LECTURE 24

In this lecture, we study O-modules for an ∞-operad $\mathsf{O}^\otimes$. Throughout this lecture, $\mathsf{O}^\otimes$ is a coherent O-operad and $q : \mathsf{C}^\otimes \to \mathsf{O}^\otimes$ is an ∞-operad map.

1. Relative tensor products

1.1. Last time we defined an ∞-category $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^\otimes$ fitting into the following commutative diagram

$$\begin{array}{ccc} \mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^\otimes & \longrightarrow & \mathsf{C}^\otimes \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \times \mathsf{O}^\otimes & \longrightarrow & \mathsf{O}^\otimes. \end{array} \tag{1.1}$$

Objects in $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^\otimes$ should be viewed as triples (A, X, M), where

- $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ is an O-algebra in C,
- $X \in \mathsf{O}^\otimes$ is a multi-color in $\mathsf{O}^\otimes$,
- M is an object in $\mathsf{C}^\otimes_X$ equipped with an O-A-module structure of type X.

The top horizontal functor in (1.1) sends (A, X, M) to the underlying object M, while the left vertical functor sends the triple to (A, X).

Theorem 1.2 (HA.3.3.3.9, 3.3.3.10)**.** *Let $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be an O-algebra in C. Then $\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^\otimes$ is an ∞-operad, and the forgetful functor*

$$\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^\otimes \to \mathsf{C}^\otimes$$

is an ∞-operad map.

Proposition 1.3 (HA.3.4.2.1)**.** *Suppose $q : \mathsf{C}^\otimes \to \mathsf{O}^\otimes$ is an O-monoidal ∞-category. Let $A := \mathbb{1}_{\mathsf{C}} \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be the unit algebra in C. Then the forgetful functor*

$$\mathsf{Mod}^{\mathsf{O}}_{\mathbb{1}_{\mathsf{C}}}(\mathsf{C})^\otimes \to \mathsf{C}^\otimes$$

is an equivalence.

Theorem 1.4 (HA.3.4.4.2)**.** *Let $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be an O-algebra in C. Suppose:*

(i) *The ∞-operad $\mathsf{O}^\otimes$ is essentially small;*
(ii) *The functor $q : \mathsf{C}^\otimes \to \mathsf{O}^\otimes$ exhibits $\mathsf{C}^\otimes$ as a presentable O-monoidal ∞-category.*

Then the functor

$$\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^\otimes \to \mathsf{O}^\otimes$$

exhibits $\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^\otimes$ as a presentable O-monoidal ∞-category.

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Remark 1.5. *Informally speaking, the above theorem says the* relative *tensor products of A-modules in an O-monoidal ∞-category exist as long as the* absolute *tensor products behave well under colimits. Moreover, these relative tensor products define an O-monoidal structure on the ∞-category of A-modules.*

Remark 1.6. *For $\mathsf{O}^{\otimes} = \mathsf{Assoc}^{\otimes}$, $\mathsf{Comm}^{\otimes}$, or more generally $\mathbb{E}_k^{\otimes}$ for $k \geq 1$, the relative tensor product $M \otimes_A N$ of two objects $M, N \in \mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C}) := \mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})^{\otimes}_{\langle 1 \rangle}$ can be calculated explicitly as a geometric realization*

$$M \otimes N \leftleftarrows M \otimes A \otimes N \; \substack{\longleftarrow\\\longleftarrow\\\longleftarrow} \; M \otimes A \otimes A \otimes N \cdots$$

known as the Bar construction*. For more details, see HA.4.4 and 5.1.3.*

Proposition 1.7 (HA.3.4.4.4)**.** *In the setting of Theorem 1.4, for any multi-color $X \in \mathsf{O}^{\otimes}$, the X-unit object in $\mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})^{\otimes}$ has underlying object in $\mathsf{C}^{\otimes}$ canonically isomorphic to $A(X)$.*

Remark 1.8. *More precisely, a morphism $M_0 \to M$ in $\mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})^{\otimes}$ lying over $\langle 0 \rangle \to X$ exhibits M as an X-unit object iff the corresponding functor*

$$\Delta^1 \underset{\mathsf{O}^{\otimes}}{\times} \mathsf{K}_{\mathsf{O}} \to \mathsf{C}^{\otimes}$$

sends the morphism between $(\langle 0 \rangle \to X) \in \mathsf{K}_{\langle 0 \rangle}$ and $(X \xrightarrow{\mathsf{id}_X} X) \in \mathsf{K}_X$ to an isomorphism in $\mathsf{C}^{\otimes}_X$.

1.9. The following result generalizes the corresponding well-known result in commutative algebra.

Proposition 1.10 (HA.3.4.1.7)**.** *Let $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be an O-algebra in C. Then there is a canonical equivalence*

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})) \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})_{\mathsf{A}/}$$

compatible with the forgetful functors to $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$.

Remark 1.11. *Informally speaking, the above equivalence is induced by restriction along a functor*

$$\mathsf{O}^{\otimes} \times \Delta^1 \to \mathsf{K}_{\mathsf{O}}$$

which sends $(X, 0)$ to the null morphism $\langle 0 \rangle \to X$, and sends $(X, 1)$ to the identity morphism $X \xrightarrow{\mathsf{id}_X} X$.

1.12. We can also describe modules in $\mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})^{\otimes}$.

Proposition 1.13 (HA.3.4.1.8)**.** *Let $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be an O-algebra in C. Then there is a canonical pullback diagram of ∞-categories:*

$$\begin{array}{ccccc}
\mathsf{Mod}^{\mathsf{O}}(\mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C}))^{\otimes} & \longrightarrow & \mathsf{Alg}_{/\mathsf{O}}(\mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})) & \xrightarrow{\simeq} & \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})_{\mathsf{A}/} \\
\downarrow & & & & \downarrow \\
\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} & & \longrightarrow & & \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}).
\end{array}$$

1.14. The meaning of the above proposition can be explained via the following corollary of it.

Corollary 1.15. *Let* $\phi : A \to B$ *be a morphism in* $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$*. Then there is a canonical equivalence*

$$\mathsf{Mod}^{\mathsf{O}}_{B}(\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C}))^{\otimes} \xrightarrow{\simeq} \mathsf{Mod}^{\mathsf{O}}_{B}(\mathsf{C})^{\otimes},$$

where in the source we view B *as an* O*-algebra in* $\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes}$*.*

2. Restrictions and limits

Theorem 2.1 (HA.3.4.3.4)**.** *The functor*

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \tag{2.1}$$

is a Cartesian fibration, and a morphism in $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}$ *is Cartesian iff its image in* $\mathsf{C}^{\otimes}$ *is an isomorphism.*

Exercise 2.2. *Let* $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$*. Show that the forgetful functor* $\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes} \to \mathsf{C}^{\otimes}$ *is conservative.*

Exercise 2.3. *Show that the straightening of* (2.1)

$$\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})^{\mathsf{op}} \to \mathsf{Cat}_{\infty}$$

canonically factors through $(\mathsf{Cat}_{\infty})_{/\mathsf{C}^{\otimes}}$*.*

Exercise 2.4. *Let* $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$*. Show that a morphism in* $\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes}$ *is inert iff its image in* $\mathsf{C}^{\otimes}$ *is so. Deduce that for a morphism* $\phi : A \to B$ *in* $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$*, the contravariant transport functor*

$$\mathsf{res}_{\phi} : \mathsf{Mod}^{\mathsf{O}}_{B}(\mathsf{C})^{\otimes} \to \mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes}$$

is an ∞*-operad map.*

Remark 2.5. *Informally speaking, the above results say there are restriction functors along homomorphisms between* O*-algebras, and these restriction functors are* laxly *compatible with relative tensor products, whenever the latter exist.*

Exercise 2.6. *What is the relationship between* res_{ϕ} *and the equivalence*

$$\mathsf{Mod}^{\mathsf{O}}_{B}(\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C}))^{\otimes} \xrightarrow{\simeq} \mathsf{Mod}^{\mathsf{O}}_{B}(\mathsf{C})^{\otimes}?$$

2.7. The next result is about limits of modules. One should compare it with [Lecture 22, Theorem 5.3], which is about limits of algebras.

Theorem 2.8 (HA.3.4.3.6)**.** *Let* $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ *be an* O*-algebra in* C*. For a multicolor* $X \in \mathsf{O}^{\otimes}$ *and a simplicial set* K*, suppose:*

- (i) *The functor* $q : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *is a coCartesian fibration.*
- (ii) *The fiber* $\mathsf{C}^{\otimes}_{X}$ *admits* K*-indexed limits.*

Then the ∞*-category* $\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes}_{X}$ *admits* K*-indexed limits and the foegetful functor*

$$\mathsf{Mod}^{\mathsf{O}}_{A}(\mathsf{C})^{\otimes}_{X} \to \mathsf{C}^{\otimes}_{X}$$

preserves and detects K*-indexed limits.*

Exercise 2.9. *In the above setting, show that for any morphism $\phi : A \to B$ in $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$, the functor*

$$\mathsf{res}_\phi : \mathsf{Mod}_B^{\mathsf{O}}(\mathsf{C})_X^{\otimes} \to \mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})_X^{\otimes}$$

preserves K-indexed limits.

Remark 2.10. *In fact, Theorem 2.1 and Theorem 2.8 can be combined into a single statement about limits in $\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes}$* relative *to the projection*

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C}) \times \mathsf{O}^{\otimes}.$$

See HA.3.4.3.1 for more details.

3. Induction and colimits

3.1. Exercise 2.9 and the adjoint functor theorem imply the following result.

Corollary 3.2. *Suppose:*

(i) *The ∞-operad $\mathsf{O}^{\otimes}$ is essentially small;*
(ii) *The functor $q : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ exhibits $\mathsf{C}^{\otimes}$ as a presentable O-monoidal ∞-category.*

Then for any multi-color $X \in \mathsf{O}^{\otimes}$, the functor

$$\mathsf{Mod}^{\mathsf{O}}(\mathsf{C})_X^{\otimes} \to \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$$

is a coCartesian fibration.

Remark 3.3. *For a morphism $\phi : A \to B$ in $\mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$, the covariant transport functor*

$$\mathsf{ind}_\phi : \mathsf{Mod}_A^{\mathsf{O}}(\mathsf{C})_X^{\otimes} \to \mathsf{Mod}_B^{\mathsf{O}}(\mathsf{C})_X^{\otimes}$$

*is called the **induction functor**.*

3.4. The following result says for $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$ the induction functor is symmetric monoidal.

Theorem 3.5 (HA.4.5.1.3)**.** *Let $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$. Suppose:*

- *The functor $q : \mathsf{C}^{\otimes} \to \mathsf{Comm}^{\otimes}$ exhibits $\mathsf{C}^{\otimes}$ as a symmetric monoidal ∞-category compatible with geometric realizations.*

Then the functor

$$\mathsf{Mod}^{\mathsf{Comm}}(\mathsf{C})^{\otimes} \to \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C}) \times \mathsf{O}^{\otimes}$$

is a coCartesian fibration.

Remark 3.6. *The induction functor*

$$\mathsf{ind}_\phi : \mathsf{Mod}_A^{\mathsf{Comm}}(\mathsf{C}) \to \mathsf{Mod}_B^{\mathsf{Comm}}(\mathsf{C})$$

sends M to $B \otimes_A M$, which can be calculated explicitly as a geometric realization

$$B \otimes M \leftleftarrows B \otimes A \otimes N \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, B \otimes A \otimes A \otimes M \cdots.$$

For more details, see HA.4.5.3.

3.7. Recall that general colimits of algebras may not be preserved by the forgetful functors. However, this would not happen for modules.

Theorem 3.8 (HA.3.4.4.6)**.** *Let $A \in \mathsf{Alg}_{/\mathsf{O}}(\mathsf{C})$ be an O-algebra in C. For a multicolor $X \in \mathsf{O}^{\otimes}$ and a simplicial set K, suppose:*

- *The functor $q : \mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ exhibits $\mathsf{C}^{\otimes}$ as an O-monoidal ∞-category compatible with K-indexed colimits*[1]*.*

Then the ∞-category $\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes}_X$ admits K-indexed colimits and the foegetful functor

$$\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes}_X \to \mathsf{C}^{\otimes}_X$$

preserves and detects K-indexed colimits.

Remark 3.9. *Explain that the above theorem is plausible using an argument similar to [Lecture 22, §5.5].*

Remark 3.10. *In fact, Theorem 1.4 and Theorem 3.8 can be combined into a single statement about colimits in $\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes}$* relative *to the projection*

$$\mathsf{Mod}^{\mathsf{O}}_A(\mathsf{C})^{\otimes} \to \mathsf{O}^{\otimes}.$$

See HA.3.4.4.3 for more details.

4. Left modules and bimodules

4.1. The following result says a Comm-module of a commutative algebra is the same as a left (or right) module of the underlying associative algebra.

Proposition 4.2 (HA.4.5.1.4)**.** *Let $\mathsf{O}^{\otimes} = \mathsf{Comm}^{\otimes}$. There is a canonical pullback diagram of ∞-categories:*

$$\begin{array}{ccc} \mathsf{Mod}^{\mathsf{Comm}}(\mathsf{C}) & \longrightarrow & \mathsf{Alg}_{\mathsf{LMod}}(\mathsf{C}) \\ \downarrow & & \downarrow \\ \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C}) & \longrightarrow & \mathsf{Alg}_{\mathsf{Assoc}}(\mathsf{C}). \end{array}$$

4.3. The following result says an Assoc-module of an associative algebra is the same as a bimodule of it.

Proposition 4.4 (HA.4.4.1.28)**.** *Let $\mathsf{O}^{\otimes} = \mathsf{Assoc}^{\otimes}$. There is a canonical pullback diagram of ∞-categories:*

$$\begin{array}{ccc} \mathsf{Mod}^{\mathsf{Assoc}}(\mathsf{C}) & \longrightarrow & \mathsf{Alg}_{\mathsf{BMod}/\mathsf{Assoc}}(\mathsf{C}) \\ \downarrow & & \downarrow \\ \mathsf{Alg}_{/\mathsf{Assoc}}(\mathsf{C}) & \longrightarrow & \mathsf{Alg}_{/\mathsf{Assoc}}(\mathsf{C}) \times \mathsf{Alg}_{/\mathsf{Assoc}}(\mathsf{C}) \end{array}$$

where the bottom horizontal functor is the diagonal functor.

[1] HA.3.4.4.6 is stated under the assumption that q is compatible with arbitrary small colimits. However, our weaker assumption suffices for the claim below, which is claim (2) in *loc.cit.*.

Appendix A. More on bimodules

A.1. Knowing an (A, B)-bimodule is equivalent to knowing a right B-module equipped with a left A-action that commutes with the B-action. In formula,

$$ {}_A\mathsf{BMod}_B(\mathsf{C}) \simeq \mathsf{LMod}_A(\mathsf{RMod}_B(\mathsf{C})). $$

A.2. The monoidal structure on $\mathsf{Mod}_A^{\mathsf{Assoc}}(\mathsf{C})$ can be generalized to operations

$$ {}_{A_0}\mathsf{BMod}_{A_1}(\mathsf{C}) \times {}_{A_1}\mathsf{BMod}_{A_2}(\mathsf{C}) \times \cdots \times {}_{A_{n-1}}\mathsf{BMod}_{A_n}(\mathsf{C}) \to {}_{A_0}\mathsf{BMod}_{A_n}(\mathsf{C}) $$

satisfying certain compatibilities.

A.3. **Suggested readings.** HA.4.3.