LECTURE 26

In this lecture, we introduce tensor products of ∞-operads. The idea is to have a symmetric monoidal structure $\odot$ on Op_∞, such that

$$\mathsf{Alg}_{\mathsf{O}_1}(\mathsf{Alg}_{\mathsf{O}_2}(\mathsf{C})) \simeq \mathsf{Alg}_{\mathsf{O}_1 \odot \mathsf{O}_2}(\mathsf{C})$$

holds for any symmetric monoidal ∞-category C.

1. Definition

1.1. Recall Op_∞ is a 1-full sub-∞-category of $(\mathsf{Cat}_\infty)_{/\mathsf{Fin}_*}$. We first construct a symmetric monoidal structure on Fin_*.

Construction 1.2. *Recall* Spc_* *has a symmetric monoidal structure* $(\mathsf{Spc}_*)^\wedge \to \mathsf{Comm}^\otimes$ *given by smash products of pointed spaces. Let* $(\mathsf{Fin}_*)^\wedge \subseteq (\mathsf{Spc}_*)^\wedge$ *be the full subcategory consisting of objects* $(X_1, \cdots, X_n)$ *such that each* X_k *is an object in* $\mathsf{Fin}_* \subseteq \mathsf{Spc}_*$. *It is easy to see* $(\mathsf{Fin}_*)^\wedge \to \mathsf{Comm}^\otimes$ *exhibits* Fin_* *as a symmetric monoidal* ordinary *category, with the multiplication functor given by*

$$- \wedge - : \mathsf{Fin}_* \times \mathsf{Fin}_* \to \mathsf{Fin}_*,\ (\langle m \rangle, \langle n \rangle) \to \langle mn \rangle.$$

In other words, we obtain an object

$$\mathsf{Fin}_* \in \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{Cat}_\infty)$$

where Cat_∞ *is equipped with the Cartesian symmetric monoidal structure.*

Exercise 1.3. *Construct the symmetric monoidal category* $(\mathsf{Fin}_*, \wedge)$ *explicitly by providing the associators, unitors and swap natural transformations.*

1.4. The next step is to construct a symmetric monoidal structure on $(\mathsf{Cat}_\infty)_{/\mathsf{Fin}_*}$ using that on Fin_*. This construction works for any slice ∞-categories.

Proposition-Construction 1.5 (HA.2.2.2.4)**.** *Let* $\mathsf{C}^\otimes \to \mathsf{Comm}^\otimes$ *be any* ∞*-operad and* $A \in \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C})$ *be a commutative algebra. Then there is a canonical* ∞*-operad* $(\mathsf{C}_{/A})^\otimes$ *equipped with an* ∞*-operad map*

$$q : (\mathsf{C}_{/A})^\otimes \to \mathsf{C}^\otimes$$

such that

1. *The* ∞*-category* $(\mathsf{C}_{/A})^\otimes_{\langle 1 \rangle}$ *is identified with* $\mathsf{C}_{/A(\langle 1 \rangle)}$.
2. *A morphism* f *in* $(\mathsf{C}_{/A})^\otimes$ *is inert iff* $q(f)$ *is inert.*
3. *If* $\mathsf{C}^\otimes$ *is symmetric monoidal, so is* $(\mathsf{C}_{/A})^\otimes$*, and* q *is a symmtric monoidal functor.*

Remark 1.6. *Informally speaking, for objects* $M \in \mathsf{C}_{/A}$ *and* $N \in \mathsf{C}_{/A}$*, their tensor product is the object in* $\mathsf{C}_{/A}$ *given by the composition* $M \otimes N \to A \otimes A \to A$.

Remark 1.7. *In fact, similar construction exists even when* $\mathsf{Comm}^\otimes$ *is replaced by a general* ∞*-operad* $\mathsf{O}^\otimes$.

Date: Dec. 24, 2024.

1.8. In particular, the object $\mathsf{Fin}_* \in \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{Cat}_\infty)$ induces a symmetric monoidal structure on $(\mathsf{Cat}_\infty)_{/\mathsf{Fin}_*}$.

Warning 1.9. *This is* not *the Cartesian symmetric monoidal structure on* $(\mathsf{Cat}_\infty)_{/\mathsf{Fin}_*}$.

Theorem-Definition 1.10 (HA.2.2.5.7[1])**.** *Let*

$$(\mathsf{Op}_\infty)^\odot \subseteq \big((\mathsf{Cat}_\infty)_{/\mathsf{Fin}_*}\big)^\otimes$$

be the 1-full sub-∞-category such that

(i) *An object $(\mathsf{C}_1,\cdots,\mathsf{C}_n)$ is contained in $(\mathsf{Op}_\infty)^\odot$ iff each $\mathsf{C}_k \to \mathsf{Fin}_*$ is an ∞-operads;*
(ii) *A morphism $(\mathsf{C}_1,\cdots,\mathsf{C}_m) \to (\mathsf{D}_1,\cdots,\mathsf{D}_n)$ defined over $\alpha : \langle m\rangle \to \langle n\rangle$ is contained in $(\mathsf{Op}_\infty)^\odot$ iff for each $j \in \langle n\rangle^\circ$, the corresponding functor*

$$\prod_{i\in\alpha^{-1}(j)} \mathsf{C}_i \to \mathsf{D}_j \tag{1.1}$$

preserves inert morphisms in each factor.

Then $(\mathsf{Op}_\infty)^\odot \to \mathsf{Comm}^\otimes$ exhibits Op_∞ with a symmetric monoidal structure. We denote the multiplication functor by

$$(\mathsf{Op}_\infty)^\odot \times (\mathsf{Op}_\infty)^\odot \to (\mathsf{Op}_\infty)^\odot,\ (\mathsf{O}^\otimes, \mathsf{O}'^\otimes) \mapsto (\mathsf{O}\odot\mathsf{O}')^\otimes.$$

Remark 1.11. *Condition (ii) is equivalent to: the functor* (1.1) *sends any morphism $(f_i)_{i\in\alpha^{-1}(j)}$ such that each f_i is an inert morphism in C_i to an inert morphism in D_j. Indeed, this follows from the fact that inert morphisms are closed under compositions.*

1.12. Similar to the case of the Lurie tensor products, for ∞-operads $\mathsf{O}_i^\otimes \in \mathsf{Op}_\infty$, there is a canonical functor

$$\prod_{i=1}^n \mathsf{O}_i^\otimes \to (\bigodot_{i=1}^n \mathsf{O}_i)^\otimes$$

defined over $\prod_{i=1}^n \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$ that preserves inert morphisms in each factor.

Exercise 1.13. *For any $\mathsf{C}^\otimes \in \mathsf{Op}_\infty$, composing with the above functor induces an equivalence between the* spaces *of*

- *∞-operads maps $(\bigodot_{i=1}^n \mathsf{O}_i)^\otimes \to \mathsf{C}^\otimes$;*
- *functors $\prod_{i=1}^n \mathsf{O}_i^\otimes \to \mathsf{C}^\otimes$ defined over $\prod_{i=1}^n \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$ that preserves inert morphisms in each factor.*

The same is true if we consider the ∞-categories *of these functors.*

2. Tensor with $\mathsf{Triv}^\otimes$, $\mathbb{E}_0^\otimes$ and $\mathsf{Comm}^\otimes$

Exercise 2.1. *Show that $\mathsf{Triv}^\otimes \in \mathsf{Op}_\infty$ is the unit object.*

Exercise 2.2. $(\mathbb{E}_0 \odot \mathbb{E}_0)^\otimes \simeq \mathbb{E}_0^\otimes$ *and* $(\mathsf{Comm}\odot\mathsf{Comm})^\otimes \simeq \mathsf{Comm}^\otimes$.

[1]In HA.2.2.5.7, Lurie actually constructed a symmetric monoidal structure on the model category of ∞-*preoperads*. Unwinding the definitions, one can show the underlying symmetric monoidal ∞-category of this symmetric monoidal model category s is equivalent to $(\mathsf{Op}_\infty)^\odot$.

Proposition 2.3 (HA.2.3.1.9)**.** *The obvious functor* $\mathsf{Triv}^{\otimes} \to \mathbb{E}_0^{\otimes}$ *makes* $\mathbb{E}_0^{\otimes}$ *an object in* $\mathsf{Alg}^{\mathsf{idem}}_{\mathsf{Comm}}(\mathsf{Op}_\infty)$*, and*

$$\mathsf{Mod}^{\mathsf{Comm}}_{\mathbb{E}_0^{\otimes}}(\mathsf{Op}_\infty) \subseteq \mathsf{Op}_\infty$$

consists exactly of unital ∞-operads. In particular, the latter are closed under $\odot$*.*

Proposition 2.4 (HA.3.2.4.6)**.** *The obvious functor* $\mathsf{Triv}^{\otimes} \to \mathsf{Comm}^{\otimes}$ *makes* $\mathsf{Comm}^{\otimes}$ *an object in* $\mathsf{Alg}^{\mathsf{idem}}_{\mathsf{Comm}}(\mathsf{Op}_\infty)$*, and we have*

$$\mathsf{Mod}^{\mathsf{Comm}}_{\mathsf{Comm}^{\otimes}}(\mathsf{Op}_\infty) \subseteq \mathsf{Op}_\infty$$

consists exactly of coCartesian ∞-operads. In particular, the latter are closed under $\odot$*.*

Exercise 2.5. *Show that the tensor product of* $\mathsf{C}^{\sqcup}$ *and* $\mathsf{D}^{\sqcup}$ *is equivalent to* $(\mathsf{C}\times\mathsf{D})^{\sqcup}$*.*

3. Algebras in algebras

3.1. Now we turn to the equivalence

$$\mathsf{Alg}_{\mathsf{O}_1}(\mathsf{Alg}_{\mathsf{O}_2}(\mathsf{C})) \simeq \mathsf{Alg}_{\mathsf{O}_1\odot\mathsf{O}_2}(\mathsf{C}).$$

The first task is to define an ∞-operad $\mathsf{Alg}_{\mathsf{O}_2}(\mathsf{C})^{\otimes}$.

Construction 3.2. *Let* $\mathsf{O}_1^{\otimes}, \mathsf{O}_2^{\otimes} \in \mathsf{Op}_\infty$ *and* $(\mathsf{O}_1 \odot \mathsf{O}_2)^{\otimes} \to \mathsf{O}^{\otimes}$ *be an ∞-operad map. Consider the correspondence*

$$\mathsf{O}_1^{\otimes} \xleftarrow{p} \mathsf{O}_1^{\otimes} \times \mathsf{O}_2^{\otimes} \xrightarrow{m} \mathsf{O}^{\otimes}.$$

For any ∞-operad $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *over* $\mathsf{O}^{\otimes}$*, let*

$$\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes} \subseteq p_* \circ m^*(\mathsf{C}^{\otimes})$$

be the full sub-∞-category consisting of objects

$$F \in \mathsf{Fun}_{\mathsf{O}_1^{\otimes}\times\mathsf{O}_2^{\otimes}}(\{X\}\times\mathsf{O}_2^{\otimes}, m^*(\mathsf{C}^{\otimes})) \simeq \mathsf{Fun}_{\mathsf{O}^{\otimes}}(\{X\}\times\mathsf{O}_2^{\otimes}, \mathsf{C}^{\otimes})$$

such that the corresponding functor $\{X\}\times\mathsf{O}_2^{\otimes} \to \mathsf{C}^{\otimes}$ *preserves inert morphisms, where* $X \in \mathsf{O}_1^{\otimes}$ *is the image of* F *under the functor* $p_* \circ m^*(\mathsf{C}^{\otimes}) \to \mathsf{O}_1^{\otimes}$*.*

Remark 3.3. *By definition, the fiber* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes}_X$ *of* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes}$ *at* $X \in \mathsf{O}_1^{\otimes}$ *is the ∞-category of* O_2*-algebras in* C *relative to* O*, where the structure functor* $\mathsf{O}_2^{\otimes} \to \mathsf{O}^{\otimes}$ *is given by*

$$\mathsf{O}_2^{\otimes} \simeq \{X\}\times\mathsf{O}_2^{\otimes} \to \mathsf{O}_1^{\otimes}\times\mathsf{O}_2^{\otimes} \to \mathsf{O}^{\otimes}.$$

Exercise 3.4. *Show that for any object* $Y \in \mathsf{O}_2^{\otimes}$*, the diagram*

$$\begin{array}{ccccc} & & \mathsf{O}_1^{\otimes} & & \\ & \swarrow & \downarrow{\scriptstyle (\mathsf{Id},\{Y\})} & \searrow & \\ \mathsf{O}_1^{\otimes} & \xleftarrow{\;p\;} & \mathsf{O}_1^{\otimes}\times\mathsf{O}_2^{\otimes} & \xrightarrow{\;m\;} & \mathsf{O}^{\otimes} \end{array}$$

induces a natural functor

$$\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{O}_1^{\otimes} \underset{\mathsf{O}^{\otimes}}{\times} \mathsf{C}^{\otimes}.$$

In particular, we obtain a functor

$$\mathsf{ev}_Y : \mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{C}^{\otimes}.$$

Proposition 3.5 (HA.3.2.4.3)**.** *In the above setting:*

(1) *The functor* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{O}_1^{\otimes}$ *exhibits* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes}$ *as an* ∞*-operad over* $\mathsf{O}_1^{\otimes}$*. Moreover, a morphism in* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes}$ *is inert iff*
   – *its image in* $\mathsf{O}_1^{\otimes}$ *is inert;*
   – *for any* $Y \in \mathsf{O}_2$*, its image under* ev_Y *is inert.*
(2) *If* $\mathsf{C}^{\otimes} \to \mathsf{O}^{\otimes}$ *is an* O*-monoidal* ∞*-category, then* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{O}_1^{\otimes}$ *is an* O_1*-monoidal* ∞*-category. Moreover, a morphism in* $\mathsf{Alg}_{\mathsf{O}_2/\mathsf{O}}(\mathsf{C})^{\otimes}$ *is coCartesian over* $\mathsf{O}_1^{\otimes}$ *iff for any* $Y \in \mathsf{O}_2$*, its image under* ev_Y *is coCartesian over* $\mathsf{O}^{\otimes}$*.*

Example 3.6. *For any* ∞*-operad* $\mathsf{O}^{\otimes}$*, we have an* ∞*-operad map* $(\mathsf{Comm} \odot \mathsf{O})^{\otimes} \to \mathsf{Comm}^{\otimes}$*. It follows that for any* ∞*-operad* $\mathsf{C}^{\otimes}$*, there is a canonical* ∞*-operad* $\mathsf{Alg}_{\mathsf{O}}(\mathsf{C})^{\otimes}$*, equipped with an* ∞*-operad map to* $\mathsf{C}^{\otimes}$*, whose underlying* ∞*-category of colors is* $\mathsf{Alg}_{\mathsf{O}}(\mathsf{C})$*. Moreover, if* C *is symmetric monoidal, so is* $\mathsf{Alg}_{\mathsf{O}}(\mathsf{C})$*, and the forgetful functor* $\mathsf{Alg}_{\mathsf{O}}(\mathsf{C})^{\otimes} \to \mathsf{C}^{\otimes}$ *is symmetric monoidal.*

Exercise 3.7. *In the setting of Construction 3.2, when* $\mathsf{O}^{\otimes} := (\mathsf{O}_1 \odot \mathsf{O}_2)^{\otimes}$*, we have a canonical equivalence*

$$\mathsf{Alg}_{\mathsf{O}_1}(\mathsf{Alg}_{\mathsf{O}_2}(\mathsf{C})) \simeq \mathsf{Alg}_{\mathsf{O}_1 \odot \mathsf{O}_2}(\mathsf{C}).$$

Proposition 3.8 (HA.3.2.4.7)**.** *Let* $\mathsf{C}^{\otimes}$ *be a symmetric monoidal* ∞*-category. Then the symmetric monoidal structure on* $\mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C})^{\otimes}$ *is coCartesian. In particular,* $\mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C})$ *admits coproducts.*

Remark 3.9. *Informally speaking, the above proposition says coproducts of commutative algebras are given by tensor products.*

4. $\mathbb{E}_k$-ALGEBRAS

Definition 4.1. *Let* $k \geq 0$ *be a nonnegative integer. Define a topological operad* ${}^{\mathsf{t}}\mathbb{E}_k^{\otimes} \to \mathsf{Fin}_*$ *such that:*

- *an object in* ${}^{\mathsf{t}}\mathbb{E}_k^{\otimes}$ *is a marked finite set* $\langle n \rangle$*;*
- *the topological space of morphisms from* $\langle m \rangle$ *to* $\langle n \rangle$ *is*

$$\coprod_{\langle m \rangle \xrightarrow{\alpha} \langle n \rangle} \prod_{j \in \langle n \rangle^{\circ}} \mathsf{Rect}(\square^k \times \alpha^{-1}(j), \square^k),$$

  where $\square := (-1,1)$ *and* $\mathsf{Rect}(-,-)$ *is the space of open embeddings* $\square^k \times \alpha^{-1}(j) \to \square^k$ *such that for each* $i \in \alpha^{-1}(j)$*, the map* $f_i : \square^k \to \square^k$ *is a* ***rectilinear embedding****, i.e., a map of the form*

$$f_i(x_1, \cdots, x_k) = (a_1 x_1 + b_1, \cdots, a_k x_k + b_k).$$

Let $\mathbb{E}_k^{\otimes}$ *be the* ∞*-operad corresponding to* ${}^{\mathsf{t}}\mathbb{E}_k^{\otimes}$ *via the functor* $\mathsf{Cat}_{\mathsf{Top}} \to \mathsf{Cat}_{\infty}$*. We call it the* ***little cube*** ∞***-operad*** *of dimension* k*.*

Exercise 4.2. *The space of (active) m-ary operators in* $\mathbb{E}_k^{\otimes}$ *can be realized as the homotopy type of* m *distinct points in* $\square^k$*. In particular:*

(0) $\mathbb{E}_k^{\otimes}$ *is* unital*, i.e., the space of nullary operators is contractible;*
(1) $\mathbb{E}_k^{\otimes}$ *is* reduced*, i.e., the space of uniary operators is contractible;*
(2) *the space of binary operators in* $\mathbb{E}_k^{\otimes}$ *is homotopic to* $\mathbb{S}^{k-1}$*.*

Exercise 4.3. *The space of (active) m-ary operators in* $\mathbb{E}_k^{\otimes}$ *is* $(k-1)$*-connective, i.e., has* $\pi_i \simeq 0$ *for* $i < k-1$*.*

Exercise 4.4. *Show that $\mathbb{E}_0^{\otimes}$ defined above coincides with our previous definition: the subcategory of Fin_* consists of injective morphisms.*

Exercise 4.5. *Show that $\mathbb{E}_1^{\otimes} \simeq \mathsf{Assoc}^{\otimes}$.*

4.6. For $0 \le k \le k'$, we have an ∞-operad map $\mathbb{E}_k^{\otimes} \to \mathbb{E}_{k'}^{\otimes}$ which sends a rectilinear embedding $f : \square^k \to \square^k$ to $(f, \mathrm{id}) : \square^k \times \square^{k'-k} \to \square^k \times \square^{k'-k}$.

Proposition 4.7 (HA.5.1.1.5)**.** *We have*

$$\operatorname*{colim}_k \mathbb{E}_k^{\otimes} \simeq \mathsf{Comm}^{\otimes},$$

where the colimit is taken in Op_∞. In particular, for any ∞-operad C, we have

$$\mathsf{Alg}_{\mathsf{Comm}}(\mathsf{C}) \simeq \lim_k \mathsf{Alg}_{\mathbb{E}_k}(\mathsf{C}).$$

4.8. Motivatived by the above result, we also write $\mathbb{E}_\infty^{\otimes} := \mathsf{Comm}^{\otimes}$.

Construction 4.9. *Let $k, k' \ge 0$ be nonnegative integers. We have a functor*

$$^{\mathsf{t}}\mathbb{E}_k^{\otimes} \times {}^{\mathsf{t}}\mathbb{E}_{k'}^{\otimes} \to^{\mathsf{t}} \mathbb{E}_{k+k'}^{\otimes}$$

defined over $\mathsf{Fin}_* \times \mathsf{Fin}_* \xrightarrow{\wedge} \mathsf{Fin}_*$ induced by $\square^k \times \square^{k'} \simeq \square^{k+k'}$. Passing to ∞-categories, we obtain a functor

$$\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$$

Exercise 4.10. *Write down the explicit formula for the above functor, and check that it preserves inert morphisms in each factor.*

4.11. The following result is known as the *Dunn Additivity Theorem.*

Theorem 4.12 (HA.5.1.2.2)**.** *Let $k, k' \ge 0$ be nonnegative integers. The functor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ induces an equivalence*

$$(\mathbb{E}_k \odot \mathbb{E}_{k'})^{\otimes} \simeq \mathbb{E}_{k+k'}^{\otimes}.$$

Remark 4.13. *Informally, the above theorem says an $\mathbb{E}_{k+k'}$-algebra is the same as an $\mathbb{E}_k$-algebra in $\mathbb{E}_{k'}$-algebras.*

Corollary 4.14. *Let $k \ge 0$ be nonnegative integers. Then $(\mathbb{E}_k \odot \mathsf{Comm})^{\otimes} \simeq \mathsf{Comm}^{\otimes}$.*

4.15. The following result, known as the *Baez-Dolan Stabilization Hypothesis*, follows from Exercise 4.3.

Proposition 4.16. *If C is a symmetric monoidal $(n,1)$-category. Then the map $\mathbb{E}_k^{\otimes} \to \mathbb{E}_\infty^{\otimes}$ induces an equivalence*

$$\mathsf{Alg}_{\mathbb{E}_\infty}(\mathsf{C}) \simeq \mathsf{Alg}_{\mathbb{E}_k}(\mathsf{C})$$

for any $k > n$.

Remark 4.17. *In particular, in an ordinary category, any structure beyond $\mathbb{E}_2$ is commutative.*

4.18. Note that ordinary categories themselves form a (2,1)-category. Hence it is possible to have interesting $\mathbb{E}_2$-monoidal ordinary categories that are not symmetric monoidal.

Proposition 4.19 (HA.5.1.2.4)**.** *An $\mathbb{E}_2$-monoidal ordinary category is just a braided monoidal category.*

Remark 4.20. *Informally speaking, the space of binary operators in $\mathbb{E}_2$ is homotopic to $\mathbb{S}^1$, which means an $\mathbb{E}_2$-monoidal ordinary category C gives a Σ_2-*equivariant *functor $\mathbb{S}^1 \times \mathsf{C} \times \mathsf{C} \to \mathsf{C}$. For any point $s \in \mathbb{S}^1$, we obtain a functor $\mathsf{mult}_s : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ and an equivalence $\mathsf{mult}_s \xrightarrow{\simeq} \mathsf{mult}_{-s} \circ \sigma$, where $\sigma(x,y) := (y,x)$. Choosing a path from $1 \in \mathbb{S}^1$ to $-1 \in \mathbb{S}^1$, we obtain an equivlence $\mathsf{mult}_1 \xrightarrow{\simeq} \mathsf{mult}_{-1}$ and therefore an equivalence $\mathsf{mult}_1 \xrightarrow{\simeq} \mathsf{mult}_1 \circ \sigma$. In other words, we obtain functorial isomorphisms $\mathsf{mult}_1(x,y) \xrightarrow{\simeq} \mathsf{mult}_1(y,x)$, which are the **braiding morphisms** for the corresponding braided monoidal category.*

Exercise 4.21. *Describe the square of the braidings in terms of the functor $\mathbb{S}^1 \times \mathsf{C} \times \mathsf{C} \to \mathsf{C}$.*

Theorem 4.22 (HA.5.1.1.1)**.** *For $k \geq 1$ or $k = \infty$, the ∞-operad $\mathbb{E}_k^{\otimes}$ is coherent.*

Theorem 4.23 (HA.5.1.3.2)**.** *For $k \geq 1$ or $k = \infty$, let $\mathsf{C}^{\otimes} \to \mathbb{E}_k^{\otimes}$ be an $\mathbb{E}_k$-monoidal ∞-category compatible with geometric realizations. Then:*

(1) *For $A \in \mathsf{Alg}_{/\mathbb{E}_k}(\mathsf{C})$, the functor*
$$\mathsf{Mod}_A^{\mathbb{E}_k}(\mathsf{C})^{\otimes} \to \mathbb{E}_k^{\otimes}$$
is an $\mathbb{E}_k$-monoidal ∞-category.

(2) *There is a canonical monoidal functor*
$$\mathsf{Mod}_A^{\mathbb{E}_k}(\mathsf{C})^{\otimes} \underset{\mathbb{E}_k^{\otimes}}{\times} \mathbb{E}_1^{\otimes} \to \mathsf{Mod}_{A'}^{\mathbb{E}_1}(\mathsf{C} \underset{\mathbb{E}_k}{\times} \mathbb{E}_1)^{\otimes},$$
where
$$\mathsf{A}' \in \mathsf{Alg}_{/\mathbb{E}_1}(\mathsf{C} \underset{\mathbb{E}_k}{\times} \mathbb{E}_1) \simeq \mathsf{Alg}_{\mathbb{E}_1/\mathbb{E}_k}(\mathsf{C})$$
is the composition $\mathbb{E}_1^{\otimes} \to \mathbb{E}_k^{\otimes} \xrightarrow{A} \mathsf{C}^{\otimes}$.

Remark 4.24. *Informally speaking, the above theorem says the relative tensor products of $\mathbb{E}_k$-modules are given by those of the underlying $\mathbb{E}_1$-modules.*

Appendix A. Factorization homology

A.1. Given any $\mathbb{E}_k$-algebra A, one can obtain an invariant $\int_M A$ for any k-manifold M, known as the *factorization homology* of A over M.

A.2. **Suggested readings.** HA.5.5., [AF20] (and the references listed there).

References

[AF20] David Ayala and John Francis. A factorization homology primer. In *Handbook of homotopy theory*, pages 39–101. Chapman and Hall/CRC, 2020.