LECTURE 23

In this lecture, we define O-modules for an ∞-operad $\mathsf{O}^{\otimes}$.

1. Outline

1.1. In the previous lectures, we constructed operads $\mathsf{CMod}^{\otimes}$, $\mathsf{LMod}^{\otimes}$, $\mathsf{RMod}^{\otimes}$ and $\mathsf{BMod}^{\otimes}$ such that the corresponding *algebra objects* in a symmetric monoidal ∞-category are pairs/triples of commutative/associative algebras and certain types of modules of them. However, these constructions are conducted in an *ad hoc* manner based on the explicit description of $\mathsf{Comm}^{\otimes}$ and $\mathsf{Assoc}^{\otimes}$. In this lecture, we give a general theory of modules for O-algebras for *coherent* ∞-operad $\mathsf{O}^{\otimes}$.

1.2. The definition of coherent ∞-operads will be given later. For now, let us treat it as a blackbox.

Let $\mathsf{O}^{\otimes}$ be a coherent ∞-operad and $p : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ be an O-monoidal ∞-category. Let $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be an O-algebra. For each color $X \in \mathsf{O}$, we will construct an ∞-category, $\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes}_X$, called the ∞-category of O-A-modules in C of color X. There will be a forgetful functor

$$\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes}_X \to \mathsf{C}^{\otimes}_X$$

which sends an O-A-module to its **underlying object**. In fact, we will allow X to be a *multi-color*, i.e., any object in $\mathsf{O}^{\otimes}$. Moreover, we will have a *parameterized* version

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \times \mathsf{O}^{\otimes} \tag{1.1}$$

such that its fiber at (A, X) is $\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})_X$. Under *good conditions*, we have the following results, details of which will be explained in the next lecture.

- For fixed $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$, the fiber of (1.1)
$$\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes} \to \mathsf{O}^{\otimes}$$
will be an O-monoidal ∞-category. This generalizes the relative tensor products $- \otimes_A -$ in commutative algebra.
- The projection
$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$$
will be a Cartesian fibration. This generalizes the restriction functors in commutative algebra.
- Note however that for a morphism $A \to B$ in $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$, the contravariant transport functor
$$\mathsf{res} : \mathsf{Mod}^{\mathsf{O}}_B(\mathsf{C})^{\otimes} \to \mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes}$$
is *not* an O-monoidal functor. Instead, it is only a *right lax* O-monoidal one, i.e., an ∞-operad map. This generalizes the corresponding fact in commutative algebra.

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- For fixed $X \in \mathsf{O}^{\otimes}$, the fiber of (1.1)
$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}_{X} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$$
will be a coCartesian fibration. This generalizes the induction functors in commutative algebra.
- Note however that the functors
$$\mathsf{ind} : \mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes}_{X} \to \mathsf{Mod}^{\mathsf{O}}_{B}(\mathsf{C})^{\otimes}_{X},\ X \in \mathsf{O}^{\otimes}$$
cannot be assembled to a functor over $\mathsf{O}^{\otimes}$. Indeed, as the left adjoint of the right lax O-monoidal functor res, the functor ind should in general only be *left lax* O-monoidal. In the commutative case, the induction functor *happens* to be O-monoidal, but this already fails for $\mathsf{O}^{\otimes} = \mathsf{Assoc}^{\otimes}$. See Exercise 1.4 below.

Warning 1.3. *For $\mathsf{O}^{\otimes} = \mathsf{Assoc}^{\otimes}$, objects in $\mathsf{Mod}^{\mathsf{Assoc}}_{A}(\mathsf{C})$ are (A,A)-bimodules rather than left or right A-modules. Indeed, to define relative tensor products over A, we need a left A-module* and *a right* A*-module.*

Exercise 1.4. *Let $A \to B$ be a homomorphism between associative rings. Show that the restriction functor*
$$\mathsf{res} :\ {}_{B}\mathsf{BMod}_{B} \to {}_{A}\mathsf{BMod}_{A}$$
has a left adjoint
$$\mathsf{ind} :\ {}_{A}\mathsf{BMod}_{A} \to {}_{B}\mathsf{BMod}_{B}$$
which is left lax monoidal but not monoidal.

2. Coherent ∞-operads

2.1. The ∞-category $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}_{X}$ will be defined as a certain full sub-∞-category
$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}_{X} \subseteq \mathsf{Fun}_{\mathsf{O}^{\otimes}}(\mathsf{K}_{X}, \mathsf{C}^{\otimes}),$$
where K_{X} is a certain ∞-category over $\mathsf{O}^{\otimes}$ that plays the role of $\mathsf{CMod}^{\otimes}$ in the Comm-case.

We will obtain the functoriality of $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}_{X}$ in X via that of K_{X}. In other words, we will define a certain ∞-category K_{O} equipped with *two* projections
$$(\mathsf{ev}_0, \mathsf{ev}_1) : \mathsf{K}_{\mathsf{O}} \to \mathsf{O}^{\otimes} \times \mathsf{O}^{\otimes}$$
and define
$$\mathsf{K}_{X} \coloneqq \{X\} \underset{\mathsf{O}^{\otimes},\mathsf{ev}_0}{\times} \mathsf{K}_{\mathsf{O}} \xrightarrow{\mathsf{ev}_1} \mathsf{O}^{\otimes}$$
where the fiber product uses ev_0, and the projection $\mathsf{K}_{X} \to \mathsf{O}^{\otimes}$ uses ev_1.

To achieve this, we will *not* make K_{X} an ∞-operad. When $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$ and $X = \langle 1 \rangle$, this means we only allow objects in $\mathsf{CMod}^{\otimes}$ with at most one $\mathfrak{m}$-term.

2.2. To define K_{O}, we need some terminologies.

Definition 2.3. *Let $\mathsf{O}^{\otimes}$ be an ∞-operad. We say a morphism $f : (X_1, \cdots, X_m) \to (Y_1, \cdots, Y_n)$ lying over $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle$ is* ***semi-inert*** *if for each $j \in \langle n \rangle^{\circ}$,* exactly *one of the following conditions holds:*

(a) *The set $\alpha^{-1}(\{j\})$ is empty.*
(b) *The set $\alpha^{-1}(\{j\})$ contains exactly one element i, and the corresponding morphism $X_i \to Y_j$ is an isomorphism.*

Exercise 2.4. *Show that f is inert iff (b) holds for each $j \in \langle n\rangle^\circ$.*

Definition 2.5. *Let*

$$\mathsf{K}_{\mathsf{O}} \subseteq \mathsf{Fun}([1], \mathsf{O}^{\otimes})$$

be the full sub-∞-category of semi-inert morphisms in $\mathsf{O}^{\otimes}$ and

$$\mathsf{K}_X := \{X\} \underset{\mathsf{O}^{\otimes}, \mathsf{ev}_0}{\times} \mathsf{K}_{\mathsf{O}}$$

be the ∞-category of semi-inert morphisms out of X.

Definition 2.6. *We say a morphism in K_{O} is* ***inert*** *iff both functors ev_0 and ev_1 send it to an inert morphism in $\mathsf{O}^{\otimes}$.*

Exercise 2.7. *For $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$ and $X = \langle 1\rangle$, describe the category K_X and construct a canonical functor $\mathsf{K}_X \to \mathsf{CMod}^{\otimes}$ defined over $\mathsf{Comm}^{\otimes}$. Show that*

$$\mathsf{Alg}_{\mathsf{CMod}}(\mathsf{C}) \to \mathsf{Fun}_{\mathsf{Comm}^{\otimes}}(\mathsf{CMod}^{\otimes}, \mathsf{C}^{\otimes}) \to \mathsf{Fun}_{\mathsf{Comm}^{\otimes}}(\mathsf{K}_X, \mathsf{C}^{\otimes})$$

is fully faithful and the essential image consists of those functors $\mathsf{K}_X \to \mathsf{C}^{\otimes}$ that preserves inert morphisms.

2.8. We are ready to give the definition of coherent ∞-operads.

Definition 2.9. *Let $\mathsf{O}^{\otimes}$ be a small ∞-operad. We say $\mathsf{O}^{\otimes}$ is* ***coherent*** *if it satisfies the following conditions*

(i) *The ∞-operad $\mathsf{O}^{\otimes}$ is unital.*
(ii) *The ∞-category O of colors is a space.*
(iii) *The base-change functor along $\mathsf{ev}_0 : \mathsf{K}_{\mathsf{O}} \to \mathsf{O}^{\otimes}$*

$$\mathsf{ev}_0^* : (\mathsf{Cat}_\infty)_{/\mathsf{O}^{\otimes}} \to (\mathsf{Cat}_\infty)_{/\mathsf{K}_{\mathsf{O}}},\ \mathsf{D} \mapsto \mathsf{D} \underset{\mathsf{O}^{\otimes}, \mathsf{ev}_0}{\times} \mathsf{K}_{\mathsf{O}}$$

admits a right adjoint $\mathsf{ev}_{0,*}$.

Proposition 2.10 (HA.3.3.1.12, 4.1.1.20)**.** *The ∞-operads $\mathsf{Comm}^{\otimes}$ and $\mathsf{Assoc}^{\otimes}$ are coherent.*

2.11. Although we have not yet defined E_k, let us record the following result.

Theorem 2.12 (HA.5.1.1.1)**.** *The ∞-operad $\mathsf{E}_k^{\otimes}$ is coherent for any $k \geq 0$.*

Remark 2.13. *Using the results in HA.B.3, one can show (iii) is equivalent to*

(iii') *The functor $\mathsf{ev}_0 : \mathsf{K}_{\mathsf{O}} \to \mathsf{O}^{\otimes}$ is a* flat categorical fibration*.*

Hence by HA.3.3.2.2, our definition of coherent ∞-operads coincides with Lurie's definition in HA.3.3.1.1. In fact, under these equivalent conditions, the adjunction

$$\mathsf{ev}_0^* : (\mathsf{Cat}_\infty)_{/\mathsf{O}^{\otimes}} \rightleftarrows (\mathsf{Cat}_\infty)_{/\mathsf{K}_{\mathsf{O}}} : \mathsf{ev}_{0,*}$$

can be realized as a Quillen adjunction between the (slice) Joyal model categories. It follows that for a categorical fibration D over K_{O}, the functor $\mathsf{ev}_{0,*}$ sends it to the quasi-category $\mathsf{ev}_{0,*}(D)$ over $\mathsf{O}^{\otimes}$ satisfying the following univresal property

$$\mathsf{Hom}_{(\mathsf{Set}_\Delta)_{/\mathsf{O}^{\otimes}}}(J, \mathsf{ev}_{0,*}(D)) \simeq \mathsf{Hom}_{(\mathsf{Set}_\Delta)_{/\mathsf{K}_{\mathsf{O}}}}(J \underset{\mathsf{O}^{\otimes}}{\times} \mathsf{K}_{\mathsf{O}}, D),$$

where the fiber product in the RHS is the naive one.

Exercise 2.14. *Let $\mathsf{D} \in (\mathsf{Cat}_\infty)_{/\mathsf{O}^{\otimes}}$, show that the fiber of $\mathsf{ev}_{0,*} \circ \mathsf{ev}_1^*(\mathsf{D})$ at $X \in \mathsf{O}^{\otimes}$ is equivalent to $\mathsf{Fun}_{\mathsf{O}^{\otimes}}(\mathsf{K}_X, \mathsf{D})$.*

3. Definition of modules

3.1. We are finally ready to construct $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}$.

Definition 3.2. *Let $\mathsf{O}^{\otimes}$ be a coherent ∞-operad and $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ be an ∞-operad map. We define*

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \subseteq \mathsf{ev}_{0,*} \circ \mathsf{ev}_1^*(\mathsf{C}^{\otimes})$$

to be the full sub-∞-category consisting of those objects such that the corresponding functor $\mathsf{K}_X \to \mathsf{C}^{\otimes}$ (see Exercise 2.14) preserves inert morphisms.

3.3. Note that one projection in (1.1) is automatic: we have a structure functor

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{O}^{\otimes}.$$

Exercise 3.4. *For $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$, identify the fiber of the above functor at $\langle 1 \rangle$ with $\mathsf{Alg}_{\mathsf{CMod}}(\mathsf{C})$.*

3.5. To obtain the other projection

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{Alg}_{\mathsf{O}}(\mathsf{C}),$$

we need more definitions.

Definition 3.6. *Let $\mathsf{O}^{\otimes}$ be an ∞-operad. We say a morphism $f : (X_1, \cdots, X_m) \to (Y_1, \cdots, Y_n)$ lying over $\langle m \rangle \xrightarrow{\alpha} \langle n \rangle$ is* ***null*** *if for each $j \in \langle n \rangle^{\circ}$,*

(a) *The set $\alpha^{-1}(\{j\})$ is empty.*

Let $\mathsf{K}^0_{\mathsf{O}} \subseteq \mathsf{K}_{\mathsf{O}}$ be the full sub-∞-category of null morphisms.

Exercise 3.7. *Show that f is null iff it factors through the object $\langle 0 \rangle \in \mathsf{O}^{\otimes}$.*

Exercise 3.8. *Show that the functor $(\mathsf{ev}^0_0, \mathsf{ev}^0_1) : \mathsf{K}^0_{\mathsf{O}} \to \mathsf{O}^{\otimes} \times \mathsf{O}^{\otimes}$ is an equivalence.*

Exercise 3.9. *Show that the diagram*

$$\begin{array}{ccccc} & & \mathsf{K}^0_{\mathsf{O}} & & \\ & \swarrow^{\mathsf{ev}^0_0} & \downarrow & \searrow^{\mathsf{ev}^0_1} & \\ \mathsf{O}^{\otimes} & \xleftarrow{\mathsf{ev}_0} & \mathsf{K}_{\mathsf{O}} & \xrightarrow{\mathsf{ev}_1} & \mathsf{O}^{\otimes} \end{array}$$

induces a natural functor

$$\mathsf{ev}_{0,*} \circ \mathsf{ev}_1^*(\mathsf{C}^{\otimes}) \to \mathsf{ev}^0_{0,*} \circ \mathsf{ev}_1^{0,*}(\mathsf{C}^{\otimes}) \simeq \mathsf{O}^{\otimes} \times \mathsf{Fun}_{\mathsf{O}^{\otimes}}(\mathsf{O}^{\otimes}, \mathsf{C}^{\otimes})$$

which restricts to a functor

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{O}^{\otimes} \times \mathsf{Alg}_{\mathsf{O}}(\mathsf{C})$$

defined over $\mathsf{O}^{\otimes}$.

3.10. Let us conclude by constructing the forgetful functor

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{C}^{\otimes}.$$

Exercise 3.11. *Show that any isomorphism in $\mathsf{O}^{\otimes}$ is semi-inert. In particular, there is a functor*

$$\Delta : \mathsf{O}^{\otimes} \to \mathsf{K}_{\mathsf{O}},\ X \mapsto \mathsf{id}_X.$$

Exercise 3.12. *Show that the diagram*

$$\begin{array}{ccccc} & & \mathsf{O}^{\otimes} & & \\ & \diagup\!\!\diagup & \downarrow & \diagdown\!\!\diagdown & \\ \mathsf{O}^{\otimes} & \xleftarrow{\mathsf{ev}_0} & \mathsf{K}_{\mathsf{O}} & \xrightarrow{\mathsf{ev}_1} & \mathsf{O}^{\otimes} \end{array}$$

induces a natural functor

$$\mathsf{ev}_{0,*} \circ \mathsf{ev}_1^*(\mathsf{C}^{\otimes}) \to \mathsf{C}^{\otimes}$$

defined over $\mathsf{O}^{\otimes}$. *Its restriction*

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{C}^{\otimes}$$

*is called the **forgetful functor**.*

Appendix A. Flat fibrations

A.1. Let $q : X \to Y$ be a categorical fibration between quasi-categories. As mentioned in Remark 2.13, the functor $q_* : (\mathsf{Cat}_\infty)_{/X} \to (\mathsf{Cat}_\infty)_{/Y}$ exists iff q is a *flat* categorical fibration. The latter condition has a nice characterization via simplicial sets, which says q is flat iff for any $\Delta^2 \to Y$, the map $X \times_Y \Lambda^2_1 \to X \times_Y \Delta^2$ is a categorical equivalence. Informally speaking, this means for any chain $u \xrightarrow{f} v \xrightarrow{g} w$ in Y, the *correspondence* between X_u and X_w over $g \circ f$ is the composition of the correspondences over f and g.

A.2. **Suggested readings.** HA.B.3.