exercise 2.19. *Show that for $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$ or $\mathsf{Assoc}^{\otimes}$, the above definition recovers the notions of (symmetric) monoidal ∞-categories and monoidal functors via straightening.*

Exercise 2.20. *What are the initial and final objects in $\mathsf{Mon}_{\mathsf{O}}(\mathsf{Cat}_{\infty})$?*

3. Algebras

3.1. One advantage of defining O-monoidal ∞-categories via coCartesian fibrations is that the definition of algebras inside them is extremely simple.

Definition 3.2. *Let $\mathsf{O}^{\otimes}$ and $\mathsf{C}^{\otimes}$ be ∞-operads. An **∞-operad map** between them is a functor $p : \mathsf{O}^{\otimes} \to \mathsf{C}^{\otimes}$ defined over Fin_* that preserves inert morphisms. We also call such a functor **an O-algebra in** C. Let*

$$\mathsf{Op}_{\infty} \subseteq (\mathsf{Cat}_{\infty})_{/\mathsf{Fin}_*}$$

be the 1-full sub-∞-category consisting of small ∞-operads and ∞-operad maps between them.

Let

$$\mathsf{Alg}_{\mathsf{O}}(\mathsf{C}) \subseteq \mathsf{Fun}_{\mathsf{Fin}_*}(\mathsf{O}^{\otimes}, \mathsf{C}^{\otimes})$$

be the full sub-∞-category of ∞-operad maps $\mathsf{O}^{\otimes} \to \mathsf{C}^{\otimes}$.

Exercise 3.3. *What are the initial and final objects in Op_{∞}?*

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[4] Lurie's books use $C_1 \oplus \cdots \oplus C_n$, which might cause confusion because C is *neither* the (absolute) product nor coproduct of C_i's.

3.4. To define associative algebra in a category, we only need to equip the latter with a monoidal structure rather than a symmetric monoidal one. More generally, we have the following relative version of Definition 3.2.

Definition 3.5. *Let*

$$\begin{array}{ccc} \mathsf{O}'^{\otimes} & & \mathsf{C}^{\otimes} \\ & \overset{q}{\searrow} \quad \swarrow & \\ & \mathsf{O}^{\otimes} & \end{array}$$

be maps between ∞*-operads. An* O'***-algebra in*** C *(relative to* O*) is an object in*

$$\mathsf{Alg}_{\mathsf{O}'/\mathsf{O}}(\mathsf{C}) \coloneqq \mathsf{Alg}_{\mathsf{O}'}(\mathsf{C}) \underset{\mathsf{Alg}_{\mathsf{O}'}(\mathsf{O})}{\times} \{q\}. \tag{3.1}$$

When q *is the identity map on* $\mathsf{O}^{\otimes}$*, we also write*

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \coloneqq \mathsf{Alg}_{\mathsf{O}/\mathsf{O}}(\mathsf{C}).$$

Warning 3.6. *The* ∞*-categories* $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ *and* $\mathsf{Alg}_{\mathsf{O}}(\mathsf{C})$ *are different.*

Remark 3.7. *If* $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *is realized as a categorical fibration, then the fiber product* (3.1) *can be calculated as the naive fiber product.*

Exercise 3.8. *Show that*

$$(\mathsf{C} \underset{\mathsf{O}}{\times} \mathsf{O}')^{\otimes} \coloneqq \mathsf{C}^{\otimes} \underset{\mathsf{O}^{\otimes}}{\times} \mathsf{O}'^{\otimes}$$

is an ∞*-operad, and*

$$\mathsf{Alg}_{\mathsf{O}'/\mathsf{O}}(\mathsf{C}) \simeq \mathsf{Alg}_{/\mathsf{O}'}(\mathsf{C} \underset{\mathsf{O}}{\times} \mathsf{O}').$$

Exercise 3.9. *Show that restriction along* $\langle 1 \rangle \in \mathsf{Fin}_*$ *gives a conservative functor*

$$\mathsf{Alg}_{\mathsf{O}'/\mathsf{O}}(\mathsf{C}) \to \mathsf{Fun}_{\mathsf{O}}(\mathsf{O}', \mathsf{C}),$$

which is called the ***forgetful functor****.*

Definition 3.10. *Let* $\mathsf{C}^{\otimes}$ *and* $\mathsf{C}'^{\otimes}$ *be* O*-monoidal* ∞*-categories. An object in* $\mathsf{Alg}_{\mathsf{C}/\mathsf{O}}(\mathsf{C}')$ *is called a* ***(right) lax*** O***-monoidal*** *functor from* $\mathsf{C}^{\otimes}$ *to* $\mathsf{C}'^{\otimes}$

Remark 3.11. *Show that for* $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$ *or* $\mathsf{Assoc}^{\otimes}$*, the above definition generalizes the notions of lax (symmetric) monoidal functors between ordinary categories.*

4. Unitality

4.1. Note that for any

We say an ∞-operad $\mathsf{O}^{\otimes}$ is unital if there is an essentially unique nullary operator into any color $X \in \mathsf{O}$.

Appendix A. Monoidal envelope

A.1. By definition, any symmetric monoidal ∞-category is an ∞-operad. The converse is false because an ∞-operad $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ might not have enough coCartesian arrows.

Exercise A.2. *Let* $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ *be any symmetric monoidal* ∞*-category and* $\mathsf{C}' \subseteq \mathsf{C}$ *be a full sub-*∞*-category. Let* $\mathsf{C}'^{\otimes} \subset \mathsf{C}^{\otimes}$ *be the full sub-*∞*-category consisting of objects* $(X_1, \cdots, X_n)$ *with* $X_i \in \mathsf{C}'$*. Show that* $\mathsf{C}'^{\otimes}$ *is an* ∞*-operad.*

Exercise A.3. *Conversely, show that any* ∞*-operad* $\mathsf{C}^{\otimes}$ *can be realized as a full sub-*∞*-category of a symmetric monoidal* ∞*-category.*

A.4. In fact, there is a universal symmetric monoidal ∞-category $\mathsf{Env}(\mathsf{C})^{\otimes}$ containing $\mathsf{C}^{\otimes}$ such that for any test symmetric monoidal ∞-category $\mathsf{D}^{\otimes}$, we have

$$\mathsf{Fun}^{\otimes}(\mathsf{Env}(\mathsf{C}), \mathsf{D}) \simeq \mathsf{Alg}_{\mathsf{C}}(\mathsf{D}).$$

This is called the *symmetric monoidal envelope* of C.

A.5. **Suggested readings.** HA.2.2.4.