# LECTURE 25

In this lecture, we define and study several symmetric monoidal $\infty$-categories.

## 1. Cartesian symmetric monoidal structure

1.1. Let $\mathsf{C}$ be an $\infty$-category that admits finite products. In this section, we endow $\mathsf{C}$ with a symmetric monoidal structure $\mathsf{C}^\times \to \mathsf{Comm}^\otimes$ such that the multiplication functor is given by

$$\mathsf{C} \times \mathsf{C} \to \mathsf{C},\ (X, Y) \mapsto X \times Y.$$

**Definition 1.2.** *We say a symmetric monoidal structure $\mathsf{C}^\otimes \to \mathsf{Comm}^\otimes$ on $\mathsf{C}$ is* ***Cartesian*** *if it satisfies the following properties:*

1. *The unit object $\mathbb{1}_{\mathsf{C}} \in \mathsf{C}$ is final.*
2. *For objects $X, Y \in \mathsf{C}$, the morphisms*

$$X \xleftarrow{\simeq} X \otimes \mathbb{1}_{\mathsf{C}} \leftarrow X \otimes Y \to \mathbb{1}_{\mathsf{C}} \otimes Y \xrightarrow{\simeq} Y$$

*exhibits $X \otimes Y$ as a product of $X$ and $Y$ in $\mathsf{C}$.*

**Theorem 1.3** (HA.2.4.1.19)**.** *The functor $\mathsf{C}^\otimes \mapsto \mathsf{C}$ defines an equivalence between*

- (i) *The $\infty$-category of (small) Cartesian symmetric monoidal $\infty$-categories, and symmetric monoidal functors between them;*
- (ii) *The $\infty$-category of (small) $\infty$-categories that admit finite products, and finite-product-preserving functors between them.*

1.4. In particular, for an $\infty$-category $\mathsf{C}$ that admits finite products, there is an essentially unique Cartesian symmetric monoidal structure on it, which we denote by $\mathsf{C}^\times \to \mathsf{Fin}_*$. Note that when using this notation, we need to equip it with an identification $\mathsf{C}^\times_{\langle 1 \rangle} \simeq \mathsf{C}$.

1.5. We can equip $\mathsf{Cat}_\infty$ with the Cartesian symmetric monoidal structure $\mathsf{Cat}_\infty^\times \to \mathsf{Fin}_*$ and consider $\mathsf{O}$-*algebra object* in it. Note that such objects are *a priori* different from $\mathsf{O}$-monoidal $\infty$-categories: the former is a functor $\mathsf{O}^\otimes \to \mathsf{Cat}_\infty^\times$ while the latter is a functor $\mathsf{O}^\otimes \to \mathsf{Cat}_\infty$ (via straightening). To compare these notions, we need a functor

$$\mathsf{Cat}_\infty^\times \to \mathsf{Cat}_\infty$$

that sends an object $(\mathsf{D}_1, \cdots, \mathsf{D}_n) \in (\mathsf{Cat}_\infty^\times)_{\langle n \rangle}$ to $\prod_{i=1}^n \mathsf{D}_i \in \mathsf{Cat}_\infty$. As a sanity check, note that the inert morphism $(\mathsf{D}_1, \cdots, \mathsf{D}_n) \to \mathsf{D}_k$ can be sent to the projection functor $\mathsf{D}_1 \times \cdots \times \mathsf{D}_n \to \mathsf{D}_k$. It turns out this can be generalized to any *Cartesian* symmetric monoidal structure $\mathsf{C}^\times$, including $\mathsf{Cat}_\infty^\times$.

**Exercise 1.6.** *Convince yourself that there is no similar functor $\mathsf{C}^\otimes \to \mathsf{C}$ for general symmetric monoidal $\infty$-category $\mathsf{C}$.*

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1.7. To give the precise construction of the desired functor $\mathsf{C}^\times \to \mathsf{C}$, we need some terminologies.

**Definition 1.8.** *Let $p : \mathsf{O}^\otimes \to \mathsf{Fin}_*$ be an $\infty$-operad and $\mathsf{D}$ be an $\infty$-category that admits finite products. Let*

$$\mathsf{Mon}_\mathsf{O}(\mathsf{D}) \subseteq \mathsf{Fun}(\mathsf{O}^\otimes, \mathsf{D})$$

*be the full sub-$\infty$-category consisting of functors $M : \mathsf{O}^\otimes \to \mathsf{D}$ such that:*

(i) *For any multi-color $X = (X_1, \cdots, X_n)$ in $\mathsf{O}^\otimes_{\langle n\rangle}$, the canonical morphisms $M(X) \to M(X_i)$ induce an equivalence*

$$M(X) \xrightarrow{\simeq} \prod_{i=1}^n M(X_i).$$

*Objects in $\mathsf{Mon}_\mathsf{O}(\mathsf{C})$ are called* $\mathsf{O}$***-monoids*** *in $\mathsf{C}$.*

**Example 1.9.** *When $\mathsf{C} = \mathsf{Cat}_\infty$, an $\mathsf{O}$-monoid in $\mathsf{Cat}_\infty$ is just an $\mathsf{O}$-monoidal $\infty$-category via straightening.*

**Definition 1.10.** *Let $p : \mathsf{O}^\otimes \to \mathsf{Fin}_*$ be a symmetric monoidal $\infty$-category and $\mathsf{D}$ be an $\infty$-category that admits finite products. Let*

$$\mathsf{Fun}^\times(\mathsf{O}^\otimes, \mathsf{D}) \subseteq \mathsf{Mon}_\mathsf{O}(\mathsf{D})$$

*be the full sub-$\infty$-category consisting of $\mathsf{O}$-monoids $M : \mathsf{O}^\otimes \to \mathsf{D}$ such that:*

(ii) *For any $p$-coCartesian morphism $X = (X_1, \cdots, X_n) \to Y$ over the unique active morphism $\langle n\rangle \to \langle 1\rangle$, the image under $M$ is an equivalence*

$$M(X) \xrightarrow{\simeq} M(Y).$$

**Remark 1.11.** *Informally, (i) and (ii) say that $M$ sends $\bigotimes_{i=1}^n X_i$ to $\prod_{i=1}^n M(X_i)$.*

**Proposition 1.12** (HA.2.4.1.6)**.** *Let $\mathsf{C}$ and $\mathsf{D}$ be $\infty$-categories that admit finite products. Then restriction along $\mathsf{C} \to \mathsf{C}^\times$ defines an equivalence*

$$\mathsf{Fun}^\times(\mathsf{C}^\times, \mathsf{D}) \xrightarrow{\simeq} \mathsf{Fun}^\times(\mathsf{C}, \mathsf{D}),$$

*where*

$$\mathsf{Fun}^\times(\mathsf{C}, \mathsf{D}) \subset \mathsf{Fun}(\mathsf{C}, \mathsf{D})$$

*is the full sub-$\infty$-category of functors $\mathsf{C} \to \mathsf{D}$ that preserves finite products.*

1.13. It follows that there is an essentially unique functor

$$\pi : \mathsf{C}^\times \to \mathsf{C}$$

in $\mathsf{Fun}^\times(\mathsf{C}^\times, \mathsf{C})$ such that the composition $\mathsf{C} \to \mathsf{C}^\times \xrightarrow{\pi} \mathsf{C}$ is the identity functor.

**Proposition 1.14** (HA.2.4.1.7)**.** *Let $\mathsf{C}$ be an $\infty$-category that admits finite products. We equip $\mathsf{C}$ with the Cartesian symmetric monoidal structure. Then:*

(1) *For any $\infty$-operad $\mathsf{O}^\otimes \to \mathsf{Fin}_*$, composing with $\pi : \mathsf{C}^\times \to \mathsf{C}$ induces an equivalence*

$$\mathsf{Alg}_\mathsf{O}(\mathsf{C}) \xrightarrow{\simeq} \mathsf{Mon}_\mathsf{O}(\mathsf{C}).$$

(2) *For any symmetric monoidal $\infty$-category $\mathsf{O}^\otimes \to \mathsf{Fin}_*$, composing with $\pi : \mathsf{C}^\times \to \mathsf{C}$ induces an equivalence*

$$\mathsf{Fun}^\otimes(\mathsf{O}, \mathsf{C}) \xrightarrow{\simeq} \mathsf{Fun}^\times(\mathsf{O}^\otimes, \mathsf{C}).$$

**Remark 1.15.** *Note that (2) implies the functor in Theorem 1.3 is fully faithful. Indeed, in HA, Lurie first gave an explicit construction of* a *Cartesian symmetric monoidal structure* $\mathsf{C}^{\times}$*, then verified it satisfies the claims in Proposition 1.12 and 1.14, and finally deduced Theorem 1.3 from them. For details, see HA.2.4.1.*

**Remark 1.16.** *In HA, Lurie used the following terminologies:*

- *Objects in* $\mathsf{Mono_O(C)}$ *are also called* lax Cartesian structures *on* $\mathsf{O}^{\otimes}$*;*
- *Objects in* $\mathsf{Fun}^{\times}(\mathsf{O}^{\otimes},\mathsf{D})$ *are called* weak Cartesian structures *on* $\mathsf{O}^{\otimes}$*.*
- *A* Cartesian structure *on* $\mathsf{O}^{\otimes}$ *is a weak one such that the composition* $\mathsf{O}\to\mathsf{O}^{\otimes}\to\mathsf{D}$ *is an equivalence.*

## 2. CoCartesian symmetric monoidal structure

2.1. Using the automorphism $\mathsf{Cat}_\infty\to\mathsf{Cat}_\infty$, $\mathsf{C}\mapsto\mathsf{C}^{\mathsf{op}}$, we obtain the dual version of Definition 1.2 and Theorem 1.3:

**Definition 2.2.** *We say a symmetric monoidal structure* $\mathsf{C}^{\otimes}\to\mathsf{Fin}_*$ *on* $\mathsf{C}$ *is* ***co-Cartesian*** *if it satisfies the following properties:*

(1) *The unit object* $\mathbb{1}_{\mathsf{C}}\in\mathsf{C}$ *is initial.*
(2) *For objects* $X,Y\in\mathsf{C}$*, the morphisms*

$$X\xrightarrow{\simeq}X\otimes\mathbb{1}_{\mathsf{C}}\to X\otimes Y\leftarrow\mathbb{1}_{\mathsf{C}}\otimes Y\xleftarrow{\simeq}Y$$

*exhibits* $X\otimes Y$ *as a coproduct of* $X$ *and* $Y$ *in* $\mathsf{C}$*.*

**Corollary 2.3.** *The functor* $\mathsf{C}^{\otimes}\mapsto\mathsf{C}$ *defines an equivalence between*

(i) *The* $\infty$*-category of (small) coCartesian symmetric monoidal* $\infty$*-categories, and symmetric monoidal functors between them;*
(ii) *The* $\infty$*-category of (small)* $\infty$*-categories that admit finite coproducts, and finite-coproduct-preserving functors between them.*

2.4. For an $\infty$-category $\mathsf{C}$ that admits finite coproducts, we denote the coCartesian symmetric monoidal structure on it by $\mathsf{C}^{\sqcup}\to\mathsf{Fin}_*$.

2.5. Note however that by Exercise 1.6, Proposition 1.12 and 1.14 cannot survive in the coCartesian setting. As a compensate, we have the following result:

**Proposition 2.6** (HA.2.4.3.9)**.** *Let* $\mathsf{C}$ *be an* $\infty$*-category that admits finite coproducts. We equip* $\mathsf{C}$ *with the coCartesian symmetric monoidal structure. Then for any* unital $\infty$*-operad* $\mathsf{O}^{\otimes}$*, the forgetful functor*

$$\mathsf{Alg_O(C)}\to\mathsf{Fun(O,C)}$$

*is an equivalence.*

**Remark 2.7.** *Let* $A:\mathsf{O}^{\otimes}\to\mathsf{C}^{\sqcup}$ *be an* $\mathsf{O}$*-algebra. The above proposition says the functor* $A$ *is determined by its restriction on* $\mathsf{O}$*. Informally speaking, for any active morphism* $X:=(X_1,\cdots,X_n)\to Y$ *in* $\mathsf{O}^{\otimes}$*, its image* $A(X)\to A(Y)$ *can be described as follows. First, this morphism essentially uniquely factors as*

$$A(X)\xrightarrow{f}\coprod_{i=1}^{n}A(X_i)\xrightarrow{g}A(Y)$$

*such that* $f$ *is coCartesian over* $\mathsf{Fin}_*$ *while* $g$ *is contained in* $\mathsf{C}$*. The morphism* $g$ *is determined by morphisms* $A(X_i)\to A(Y)$*, which by definition, must be the image of* $X_i\to(X_1,\cdots,X_n)\to Y$ *under* $A$*. Here the morphism* $X_i\to(X_1,\cdots,X_n)$ *is guaranteed by the assumption that* $\mathsf{O}^{\otimes}$ *is unit.*

**Exercise 2.8.** *Describe the essential image of* $\mathsf{Fun}^{\otimes}(\mathsf{O},\mathsf{C}) \subseteq \mathsf{Alg}_{\mathsf{O}}(\mathsf{C})$ *in* $\mathsf{Fun}(\mathsf{O},\mathsf{C})$.

**Remark 2.9.** *In fact, for* any $\infty$*-category* $\mathsf{O}$ *that not necessarilly admits finite coproducts, there is a canonical unital* $\infty$*-operad* $\mathsf{O}^{\sqcup}$ *such that the analogue of Proposition 2.6 is true. Such* $\infty$*-operads are called* ***coCartesian*** $\infty$***-operads****. For more details, in HA.2.4.3.*

## 3. Tensor products of $\infty$-categories

**Theorem-Definition 3.1** (HA.4.8.1.15)**.** *Let* $\widehat{\mathsf{Cat}_\infty}$ *be equiped with the Cartesian symmetric monoidal structure. Consider the 1-full sub-*$\infty$*-category*

$$(\mathsf{Pr}^{\mathsf{L}})^{\otimes} \subseteq (\widehat{\mathsf{Cat}_\infty})^{\times}$$

*defined as follows:*

- *An object* $(\mathsf{C}_1, \cdots, \mathsf{C}_n)$ *is contained in* $(\mathsf{Pr}^{\mathsf{L}})^{\otimes}$ *iff each* $\mathsf{C}_k$ *is presentable.*
- *A morphism* $(\mathsf{C}_1, \cdots, \mathsf{C}_m) \to (\mathsf{D}_1, \cdots, \mathsf{D}_n)$ *defined over* $\alpha : \langle m\rangle \to \langle n\rangle$ *is contained in* $(\mathsf{Pr}^{\mathsf{L}})^{\otimes}$ *iff for each* $j \in \langle n\rangle^{\circ}$*, the corresponding functor*

  $$\prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i \to \mathsf{D}_j$$

  *preserves colimits in each factor.*

*Then* $(\mathsf{Pr}^{\mathsf{L}})^{\otimes} \to \mathsf{Fin}_*$ *defines a symmetric monoidal structure on* $\mathsf{Pr}^{\mathsf{L}}$*. We call the multiplication functor*

$$- \otimes - : \mathsf{Pr}^{\mathsf{L}} \times \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$$

*the* ***Lurie tensor product****.*

3.2. By definition, for $\mathsf{C}_i \in \mathsf{Pr}^{\mathsf{L}}$, there is a coCartesian morphism $(\mathsf{C}_1, \cdots, \mathsf{C}_n) \to \bigotimes_{i=1}^n \mathsf{C}_i$ in $(\mathsf{Pr}^{\mathsf{L}})^{\otimes} \to \mathsf{Fin}_*$. This morphism is almost *never* coCartesian in the bigger fibration $(\widehat{\mathsf{Cat}_\infty})^{\times} \to \mathsf{Fin}_*$. It follows that we have an essentially unique factorization in $(\widehat{\mathsf{Cat}_\infty})^{\times}$

$$(\mathsf{C}_1, \cdots, \mathsf{C}_n) \to \prod_{i=1}^n \mathsf{C}_i \to \bigotimes_{i=1}^n \mathsf{C}_i.$$

such that the first morphism is coCartesian and the second is contained in $\widehat{\mathsf{Cat}_\infty}$.

**Exercise 3.3.** *For any* $\mathsf{D} \in \mathsf{Pr}^{\mathsf{L}}$*, precomposing with* $\prod_{i=1}^n \mathsf{C}_i \to \bigotimes_{i=1}^n \mathsf{C}$ *induces an equivalence between the* spaces *of*

- colimit-preserving functors $\bigotimes_{i=1}^n \mathsf{C} \to \mathsf{D}$*;*
- *functors* $\prod_{i=1}^n \mathsf{C}_i \to \mathsf{D}$ *that* preserve colimits in each factor.

*The same is true if we consider* $\infty$-categories *of these functors.*

**Exercise 3.4.** *Show that* $\mathsf{Spc} \in \mathsf{Pr}^{\mathsf{L}}$ *is the unit object. What is the functor*

$$\mathsf{Spc} \times \mathsf{Spc} \to \mathsf{Spc} \otimes \mathsf{Spc} \simeq \mathsf{Spc}?$$

**Exercise 3.5.** *Show that for any* $\mathsf{C}, \mathsf{D}, \mathsf{E} \in \mathsf{Pr}^{\mathsf{L}}$

$$\mathsf{LFun}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \simeq \mathsf{LFun}(\mathsf{C}, \mathsf{LFun}(\mathsf{D}, \mathsf{E})).$$

*In particular,* $\mathsf{LFun}(-,-)$ *calculates the* ***internal mapping objects*** *with respect to the Lurie tensor product.*

**Exercise 3.6.** *For small* $\infty$*-categories* $\mathsf{C}_0$ *and* $\mathsf{D}_0$*, show that*

$$\mathsf{PShv}(\mathsf{C}_0) \otimes \mathsf{PShv}(\mathsf{D}_0) \simeq \mathsf{PShv}(\mathsf{C}_0 \times \mathsf{D}_0).$$

**Theorem 3.7** (HA.4.8.1.17)**.** *For $\mathsf{C}, \mathsf{D} \in \mathsf{Pr}^{\mathsf{L}}$, there is a canonical equivalence*

$$\mathsf{C} \otimes \mathsf{D} \simeq \mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}),$$

*where $\mathsf{RFun}(-,-)$ consists of functors that preserve small limits.*

**Remark 3.8.** *Roughly speaking, the above equivalence is obtained as follows. For any testing object $\mathsf{E} \in \mathsf{Pr}^{\mathsf{L}}$, we have functors functorial in $\mathsf{E}$:*

$$\begin{array}{rl} & \mathsf{LFun}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E}) \\ \simeq & \mathsf{LFun}(\mathsf{C}, \mathsf{LFun}(\mathsf{D}, \mathsf{E})) \\ \to & \mathsf{LFun}(\mathsf{C}, \mathsf{RFun}(\mathsf{E}, \mathsf{D})^{\mathsf{op}}) \\ \simeq & \mathsf{LFun}(\mathsf{C}, \mathsf{LFun}(\mathsf{E}^{\mathsf{op}}, \mathsf{D}^{\mathsf{op}})) \\ \simeq & \mathsf{LFun}(\mathsf{E}^{\mathsf{op}}, \mathsf{LFun}(\mathsf{C}, \mathsf{D}^{\mathsf{op}})) \\ \simeq & \mathsf{RFun}(\mathsf{E}, \mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}))^{\mathsf{op}} \\ \leftarrow & \mathsf{LFun}(\mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}), \mathsf{E}), \end{array}$$

*where the non-invertible functors are obtained by passing to right adjoints:*

$$\mathsf{LFun}(X, Y) \to \mathsf{RFun}(Y, X)^{\mathsf{op}}, \ F \mapsto F^{\mathsf{R}}.$$

*Such functors is always fully faithful and its essential image contains of* accessible *right functor $Y \to X$. A careful study shows that the essential images of $\mathsf{LFun}(\mathsf{C} \otimes \mathsf{D}, \mathsf{E})$ and $\mathsf{LFun}(\mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}), \mathsf{E})$ under these embeddings coincide.*

**Warning 3.9.** *The equivalence $\mathsf{C} \otimes \mathsf{D} \simeq \mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D})$ is* not *functorial in $\mathsf{C}$ and $\mathsf{D}$ in the naive way.*

**Exercise 3.10.** *Let $F : \mathsf{C}_1 \to \mathsf{C}_2$ be a morphism in $\mathsf{Pr}^{\mathsf{L}}$. Show that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{C}_2 \otimes \mathsf{D} & \xrightarrow{\simeq} & \mathsf{RFun}(\mathsf{C}_2^{\mathsf{op}}, \mathsf{D}) \\ \downarrow {\scriptstyle (F \otimes \mathsf{Id}_{\mathsf{D}})^{\mathsf{R}}} & & \downarrow {\scriptstyle - \circ F^{\mathsf{op}}} \\ \mathsf{C}_1 \otimes \mathsf{D} & \xrightarrow{\simeq} & \mathsf{RFun}(\mathsf{C}_2^{\mathsf{op}}, \mathsf{D}). \end{array}$$

**Exercise 3.11.** *Let $G : \mathsf{D}_1 \to \mathsf{D}_2$ be a morphism in $\mathsf{Pr}^{\mathsf{L}}$. Show that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{C} \otimes \mathsf{D}_2 & \xrightarrow{\simeq} & \mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}_2) \\ \downarrow {\scriptstyle (\mathsf{Id}_{\mathsf{C}} \otimes G)^{\mathsf{R}}} & & \downarrow {\scriptstyle G^{\mathsf{R}} \circ -} \\ \mathsf{C} \otimes \mathsf{D}_1 & \xrightarrow{\simeq} & \mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}_1). \end{array}$$

**Exercise 3.12.** *Use either Exercise 3.10 or 3.11 to deduce that the Lurie tensor product $- \otimes - : \mathsf{Pr}^{\mathsf{L}} \times \mathsf{Pr}^{\mathsf{L}} \to \mathsf{Pr}^{\mathsf{L}}$ preserves colimits in each factor.*

**Exercise 3.13.** *Let $\mathsf{C}, \mathsf{D} \in \mathsf{Pr}^{\mathsf{L}}$. Recall we have adjunctions $\mathsf{D} \rightleftarrows \mathsf{D}_*$, where the left adjoint sends $X$ to $X_+ := X \sqcup \{*\}$.*

1. *Show that there is a canonical equivalence $\mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D}_*) \to \mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D})_*$ compatible with forgetful functors to $\mathsf{RFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{D})$.*
2. *Deduce that there is a canonical equivalence $\mathsf{C} \otimes \mathsf{D}_* \simeq (\mathsf{C} \otimes \mathsf{D})_*$ compatible with the functors out of $\mathsf{C} \otimes \mathsf{D}$. In particular, $\mathsf{Spc}_* \otimes \mathsf{Spc}_* \simeq \mathsf{Spc}_*$.*

**Exercise 3.14.** *Use the above exercise to show that for any* $\mathsf{C} \in \mathsf{Pr}^{\mathsf{L}}$

$$\mathsf{C} \otimes \mathsf{Sptr} \simeq \mathsf{Sptr}(\mathsf{C}_*).$$

*In particular,* $\mathsf{Sptr} \otimes \mathsf{Sptr} \simeq \mathsf{Sptr}$.

## 4. Smash products

**Definition 4.1.** *Let* $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ *be a symmetric monoidal* $\infty$*-category. We say a commutative algebra* $A \in \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C})$ *is* ***idempotent*** *if* $A \otimes A \to A$ *is an equivalence.*

*Let*

$$\mathsf{Alg}^{\mathsf{idem}}_{\mathsf{Comm}}(\mathsf{C}) \subseteq \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C})$$

*be the full sub-*$\infty$*-category of idempotent commutative algebras in* $\mathsf{C}$.

**Proposition 4.2** (HA.4.8.2.9)**.** *Let* $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ *be a symmetric monoidal* $\infty$*-category. Then the forgetful functor*

$$\mathsf{Alg}^{\mathsf{idem}}_{\mathsf{Comm}}(\mathsf{C}) \to \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C}) \simeq \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C})_{\mathbb{1}_{\mathsf{C}}/} \to \mathsf{C}_{\mathbb{1}_{\mathsf{C}}/}$$

*is fully faithful, and its essential image contains those objects* $\mathbb{1}_{\mathsf{C}} \to X$ *such that*

$$X \simeq \mathbb{1}_{\mathsf{C}} \otimes X \to X \otimes X$$

*is an isomorphism.*

**Exercise 4.3.** *Let* $\mathsf{C} := \mathsf{Pr}^{\mathsf{L}}$ *be equiped with the Lurie symmetric monoidal structure. Show that the morphisms* $\mathbb{1}_{\mathsf{C}} \simeq \mathsf{Spc} \to \mathsf{Spc}_*$ *and* $\mathbb{1}_{\mathsf{C}} \simeq \mathsf{Spc} \to \mathsf{Sptr}$ *satisfy the conditions in the proposition. Deduce that we have essentially unique upgrades of* $\mathsf{Sptr}$ *and* $\mathsf{Spc}_*$ *to objects in*

$$\mathsf{Alg}_{\mathsf{Comm}}(\widehat{\mathsf{Cat}_{\infty}}) \simeq \mathsf{Mon}_{\mathsf{Comm}}(\widehat{\mathsf{Cat}_{\infty}})$$

*such that*

- *These symmetric monoidal structures are compatible with small colimits;*
- *The unit objects are given respectively by* $\mathbb{S} \in \mathsf{Sptr}$ *and* $\mathbb{S}^0 \in \mathsf{Spc}_*$.

**Definition 4.4.** *We denote the multiplication functors of the above symmetric monoidal structures by*

$$- \wedge - : \mathsf{Spc}_* \times \mathsf{Spc}_* \to \mathsf{Spc}_*$$

$$- \otimes - : \mathsf{Sptr} \times \mathsf{Sptr} \to \mathsf{Sptr},$$

*and call them the* ***smash products****.*

**Exercise 4.5.** *Show that* $\mathsf{Spc} \to \mathsf{Spc}_* \to \mathsf{Sptr}$ *are natually symmetric monoidal functors.*

**Proposition 4.6** (HA.4.8.2.10)**.** *Let* $\mathsf{C}^{\otimes} \to \mathsf{Fin}_*$ *be a symmetric monoidal* $\infty$*-category and* $A \in \mathsf{Alg}^{\mathsf{idem}}_{\mathsf{Comm}}(\mathsf{C})$*. Then* $\mathsf{Mod}^{\mathsf{Comm}}_A(\mathsf{C})^{\otimes}$ *is a symmetric monoidal* $\infty$*-category and the forgetful functor*

$$\mathsf{Mod}^{\mathsf{Comm}}_A(\mathsf{C})^{\otimes} \to \mathsf{C}^{\otimes}$$

*is fully faithful. Moreover, an object* $M \in \mathsf{C}$ *is contained in the essential image iff*

$$M \simeq \mathbb{1}_{\mathsf{C}} \otimes M \to A \otimes M$$

*is an isomorphism.*

**Corollary 4.7.** *The functor*

$$\mathsf{Mod}^{\mathsf{Comm}}_{\mathsf{Spc}_*}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Pr}^{\mathsf{L}}$$

*is fully faithful with its essential image consists exactly of pointed presentable $\infty$-categories. Moreover, these $\infty$-categories are stable under the Lurie tensor product.*

**Corollary 4.8.** *The functor*

$$\mathsf{Mod}^{\mathsf{Comm}}_{\mathsf{Sptr}}(\mathsf{Pr}^{\mathsf{L}}) \to \mathsf{Pr}^{\mathsf{L}}$$

*is fully faithful with its essential image consists exactly of stable presentable $\infty$-categories. Moreover, these $\infty$-categories are stable under the Lurie tensor product.*

**Remark 4.9.** *Note that the above corollary implies* $\mathsf{Sptr}$ *is the initial object in* $\mathsf{Alg}_{\mathsf{Comm}}(\mathsf{PrSt}^{\mathsf{L}})$*, which gives an intrinsic characterization of the smash products of spectra.*

## Appendix A. More on smash products

A.1. The smash product on the $\infty$-*category* $\mathsf{Sptr}$ can be rectified to a symmetric monoidal structure on a simplicial model category. The most successful one is the model category of *symmetric spectra*.

A.2. **Suggested readings.** [HSS00].

## References

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *Journal of the American Mathematical Society*, 13(1):149–208, 2000.