LECTURE 4

For the purpose of this course, we do not give proofs in this lecture, nor present the results in the logical order.

1. Quasi-categories

1.1. Last time, we constructed a Quillen equivalence

$$|-| : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Top} : \mathsf{Sing}.$$

This implies:

- Any ∞-groupoid, which can be identified with a homotopy type by the Homotopy Hypothesis, can also be realized as a bifibrant object in Set_Δ, i.e., a **Kan complex**.
- Any functor between ∞-groupoids can be realized as a *homotopy class* of morphisms between the corresponding Kan complexes. Here two morphisms $f, g : X \to Y$ are homotopic iff there exists a morphism $H : X \times \Delta^1 \to Y$ with $H(-, 0) = f$ and $H(-, 1) = g$.

1.2. Recall a simplicial set X is a Kan complex iff it has the right lifting property against *all* horn inclusions with $n \geq 2$

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n. & & \end{array}$$

Let us inspect these lifting properties for different choices of (n, i) using the *language* of higher category theory.

1.3. For $n = 2, i = 0$, the lifting property says that for a given morphism $f : x_0 \to x_1$, any morphism of the form $x_0 \to x_2$ can factor through f.

For $n = 3, i = 0$, the lifting property says that for a given morphism $f : x_0 \to x_1$, any commutative diagram

$$\begin{array}{ccc} & x_2 & \\ \nearrow & & \searrow \\ x_0 & \longrightarrow & x_3, \end{array}$$

or more precisely, an invertible 2-morphism *witnessing* such a diagram, can factor through f. This means the above 2-morphism is equivalent to the horizontal

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composition of a 2-morphism

$$
\begin{array}{ccc}
 & x_2 & \\
\nearrow & & \searrow \\
x_1 & \longrightarrow & x_3
\end{array}
$$
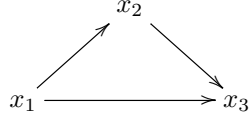
with the identity 2-morphism id_f.

In general, the lifting properties for $i = 0$ imply that for a given morphism $f : x \to y$, any *homotopy coherent* diagram with *initial* vertex x can factor through f. Note that this is a feature of ∞-groupoids rather than general ∞-categories: it implies the given morphism f has a left inverse.

1.4. Dually, the lifting properties for $i = n$ imply that for a given morphism $f : x \to y$, any homotopy coherent diagram with *final* vertex y can factor through f. Again, this is a feature of ∞-groupoids rather than general ∞-categories: it implies the given morphism f has a right inverse.

1.5. On the other hand, the lifting properties for $0 < i < n$ imply that for a given object x, one can compose any homotopy coherent diagram with final vertex x with another homotopy coherent diagram with initial vertex x. This requirement should be satisfied by any category-like entity in mathematics.

1.6. The above discussion motivates the following definition:

Definition 1.7 (Boardman–Vogt, 1973)**.** *A simplicial set X is called a **quasi-category** if it has the right lifting property against* inner *horn inclusions:*

$$
\begin{array}{ccc}
\Lambda_i^n & \longrightarrow & X \\
\downarrow & \nearrow & \\
\Delta^n & & \qquad \text{for } 0 < i < n.
\end{array}
$$

*Let X be a quasi-category. We call a 0-simplex $x \in X_0$ an **object** of X, and a 1-simplex $e \in X_0$ a **morphism** from $d_0^1(e)$ to $d_1^1(e)$.*

*Let X and Y be quasi-categories, we call a morphism $F : X \to Y$ in Set_Δ a **functor** from X to Y.*

Let $\mathsf{QCat} \subset \mathsf{Set}_\Delta$ be the full subcategory of quasi-categories.

Example 1.8. *Kan complexes are quasi-categories.*

Exercise 1.9. *Nerves of categories are quasi-categories.*

Exercise 1.10. *The opposite X^{op} of a quasi-category is a quasi-category. (See [Lecture 3, Construction 1.14]).*

1.11. In the 1980s, Joyal proposed:

> *Quasi-categories are models for $(\infty, 1)$-categories.*

In this lecture, we take the above proposal as acknowledged, and use it to develop basic languages about $(\infty, 1)$-categories. We will justify this proposal in the next lecture.

2. Composition of morphisms

Definition 2.1. *Let $\mathcal{C}$ be a quasi-category. For a 2-simplex σ of the form*

$$
\begin{array}{ccc}
 & y & \\
f \nearrow & & \searrow g \\
x & \xrightarrow{\quad h \quad} & z,
\end{array}
$$

*we say σ **witnesses h as a composition of f and** g.*

Remark 2.2. *Note that in the above definition, the morphism h is not determined by f and g, nor is σ determined by f, g and h. We may say:*

> Composition of morphisms in quasi-categories is not *concrete*.

Nevertheless, we will soon see that composition of morphisms in the ∞-category represented by $\mathcal{C}$ is well-defined, as long as we interpret well-definedness correctly.

3. Homotopy between morphisms

Definition 3.1. *Let $\mathcal{C}$ be a quasi-category. For a 2-simplex σ of the form*

$$
\begin{array}{ccc}
 & y & \\
f \nearrow & & \searrow \mathrm{id} \\
x & \xrightarrow{\quad g \quad} & y,
\end{array}
$$

*we say σ is a **left pinched homotopy**[1], or **left pinched 2-morphism**, from f to g.*

Dually, a 2-simplex of the form

$$
\begin{array}{ccc}
 & x & \\
\mathrm{id} \nearrow & & \searrow f \\
x & \xrightarrow{\quad g \quad} & y.
\end{array}
$$

*is called a **right pinched homotopy**, or **right pinched 2-morphism**, from f to g.*

Exercise 3.2. *The following conditions are equivalent:*

- *There exists a left pinched homotopy from f to g,*
- *There exists a right pinched homotopy from f to g,*

Moreover, these conditions define an equivalent relation on the set of morphisms from x to y. Hint: horn inclusions $\Lambda^3_1 \to \Delta^3$ and $\Lambda^3_2 \to \Delta^3$.

Definition 3.3. *Let $\mathcal{C}$ be a quasi-category and $f, g : x \to y$ be morphisms. We say f is **homotopic** to g if they satisfy the conditions in the above exercise.*

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[1]This terminology is not standard. For instance, [Rez22] called it a *right* homotopy.

4. Homotopy category

Exercise 4.1. *Let $\mathcal{C}$ be a quasi-category. Show that the following construction defines a category $\mathsf{h}\mathcal{C}$:*

- *Objects of $\mathsf{h}\mathcal{C}$ are objects of $\mathcal{C}$;*
- *Morphisms of $\mathsf{h}\mathcal{C}$ are homotopy classes of morphisms of $\mathcal{C}$;*
- *Composition of $[f]$ and $[g]$ is equal to $[h]$ in $\mathsf{h}\mathcal{C}$ iff h is a composition of f and g in $\mathcal{C}$.*

Definition 4.2. *Let $\mathcal{C}$ be a quasi-category. We call the category in the above exercise the **homotopy category of** $\mathcal{C}$, and denote it by $\mathsf{h}\mathcal{C}$.*

Exercise 4.3. *We have an adjoint pair*

$$\mathsf{h} : \mathsf{QCat} \rightleftarrows \mathsf{Cat} : \mathsf{N}_\bullet.$$

Remark 4.4. *The above exercise says that $\mathsf{h}\mathcal{C}$ is the category obtained by forcing all the higher morphisms in the ∞-category represented by $\mathcal{C}$ to be identities. Note that this is not the same as* abandoning *all the higher morphisms. The latter construction is evil because it violates the principle of isomorphism.*

4.5. As indicated by their names, homotopy categories of quasi-categories are related to homotopy categories of model categories. We will explain such relations in future lectures.

5. Isomorphisms

Definition 5.1. *Let $\mathcal{C}$ be a quasi-category. We say a morphism f is an **isomorphism** iff the morphism $[f]$ in $\mathsf{h}\mathcal{C}$ is an isomorphism.*

5.2. In other words, $f : x \to y$ is an isomorphism iff there exists a morphism $g : y \to x$ such that $g \circ f$ is homotopic to id_x and $f \circ g$ is homotopic to id_y.

Exercise 5.3. *A quasi-category is a Kan complex iff all morphisms of it are isomorphisms.*

Exercise 5.4. *For a quasi-category $\mathcal{C}$, there is a unique Kan complex $\mathcal{C}^\simeq \subset \mathcal{C}$, called the **core of** $\mathcal{C}$, that contains all the isomorphisms of $\mathcal{C}$. The obtained functor $(-)^\simeq : \mathsf{QCat} \to \mathsf{Kan}$ is right adjoint to the embedding functor $\mathsf{Kan} \to \mathsf{QCat}$.*

6. Morphism spaces: idea

6.1. As a sanity check for Joyal's proposal, for a pair of objects x, y in a quasi-category $\mathcal{C}$, we should construct an ∞-groupoid $\mathsf{Maps}_\mathcal{C}(x, y)$ such that:

- Objects in $\mathsf{Maps}_\mathcal{C}(x, y)$ are morphisms from x to y;
- Morphisms in $\mathsf{Maps}_\mathcal{C}(x, y)$ are 2-morphisms between morphisms from x to y;
- ...

In this lecture, we provide three constructions of Kan complexes that represent this ∞-groupoid (see [Lecture 3, Definition 8.3] for what this means). These three Kan complexes are *weak homotopy equivalent* but not *isomorphic* to each other.

6.2.   In a *model-independent* or *axiomatic* theory[2] of $(\infty, 1)$-categories, these three constructions should correspond respectively to the following ∞-groupoids:

(B) The fiber product

$$\{x\} \underset{\mathsf{Fun}(\{0\},\mathcal{C})}{\times} \mathsf{Fun}([1],\mathcal{C}) \underset{\mathsf{Fun}(\{1\},\mathcal{C})}{\times} \{y\},$$

where $\mathsf{Fun}(\mathcal{D},\mathcal{C})$ is the $(\infty, 1)$**-category of functors from $\mathcal{D}$ to $\mathcal{C}$**.

(L) The fiber product

$$\mathcal{C}_{x/} \underset{\mathcal{C}}{\times} \{y\},$$

where $\mathcal{C}_{x/}$ is the **coslice** $(\infty, 1)$**-category of $\mathcal{C}$ under** x;

(R) The fiber product

$$\{x\} \underset{\mathcal{C}}{\times} \mathcal{C}_{/y},$$

where $\mathcal{C}_{/y}$ is the **slice** $(\infty, 1)$**-category of $\mathcal{C}$ over** y;

Exercise 6.3. *Convince yourself that the above three ∞-groupoids should be canonically equivalent.*

7. FUNCTOR QUASI-CATEGORIES

7.1.   To realize (B) via quasi-categories, we need to define the quasi-category of functors between two quasi-categories.

Definition 7.2. *We say a category $\mathcal{C}$ is **Cartesian closed** if it satisfies the following conditions:*

(1) *Finite products exist in $\mathcal{C}$;*

(2) *For any objects X and Y in $\mathcal{C}$, the functor*

$$\mathsf{Hom}(X \times -, Y) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$$

*is corepresentable, i.e., there exists an object Y^X, called the **exponential object**, satisfying the following universal property:*

$$\mathsf{Hom}(X \times Z, Y) \simeq \mathsf{Hom}(Z, Y^X).$$

Remark 7.3. *Condition (2) is equivalent to*

(2′) *For any object X, the functor $X \times - : \mathcal{C} \to \mathcal{C}$ has a right adjoint.*

Indeed, this right adjoint is given by $(-)^X : \mathcal{C} \to \mathcal{C}$.

Proposition-Definition 7.4. *The category Set_Δ is Cartesian closed. For simplicial sets X and Y, we denote the exponential object by*

$$\mathsf{Fun}(X,Y) \overset{\text{def}}{=} Y^X \in \mathsf{Set}_\Delta.$$

Exercise 7.5. $\mathsf{Fun}(X^{\mathsf{op}}, Y^{\mathsf{op}}) \simeq \mathsf{Fun}(X,Y)^{\mathsf{op}}$.

Exercise 7.6. *Prove that the nerve functor $\mathsf{N}_\bullet : \mathsf{Cat} \to \mathsf{Set}_\Delta$ is compatible with products and exponentials, i.e.,*

$$\mathsf{N}_\bullet(\mathcal{C} \times \mathcal{D}) \overset{\simeq}{\to} \mathsf{N}_\bullet(\mathcal{C}) \times \mathsf{N}_\bullet(\mathcal{D}),$$

$$\mathsf{N}_\bullet(\mathsf{Fun}(\mathcal{C},\mathcal{D})) \overset{\simeq}{\to} \mathsf{Fun}(\mathsf{N}_\bullet(\mathcal{C}), \mathsf{N}_\bullet(\mathcal{D})).$$

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[2]I am not aware of the existence of such a theory, hence the discussion below only serves as motivations.

Proposition 7.7 (Ker.00TN)**.** *For* $X \in \mathsf{Set}_\Delta$ *and* $Y \in \mathsf{Kan}$, *we have* $\mathsf{Fun}(X, Y) \in \mathsf{Kan}$.

Proposition 7.8 (Ker.0066)**.** *For* $X \in \mathsf{Set}_\Delta$ *and* $Y \in \mathsf{QCat}$, *we have* $\mathsf{Fun}(X, Y) \in \mathsf{QCat}$.

Corollary 7.9. *The category* QCat *is Cartesian closed.*

Proposition-Definition 7.10 (Ker.01JC)**.** *Let* $\mathcal{C}$ *be a quasi-category and* x, y *be be objects in* $\mathcal{C}$. *Then the simplicial set*

$$\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x, y) \overset{\mathrm{def}}{=} \{x\} \underset{\mathsf{Fun}(\{0\}, \mathcal{C})}{\times} \mathsf{Fun}(\Delta^1, \mathcal{C}) \underset{\mathsf{Fun}(\{1\}, \mathcal{C})}{\times} \{y\}$$

*is a Kan complex, which is called the **balanced complex of morphisms**[3] from* x *to* y.

7.11. By definition, for a testing simplicial set K, we have a bijection between the sets of

- morphisms $K \to \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x, y)$
- morphisms $K \times \Delta^1 \to \mathcal{C}$ with constant value x on $K \times \{0\}$ and constant value y on $K \times \{1\}$.

7.12. In particular, a 1-simplex in $\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x, y)$ corresponds to a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

We call such a diagram a **balanced homotopy**[4], or **balanced 2-morphism**, from f to g.

Exercise 7.13. *Let* $\mathcal{C}$ *be a quasi-category and* $f, g : x \to y$ *be morphisms. Show that* f *is homotopic to* g *iff there exists a balanced homotopy from* f *to* g.

Exercise 7.14. *Constrct a canonical isomorphism* $\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x, y)^{\mathsf{op}} \overset{\simeq}{\to} \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}^{\mathsf{op}}}(y, x)$.

8. Slice and coslice

8.1. To realize (L) and (R) via quasi-categories, we need to generalize the slice/coslice categories to quasi-categories.

Definition 8.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories. The **join** of* $\mathcal{C}$ *and* $\mathcal{D}$, *denoted by* $\mathcal{C} \star \mathcal{D}$, *is the category defined by*

$$\mathsf{Ob}(\mathcal{C} \star \mathcal{D}) :\overset{\mathrm{def}}{=} \mathsf{Ob}(\mathcal{C}) \sqcup \mathsf{Ob}(\mathcal{D})$$

$$\mathsf{Hom}_{\mathcal{C} \star \mathcal{D}}(x, y) \overset{\mathrm{def}}{=} \begin{cases} \mathsf{Hom}_{\mathcal{C}}(x, y) & \text{if } x, y \in \mathcal{C} \\ \mathsf{Hom}_{\mathcal{D}}(x, y) & \text{if } x, y \in \mathcal{D} \\ \{*\} & \text{if } x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing & \text{if } x \in \mathcal{D}, y \in \mathcal{C}. \end{cases}$$

Example 8.3. *We have* $[i] \star [n - i - 1] \simeq [n]$.

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[3]This terminology is not standard. For instance, Lurie called it the *space of morphisms*.

[4]This terminology is not standard.

8.4.   The above definition can be generalized to simplicial sets.

Definition 8.5. *Let X and Y be simplicial sets. The **join** of X and Y, denoted by $X \star Y$, is the simplicial set defined by*

$$(X \star Y)([n]) \overset{\text{def}}{=\joinrel=} \bigsqcup_{-1 \leq i \leq n} (X([i]) \times Y([n-i-1])),$$

where we use the convention $[-1] = \varnothing$ and thereby $X[-1] = Y[-1] = \{*\}$. In the above definition, the functoriality in $[n]$ is provided by the equivalences $[i] \star [n-i-1] \simeq [n]$.

Example 8.6. *We have $\Delta^i \star \Delta^{n-i-1} \simeq \Delta^n$.*

Proposition 8.7 (Ker.02QV). *Let X and Y be quasi-categories. Then the join $X \star Y$ is also a quasi-category.*

Definition 8.8. *Let X be a simplicial set. We define the **left cone** and **right cone of** X to be*

$$X^\triangleleft \overset{\text{def}}{=\joinrel=} \Delta^0 \star X, \ X^\triangleright \overset{\text{def}}{=\joinrel=} X \star \Delta^0.$$

Definition 8.9. *Let X be a simplicial set and x be a 0-simplex in X. The **slice simplicial set of** X **over** x is the simplicial set $X_{/x}$ defined by*

$$\mathsf{Hom}(Y, X_{/x}) \simeq \mathsf{Hom}(Y^\triangleright, X) \underset{\mathsf{Hom}(\Delta^0, X)}{\times} \{x\}.$$

*Dually, the **coslice simplicial set of** X **under** x is defined to be the simplicial set $X_{x/}$ satisfying the following universal property*

$$\mathsf{Hom}(Y, X_{x/}) \simeq \mathsf{Hom}(Y^\triangleleft, X) \underset{\mathsf{Hom}(\Delta^0, X)}{\times} \{x\}.$$

Exercise 8.10. *Show that $X_{x/}$ and $X_{/x}$ are quasi-categories if X is so.*

8.11.   The embeddings $Y \to Y^\triangleleft$ and $Y \to Y^\triangleright$ induce forgetful morphisms

$$X_{x/} \to X, \ X_{/x} \to X.$$

Proposition-Definition 8.12 (Ker.01L0). *Let $\mathcal{C}$ be a quasi-category and x, y be objects in X. The simplicial sets[5]*

$$\mathsf{Hom}_{\mathcal{C}}^{\mathsf{L}}(x, y) \overset{\text{def}}{=\joinrel=} \mathcal{C}_{x/} \underset{\mathcal{C}}{\times} \{y\}$$

$$\mathsf{Hom}_{\mathcal{C}}^{\mathsf{R}}(x, y) \overset{\text{def}}{=\joinrel=} \{x\} \underset{\mathcal{C}}{\times} \mathcal{C}_{/y}$$

*are Kan complexes, which are called the **left/right pinched complex of morphisms** from x to y.*

8.13.   By definition, for a testing simplicial set K, we have a bijection between the sets of

- morphisms $K \to \mathsf{Hom}_{\mathcal{C}}^{\mathsf{L}}(x, y)$
- morphisms $K^\triangleleft \to \mathcal{C}$ with value x on the apex $\Delta^0 \subset K^\triangleleft$ and constant value y on the base $K \subset K^\triangleleft$.

Dually, we have a bijection between the sets of

- morphisms $K \to \mathsf{Hom}_{\mathcal{C}}^{\mathsf{R}}(x, y)$

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[5]In below, $\{x\}$ stands for a simplicial set Δ^0 with the 0-simplex labelled by x.

- morphisms $K^{\triangleright} \to \mathcal{C}$ with value y on the apex $\Delta^0 \subset K^{\triangleright}$ and constant value x on the base $K \subset K^{\triangleright}$.

In particular, morphisms in $\mathsf{Hom}_{\mathcal{C}}^{\mathsf{L}}(x, y)$ and $\mathsf{Hom}_{\mathcal{C}}^{\mathsf{R}}(x, y)$ represents respectively left/right pinched 2-morphisms in $\mathcal{C}$.

Exercise 8.14. *Construct a canonical isomorphism* $\mathsf{Hom}_{\mathcal{C}}^{\mathsf{L}}(x, y)^{\mathsf{op}} \xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{C}^{\mathsf{op}}}^{\mathsf{R}}(y, x)$.

9. Morphism spaces

9.1. For a testing simplicial set K, we have natural maps

$$K^{\triangleleft} \leftarrow K \times \Delta^1 \to K^{\triangleright},$$

where the leftward (resp. rightward) morphism collapse $K \times \{0\}$ (resp. $K \times \{1\}$) onto the apex of the left (resp right) cone. By §8.13 and §7.11, we obtain morphisms

$$(9.1) \qquad\qquad \mathsf{Hom}_{\mathcal{C}}^{\mathsf{L}}(x, y) \to \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, y) \leftarrow \mathsf{Hom}_{\mathcal{C}}^{\mathsf{R}}(x, y).$$

Proposition 9.2 (Ker.01L5). *Morphisms in* (9.1) *are weak homotopy equivalences.*

9.3. It follows that the ∞-groupoids/spaces represented by the Kan complexes in (9.1) are *canonically* equivalent.

Definition 9.4. *Let $\mathcal{C}$ be a quasi-category and x, y be objects in X. Let*

$$\mathsf{Maps}_{\mathcal{C}}(x, y) \in \mathsf{hGrpd}_{\infty} \overset{\mathrm{def}}{=\joinrel=} \mathsf{Set}_{\Delta}[W^{-1}]$$

be the object represented by (9.1)[6]. *We call it the **mapping space** from x to y.*

9.5. Whatever the following means, we call a morphism in the ∞-groupoid $\mathsf{Maps}_{\mathcal{C}}(x, y)$ a **2-morphism in the ∞-category represented by** $\mathcal{C}$.

Exercise 9.6. *Construct a canonical isomorphism* $\mathsf{Maps}_{\mathcal{C}}(x, y) \xrightarrow{\simeq} \mathsf{Maps}_{\mathcal{C}^{\mathsf{op}}}(x, y)$. *Hint: [Lecture 3, Exercise 6.5].*

10. Composition in ∞-categories

10.1. The following result, due to Joyal, is combinatorial but fundamental. See Ker.0079 for a proof.

Theorem 10.2. *A simplicial set X is a quasi-category iff*

$$\mathsf{Fun}(\Delta^2, X) \to \mathsf{Fun}(\Lambda_1^2, X)$$

is an acyclic Kan fibration in Set_{Δ}.

10.3. Let $\mathcal{C}$ be a quasi-category. By definition, a 0-simplex in $\mathsf{Fun}(\Lambda_1^2, \mathcal{C})$ is a pair of composable morphisms in $\mathcal{C}$:

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[6]See §11 for what this actually means.

and a lift of this 0-simplex to $\mathsf{Fun}(\Delta^2, \mathcal{C})$ is a 2-simplex σ

(10.1)

witnessing h as a composition of f and g.

Now the above theorem implies the simplicial set of such lifts is a **weakly contractible Kan complex**. In other words, a composition of f and g, **when understood as a 2-simplex**, is defined up to a contractible space of choices. By Slogan 11.4 below, we can say:

> *The composition $g \circ f$ is well-defind as an object in the mapping ∞-groupoid* $\mathsf{Maps}_{\mathcal{C}}(x, y)$.

10.4. To obtain the above result, it is crucial that we incorporate the witness 2-simplex as part of the data in a composition. We can say:

Slogan 10.5. *In an ∞-category, composition is a* structure *rather than a* property.

10.6. Since this is a quite important point, let us rephrase it as follows. In an ∞-category, when saying h is the composition of f and g, we *always* mean we supply a commutative diagram (10.1). Otherwise, the words *the composition* do not make sense.

Exercise 10.7. *Alternatively, one can justify the above claim as follows. Let $\mathcal{C}$ be a quasi-category.*

(1) *Use Theorem 10.2 to deduce that*

$$\mathsf{Fun}(\Delta^2, \mathcal{C}) \underset{\mathcal{C}^{\times 3}}{\times} \{(x, y, z)\} \to \mathsf{Fun}(\Lambda_1^2, X) \underset{\mathcal{C}^{\times 3}}{\times} \{(x, y, z)\}$$

is an acyclic Kan fibration in Set_Δ.

(2) *Identify the target of the above morphism with*

$$\mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, y) \times \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(y, z)$$

(3) *Construct a morphism*

$$\mathsf{Fun}(\Delta^2, \mathcal{C}) \underset{\mathcal{C}^{\times 3}}{\times} \{(x, y, z)\} \to \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, z)$$

in Set_Δ.

(4) *Construct a morphism*

$$\mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, y) \times \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(y, z) \to \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, z)$$

in $\mathsf{hGrpd}_\infty \overset{\mathrm{def}}{=} \mathsf{Set}_\Delta[W^{-1}]$.

(5) *Show that the composition in (4) satisfies* strict *identity and associativity axioms.*

Exercise 10.8. *In an ∞-category C, composing with an isomorphism induces equivalence betwen mapping spaces. In other words, for a quasi-category $\mathcal{C}$,*

(1) *If $f : x \to y$ is an isomorphism, then it induces a weak homotopy equivalence between Kan complexes:*

$$- \circ f : \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(y, z) \overset{\sim}{\to} \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, z).$$

(2) *If $g : y \to z$ is an isomorphism, then it induces a weak homotopy equivalence between Kan complexes:*

$$g \circ - : \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}^{\mathsf{B}}(x, z).$$

11. When is an object well-defined?

11.1.   This section serves to answer the following questions:

Why is $g \circ f$ well-defined as an object in the ∞-groupoid $\mathsf{Maps}_{\mathcal{C}}(x, y)$?

11.2.   First, let us make it clear: we do not claim $g \circ f$ is a well-defined 1-simplex in the quasi-category $\mathcal{C}$, nor a well-defined 0-simplex in any of the Kan complexes in (9.1). Instead, we claim it is well-defined as a morphism in the ∞-category modeled by $\mathcal{C}$, or equivalently, well-defined as an object in the ∞-groupoid modeled by (9.1).

Since we have already sketched a *mathematical proof* of this claim in Exercise 10.7, now we will provide a more metaphysical/ideological argument for this claim.

11.3.   In classical category theory, *construction* often means

- specifying an *element in a given set*;

However, in infinity category theory, we want to keep track of automorphisms between objects, as well as automorphisms between automorphisms, etc.. Hence *construction* should mean

- specifying an *object in a given ∞-groupoid,*

or equivalently

- specifying a *point of a homotopy type.*

Note that we cannot distinguish the homotopy type of a point with that of any contractible space. Hence we may say:

Slogan 11.4. *In an ∞-groupoid, an object is well-defined if it is defined up to a contractible space of choices[7].*

11.5.   From now on,

essentially unique   =   unique up to a contractible space of choices.

Appendix A. Inverse

A.1.   The lifting property in §1.3 actually implies f is an isomorphism (rather than just has a left inverse), as long as the simplicial set is a quasi-category. In more precise words:

Exercise A.2. *Let $\mathcal{C}$ be a quasi-category and f be a morphism. Suppose*

$$
\begin{array}{ccc}
\Lambda_0^n & \xrightarrow{\phi} & X \\
\downarrow & \nearrow & \\
\Delta^n & & n \geq 2
\end{array}
$$

has a solution for any ϕ such that $\phi|_{\Delta^{\{0,1\}}} = f$. Show that f is an isomorphism.

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[7]A radical consequence of this philosophy is the so-called *univalent foundations* of mathematics.

A.3.   The following exercises say the left inverse, right inverse, and inverse of an isomorphism f is well-defined, i.e., defined up to a contractible space of choices. However, one has to be careful about the meanings of these words.

Exercise A.4. *Let $\mathcal{C}$ be a quasi-category and f be a morphism. Find the correct meaning of the words*

- *A 2-simplex σ witnesses g as a left/right inverse of f.*

If f is an isomorphism, show that the *left/right inverse of f is well-defined by imitating §10.*

Exercise A.5. *Let $\mathcal{C}$ be a quasi-category and f be an isomorphism. Find a simplicial set $\mathcal{I}$ such that the words*

- *A morphism $\mathcal{I} \to \mathcal{C}$ witnesses g as an inverse of f*

is the correct notion. Show that the *inverse of f is well-defined by imitating §10. Hint: $\mathcal{I}$ should be the nerve of an ordinary category.*

Exercise A.6. *Make sense of the following words: in an ∞-category, the left inverse and right inverse of an isomorphism are* canonically *equivalent.*

References

[Rez22] Charles Rezk. Introduction to quasicategories. *Lecture Notes for course at University of Illinois at Urbana-Champaign*, 2022.