# LECTURE 22

In this lecture, we study $\mathsf{O}$-algebras for an $\infty$-operad $\mathsf{O}^{\otimes}$.

## 1. $n$-ary operators of an algebra

1.1. Last time, for any $\infty$-operads $\mathsf{O}^{\otimes}$ and $\mathsf{C}^{\otimes}$, we defined an $\mathsf{O}$-algebra in $\mathsf{C}$ as an $\infty$-operad map $p : \mathsf{O}^{\otimes} \to \mathsf{C}^{\otimes}$, i.e., a functor defined over $\mathsf{Fin}_*$ that preserves inert morphisms. Let us first justify this definition.

1.2. Let $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ be a symmetric monoidal category and $p : \mathsf{O}^{\otimes} \to \mathsf{C}^{\otimes}$ be an $\mathsf{O}$-algebra in $\mathsf{C}$. For any $n$-ary operator in $\mathsf{O}^{\otimes}$, i.e., an active morphoism $f : X = (X_1, \cdots, X_n) \to Y$ lying over the unique active morphism $\mathsf{mult}_n : \langle n \rangle \to \langle 1 \rangle$ in $\mathsf{Fin}_*$, we can construct a morphism

$$p(X_1) \otimes \cdots \otimes p(X_n) \to p(Y) \tag{1.1}$$

in $\mathsf{C}$ as follows. Since $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ is coCartesian, there exists an essential unique factorization of $p(f)$ as $p(X) \xrightarrow{g} Z \xrightarrow{h} p(Y)$ such that $g$ is a coCartesian arrow over the active morphism $\mathsf{mult}_n$ and $h$ is contained in $\mathsf{C}^{\otimes}_{\langle 1 \rangle} =: \mathsf{C}$. Using the assumption that $p$ preserves inert morphisms, one can show $p(X) \simeq (p(X_1), \cdots, p(X_n))$. It follows that

$$Z \simeq \mathsf{mult}_{n,\dagger}(p(X_1), \cdots, p(X_n)) \simeq p(X_1) \otimes \cdots \otimes p(X_n),$$

where the last isomorphism can be seen via straightening. Now the morphism $h : Z \to p(Y)$ gives the desired morphism (1.1).

**Exercise 1.3.** *Generalize the above construction to obtain a functor*

$$\mathsf{O}^{\otimes}_{\mathsf{active}} \to \mathsf{C}, \ (X_1, \cdots, X_n) \mapsto p(X_1) \otimes \cdots \otimes p(X_n)$$

*where* $\mathsf{O}^{\otimes}_{\mathsf{active}}$ *is the 1-full sub-$\infty$-category of* $\mathsf{O}^{\otimes}$ *consisting of active morphisms.*

## 2. $\mathsf{Triv}$-algebras and $\mathbb{E}_0$-algebras

2.1. Last time we introduced the ordinary operads $\mathsf{Assoc}^{\otimes}$ and $\mathsf{Comm}^{\otimes}$. We will study the corresponding algebras and modules in details in future lectures. For now, let us consider some even simpler operads.

**Exercise 2.2.** *Let* $\mathsf{Triv}^{\otimes}$ *be the subcategory of* $\mathsf{Fin}_*$ *consists of inert morphisms. Show that* $\mathsf{Triv}^{\otimes}$ *is an operad, and evaluation at* $\langle 1 \rangle \in \mathsf{Triv}^{\otimes}$ *induces an equivalence*

$$\mathsf{Alg}_{\mathsf{Triv}}(\mathsf{C}) \xrightarrow{\simeq} \mathsf{C}$$

*for any* $\mathsf{C}^{\otimes} \in \mathsf{Op}_{\infty}$.

**Exercise 2.3.** *Let* $\mathbb{E}_0^{\otimes}$ *be the subcategory of* $\mathsf{Fin}_*$ *consists of morphisms* $f : \langle n \rangle \to \langle m \rangle$ *such that* $f^{-1}\{j\}$ *is either empty or a singleton. Show that*

(1) *The functor* $\mathbb{E}_0^{\otimes} \to \mathsf{Fin}_*$ *makes* $\mathbb{E}_0^{\otimes}$ *an operad.*

---

*Date*: Dec. 10, 2024.

(2) *For any $\infty$-operad* $\mathsf{C}^{\otimes}$ *over* $\mathbb{E}_0^{\otimes}$*, evaluating at the unique morphism* $\langle 0\rangle \to \langle 1\rangle$ *in* $\mathsf{E}_0^{\otimes}$ *induces an equivalence*

$$\mathsf{Alg}_{/\mathbb{E}_0}(\mathsf{C}) \xrightarrow{\simeq} \mathsf{Fun}_{\mathbb{E}_0^{\otimes}}(\Delta^1, \mathsf{C}^{\otimes}).$$

(3) *Knwoing an* $\mathbb{E}_0$*-monoidal* $\infty$*-category* $\mathsf{C}^{\otimes}$ *is equivalent to knowing an* $\infty$*-category* $\mathsf{C}$ *and an object* $\mathbb{1}$ *in it. More precisely, there is an equivalence*

$$\mathsf{Mon}_{\mathsf{E}_0}(\mathsf{Cat}_{\infty}) \xrightarrow{\simeq} (\mathsf{Cat}_{\infty})_{*} \coloneqq (\mathsf{Cat}_{\infty})_{\{*\}/}$$

*given by the covariant transport along the morphism* $\langle 0\rangle \to \langle 1\rangle$*.*

(4) *For an* $\mathbb{E}_0$*-monoidal* $\infty$*-category* $\mathsf{C}^{\otimes}$*,*

$$\mathsf{Alg}_{/\mathbb{E}_0}(\mathsf{C}) \simeq \mathsf{C}_{\mathbb{1}/}.$$

**Remark 2.4.** *In future lectures, we will fill in more terms in the sequence*

$$\mathsf{Triv}^{\otimes} \to \mathbb{E}_0^{\otimes} \to \mathsf{Assoc}^{\otimes} \to \mathsf{Comm}^{\otimes}.$$

## 3. Unitality

3.1. Exercise 2.2 and Exercise 2.3 suggest $\mathbb{E}_0^{\otimes}$ is the *unital* version of $\mathsf{Triv}^{\otimes}$. The difference can be better formulated via the following exercise.

**Exercise 3.2.** *Show that there is no nullary operators in* $\mathsf{Triv}^{\otimes}$*, while there is a unique nullary operator in* $\mathbb{E}_0^{\otimes}$*.*

3.3. This motivates the following definition.

**Definition 3.4.** *Let* $\mathsf{O}^{\otimes}$ *be an* $\infty$*-operad and* $\langle 0\rangle \in \mathsf{O}^{\otimes}_{\langle 0\rangle}$ *be the essentially unique object.*

- *We say* $\mathsf{O}^{\otimes}$ *is* ***unital*** *if for any color* $X \in \mathsf{O}$*, the mapping space* $\mathsf{Maps}_{\mathsf{O}^{\otimes}}(\langle 0\rangle, X)$ *is contractible.*
- *We say* $\mathsf{O}^{\otimes}$ *is* ***non-unital*** *if for any color* $X \in \mathsf{O}$*, the mapping space* $\mathsf{Maps}_{\mathsf{O}^{\otimes}}(\langle 0\rangle, X)$ *is empty.*

*Let* $\mathsf{Op}_{\infty}^{\mathsf{unit}}$ *(resp.* $\mathsf{Op}_{\infty}^{\mathsf{nu}}$*) be the full sub-*$\infty$*-categories of small unital (resp. non-unital)* $\infty$*-operads.*

**Warning 3.5.** *An* $\infty$*-operad maybe neither unital nor non-unital.*

**Warning 3.6.** *A (symmetric) monoidal* $\infty$*-category* $\mathsf{C}^{\otimes}$ *may* not *be unital as an* $\infty$*-operad. See Exercise 3.13 for a criterion for* $\mathsf{C}^{\otimes}$ *being unital as an* $\infty$*-operad.*

3.7. Lurie denoted the object $\langle 0\rangle \in \mathsf{O}^{\otimes}_{\langle 0\rangle}$ by $\varnothing$. Note however it is in general not an initial object in the entire $\mathsf{O}^{\otimes}$. Nevertheless, we have the following results.

**Exercise 3.8.** *Let* $\mathsf{O}^{\otimes}$ *be any* $\infty$*-operad, show that* $\langle 0\rangle \in \mathsf{O}^{\otimes}$ *is a final object.*

**Exercise 3.9.** *An* $\infty$*-operad* $\mathsf{O}^{\otimes}$ *is unital iff* $\langle 0\rangle \in \mathsf{O}^{\otimes}$ *is an initial object.*

**Definition 3.10.** *Let* $\mathsf{O}^{\otimes}$ *be a unital* $\infty$*-operad and* $p : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *be an* $\mathsf{O}$*-monoidal* $\infty$*-category. We say a morphism* $Y_0 \to Y$ *in* $\mathsf{C}^{\otimes}$ ***exhibits*** $Y$ ***as an*** $X$***-unit object of*** $\mathsf{C}$ *if it is a* $p$*-coCartesian edge lying over* $\langle 0\rangle \to X$*.*

3.11. Since $\langle 0\rangle \in \mathsf{O}^{\otimes}$ is initial and $\mathsf{C}^{\otimes}_{\langle 0\rangle} \simeq \mathsf{C}^{\times 0}$ is contractible, for fixed $X \in \mathsf{C}^{\otimes}$, morphisms $Y_0 \to Y$ as above are essentially unique and $Y_0 \simeq \langle 0\rangle$. Hence it makes sense to talk about *the* $X$-unit object $\mathbb{1}_X$ of $\mathsf{C}$ and treat the coCartesian edge $\langle 0\rangle \to \mathbb{1}_X$ as implicit.

**Warning 3.12.** *The object $\langle 0\rangle \in \mathsf{C}^{\otimes}$ may not be initial, and coCartesian edges $\langle 0\rangle \to \mathbb{1}_X$ are not essentially unique. In fact, one can identify the space of such edges with that of automorphisms on $\mathbb{1}_X$ (Exercise).*

**Exercise 3.13.** *Let $\mathsf{O}^{\otimes}$ be a unital $\infty$-operad. Show that an $\mathsf{O}$-monoidal $\infty$-category $\mathsf{C}^{\otimes}$ is unital as an $\infty$-operad iff for any color $X \in \mathsf{O}$, the object $\mathbb{1}_X \in \mathsf{C}^{\otimes}_X$ is initial.*

3.14. In classical category theory, for a (symmetric) monoidal category $\mathsf{C}$, there is a canonical (commutative) algebra object in $\mathsf{C}$ whose underlying object is the monoidal unit $\mathbb{1}$. Moreover, this (commutative) algebra is initial in the category of all (commutative) algebras in $\mathsf{C}$. The following result generalize this to any $\infty$-operads.

**Proposition-Definition 3.15** (HA.3.2.1.8[1])**.** *Let $\mathsf{O}^{\otimes}$ be a unital $\infty$-operad and $p : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ be an $\mathsf{O}$-monoidal $\infty$-category. For an $\mathsf{O}$-algebra $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$, the following conditions are equivalent:*

(1) *The object $A$ is initial in $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$;*
(2) *For any object $X \in \mathsf{O}^{\otimes}$, the morphism $A(\langle 0\rangle) \to A(X)$ exhibits $A(X)$ as an $X$-unit object of $\mathsf{C}$;*
(3) *For any color $X \in \mathsf{O}$, the morphism $A(\langle 0\rangle) \to A(X)$ exhibits $A(X)$ as an $X$-unit object of $\mathsf{C}$.*

*We call $A$ the **unit $\mathsf{O}$-algebra**[2] in $\mathsf{C}$, and denote it by $\mathbb{1}_{\mathsf{C}}$.*

3.16. Let us also record the following result.

**Proposition 3.17** (HA.2.3.1.9)**.** *The embedding $\mathsf{Op}_{\infty}^{\mathsf{unit}} \subseteq \mathsf{Op}_{\infty}$ admits a left adjoint.*

**Remark 3.18.** *In future lectures, we will endow $\mathsf{Op}_{\infty}$ with a symmetric monoidal structure, i.e., construct a coCartesian fibration of $\infty$-operads $\mathsf{Op}_{\infty}^{\otimes} \to \mathsf{Comm}^{\otimes}$ whose fiber over $\langle 1\rangle$ is $\mathsf{Op}_{\infty}$. The monoidal unit of $\mathsf{Op}_{\infty}$ is given by $\mathsf{Triv}^{\otimes}$. Moreover, by HA.2.3.1.9, the composition*

$$\mathsf{Op}_{\infty} \to \mathsf{Op}_{\infty}^{\mathsf{unit}} \to \mathsf{Op}_{\infty}$$

*is identified with the functor $\mathbb{E}_0 \otimes -$. It follows that an $\infty$-operad $\mathsf{O}^{\otimes}$ is unital iff*

$$\mathsf{O}^{\otimes} \to \mathsf{Triv}^{\otimes} \otimes \mathsf{O}^{\otimes} \to \mathbb{E}_0^{\otimes} \otimes \mathsf{O}^{\otimes}$$

*is an equivalence. In particular, $\mathsf{Op}_{\infty}^{\mathsf{unit}}$ inherits a symmetric monoidal structure from $\mathsf{Op}_{\infty}$, with monoidal unit $\mathbb{E}_0^{\otimes}$*

## 4. Non-unital algebras

4.1. Given an $\infty$-operad $\mathsf{O}^{\otimes}$, we can produce a non-unital $\infty$-operad $\mathsf{O}^{\otimes}_{\mathsf{nu}}$ satisfying certain universal property.

[1] Lurie proved (1)⇔(2). The equivalence (2)⇔(3) follows from the axioms of $\infty$-operads.

[2] Lurie called it the **trivial O-algebra** in $\mathsf{C}$.

**Exercise 4.2.** *Let* $\mathsf{Fin}_*^{\mathsf{surj}} \subseteq \mathsf{Fin}_*$ *be the subcategory consisting of surjective morphisms. Show that for any* $\infty$*-operad* $\mathsf{O}^{\otimes}$*, the fiber product*

$$\mathsf{O}_{\mathsf{nu}}^{\otimes} := \mathsf{O}^{\otimes} \underset{\mathsf{Fin}_*}{\times} \mathsf{Fin}_*^{\mathsf{surj}}$$

*is a non-unital* $\infty$*-operad.*

**Exercise 4.3.** *Show that* $(\mathbb{E}_0^{\otimes})_{\mathsf{nu}} \simeq \mathsf{Triv}^{\otimes}$.

**Exercise 4.4.** *Show that the functor*

$$\mathsf{Op}_\infty \to \mathsf{Op}_\infty^{\mathsf{nu}},\ \mathsf{O}^{\otimes} \mapsto \mathsf{O}_{\mathsf{nu}}^{\otimes}$$

*is right adjoint to the embedding functor. In other words,*

$$\mathsf{Alg}_{\mathsf{D}}(\mathsf{O}_{\mathsf{nu}}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathsf{D}}(\mathsf{O})$$

*for any non-unital* $\mathsf{D}^{\otimes}$.

**Definition 4.5.** *Let*

$$\mathsf{O}'^{\otimes} \to \mathsf{O}^{\otimes} \leftarrow \mathsf{C}^{\otimes}$$

*be maps between* $\infty$*-operads. A* ***non-unital*** $\mathsf{O}'$***-algebra in*** $\mathsf{C}$ *(relative to* $\mathsf{O}$*) is an object in*

$$\mathsf{Alg}_{\mathsf{O}'/\mathsf{O}}^{\mathsf{nu}}(\mathsf{C}) := \mathsf{Alg}_{\mathsf{O}'_{\mathsf{nu}}/\mathsf{O}}(\mathsf{C}).$$

4.6. Recall for a non-unital (commutative) ring, being unital is a *property rather than a structure*: the unit element is unique. This can be generalized to the $\infty$-setting:

**Theorem 4.7** (HA.5.4.4.5, 5.4.4.7)**.** *Let* $\mathsf{O}^{\otimes}$ *be* $\mathsf{Assoc}^{\otimes}$ *or* $\mathsf{Comm}^{\otimes}$ *and* $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *be an* $\mathsf{O}$*-monoidal* $\infty$*-category. Then* $\mathsf{O}_{\mathsf{nu}}^{\otimes} \to \mathsf{O}^{\otimes}$ *induces a fully faithful functor*

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \to \mathsf{Alg}_{/\mathsf{O}_{\mathsf{nu}}}(\mathsf{C}).$$

**Remark 4.8.** *In fact,* loc.cit. *proved that the essential image of the above embedding consists exactly those non-unital associative (resp. commutative) algebras in* $\mathsf{C}$ *whose images in* $\mathsf{hC}$ *is unital.*

**Remark 4.9.** *Similar claim remains true for the* $\infty$*-operads* $\mathbb{E}_k^{\otimes}$ *with* $1 \leq k \leq \infty$.

**Warning 4.10.** *Similar claim is* false *for general* $\infty$*-operads.*

**Exercise 4.11.** *Show that* $\mathsf{Alg}_{/\mathbb{E}_0}(\mathsf{C}) \to \mathsf{Alg}_{/(\mathbb{E}_0)_{\mathsf{nu}}}(\mathsf{C})$ *may not be fully faithful for an* $\mathbb{E}_0$*-monoidal* $\infty$*-category* $\mathsf{C}$.

## 5. Limits and colimits of algebras

5.1. In this section, we study limits and colimits of $\mathsf{O}$-algebras in an $\mathsf{O}$-monoidal $\infty$-category $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$.

5.2. In classical algebra, the limit $\mathsf{lim} A_i$ of the underlying abelian groups of a diagram of rings $i \mapsto A_i$ admits a ring structure given by

$$(\lim_{i\in K} A_i) \otimes (\lim_{i\in K} A_i) \to \lim_{(i,j)\in K^2} (A_i \otimes A_j) \to \lim_{i\in K} (A_i \otimes A_i) \to \lim_{i\in K} A_i.$$

This can be genealized to any $\infty$-operad $\mathsf{O}^{\otimes}$.

**Theorem 5.3** (HA.3.2.2.5)**.** *Suppose $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ is an $\mathsf{O}$-monoidal $\infty$-category such that each fiber $\mathsf{C}^{\otimes}_X$ admits $K$-indexed limits for a fixed index simplicial set $K$. Then the $\infty$-category $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ admits $K$-indexed limits, and the forgetful functors*

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \to \mathsf{C}^{\otimes}_X,\ X \in \mathsf{O}$$

*preserve and detect $K$-indexed limits.*

**Remark 5.4.** *Note that in the above theorem, we do* not *assume the covariant transport functors $\mathsf{C}^{\otimes}_X \to \mathsf{C}^{\otimes}_Y$ preserves $K$-indexed limits.*

5.5. As we have seen in the case $K = \varnothing$, the analogue of Theorem 5.3 for colimits is false in general. Indeed, the above composition would become a correspondence

$$(\operatorname{colim}_{i\in K} A_i) \otimes (\operatorname{colim}_{i\in K} A_i) \leftarrow \operatorname{colim}_{(i,j)\in K^2} (A_i \otimes A_j) \leftarrow \operatorname{colim}_{i\in K} (A_i \otimes A_i) \to \operatorname{colim}_{i\in K} A_i.$$

This suggests the similar claim would be true if the two leftward morphisms are invertible. To give a precise statement, we need some definitions.

**Definition 5.6.** *Let $\mathsf{O}^{\otimes}$ be any $\infty$-operad and $p : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ be an $\mathsf{O}$-monoidal category. For a simplicial set $K$, we say $p$ is **compatible with $K$-indexed (co)limits** if*

- *For any object $X \in \mathsf{O}^{\otimes}$, the fiber $\mathsf{C}^{\otimes}_X$ admits $K$-indexed (co)limits;*
- *For any morphism $X \to X'$ in $\mathsf{O}^{\otimes}$, the covariant transport functor $\mathsf{C}^{\otimes}_X \to \mathsf{C}^{\otimes}_{X'}$ preserves $K$-indexed (co)limits.*

*We say $\mathsf{C}$ is a **presentable $\mathsf{O}$-monoidal $\infty$-category** if each $\mathsf{C}^{\otimes}_X$ is presentable and $p$ is compatible with small colimits.*

**Theorem 5.7** (HA.3.2.3.1)**.** *Let $K$ be a sifted simplicial set and $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ be an $\mathsf{O}$-monoidal $\infty$-category compatible with $K$-indexed colimits. Then the $\infty$-category $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ admits $K$-indexed colimits, and the forgetful functors*

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \to \mathsf{C}^{\otimes}_X,\ X \in \mathsf{O}$$

*preserve and detect $K$-indexed colimits.*

5.8. For general colimits, the first claim remains true, but the second would fail.

**Theorem 5.9** (HA. 3.1.3.5, 3.2.3.3)**.** *Let $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ be an $\mathsf{O}$-monoidal $\infty$-category compatible with small colimits. Suppose $\mathsf{O}^{\otimes}$ is essentially small. Then*

(i) *The forgetful functor*

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \to \mathsf{Fun}_{\mathsf{O}}(\mathsf{O}, \mathsf{C})$$

*admits a left adjoint*

$$\mathsf{Free}_{\mathsf{O}} : \mathsf{Fun}_{\mathsf{O}}(\mathsf{O}, \mathsf{C}) \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}).$$

(ii) *The $\infty$-category $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ admits small colimits.*

**Remark 5.10.** *As in classical algebra, the finite coproducts of* $\mathsf{O}$*-algebras can be calculated by taking simplicial resolutions of them by free* $\mathsf{O}$*-algebras. Namely, for an* $\mathsf{O}$*-algebra* $A$*, we have a simplicial object*

$$\Delta^{\mathsf{op}} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}),\ [n] \mapsto \mathsf{Free}_{\mathsf{O}}^{\circ(n+1)}(A)$$

*whose colimit is equivalent to* $A$*. This will be discussed in future lectures.*

**Remark 5.11.** *In future lectures, we will identify finite coproducts of* $\mathsf{Comm}$*-algebras as tensor products of them.*

5.12. Let us also record the following result.

**Proposition 5.13** (HA.3.2.3.5)**.** *Let* $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *be a presentable* $\mathsf{O}$*-monoidal* $\infty$*-category such that* $\mathsf{O}^{\otimes}$ *is essentially small. Then* $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ *is presentable.*

## Appendix A. Augmented algebras

A.1. Recall for a non-unital (commutative) ring $A$, there is a canonical unital (commutative) ring structure on $\mathbb{Z} \oplus A$. Moreover, this construction gives a equivalence between the theory of *non-unital* (commutative) rings and their modules and *augmented* (commutative) rings and their modules. These can be generalized to any unital $\infty$-operad $\mathsf{O}^{\otimes}$ and an $\mathsf{O}$-monoidal $\infty$-category $\mathsf{C}^{\otimes}$ satisfies certain conditions.

A.2. **Suggested readings.** HA.5.4.4.