

LECTURE 8

In this lecture, we use model categories to compute limits and colimits.

1. ∞ -LIMITS AS HOMOTOPY LIMITS

1.1. Recall we have a Quillen equivalence

$$\mathcal{C} : \text{Set}_{\Delta}^{\text{Joyal}} \rightleftarrows \text{Cat}_{\Delta} : \mathfrak{N}_{\bullet}$$

In this section, we will explain the following result:

Theorem 1.2 (HTT.4.2.4.1). *Let \mathbb{C} be a combinatorial simplicial model category. Then for a small fibrant simplicial category \mathbb{J} , a diagram*

$$\mathbb{J}^{\triangleleft} \rightarrow \mathbb{C}^{\circ}$$

is a homotopy limit diagram in \mathbb{C} iff the corresponding diagram

$$\mathfrak{N}_{\bullet}(\mathbb{J})^{\triangleleft} \rightarrow \mathfrak{N}_{\bullet}(\mathbb{C}^{\circ})$$

is a limit diagram in the quasi-category $\mathfrak{N}_{\bullet}(\mathbb{C}^{\circ})$.

Variant 1.3. *Dually, a diagram $\mathbb{J}^{\triangleright} \rightarrow \mathbb{C}^{\circ}$ is a homotopy colimit diagram iff $\mathfrak{N}_{\bullet}(\mathbb{J})^{\triangleright} \rightarrow \mathfrak{N}_{\bullet}(\mathbb{C}^{\circ})$ is a limit diagram.*

1.4. We will soon give the precise definitions of the undefined notions in Theorem 1.2. For now, let us be satisfied by the following informal words.

A *simplicial model category* is a model category \mathbb{C} equipped with a compatible simplicial enrichment. Here the compatibility condition guarantees

- (1) There is a canonical equivalence

$$\text{h}\mathbb{C} \simeq \pi_0\mathbb{C},$$

where $\text{h}\mathbb{C}$ is the homotopy category of the model category \mathbb{C} (see [Lecture 2, Definition 2.20]), while $\pi_0\mathbb{C}$ is the homotopy category of the simplicial category \mathbb{C} (see [Lecture 5, Definition 5.3]).

- (2) For bifibrant objects $x, y \in \mathbb{C}$, the simplicial set $\text{Hom}_{\mathbb{C}}(x, y)$ is a Kan complex. In other words, \mathbb{C}° is a fibrant object in Cat_{Δ} .

For a model category, being combinatorial is a technical set-theoretical size condition, which can be ignored for now.

For a simplicial model category \mathbb{C} , the *homotopy limit* of a diagram $u : \mathbb{J} \rightarrow \mathbb{C}$ is the value of the right derived functor of the naive limit functor. In other words,

$$\text{holim } u := \mathbb{R}\text{lim}(u),$$

where the functor lim is the right adjoint in a Quillen adjunction

$$\text{const} : \mathbb{C} \rightleftarrows \text{Fun}(\mathbb{J}, \mathbb{C}) : \text{lim}.$$

Here $\text{Fun}(\mathbb{J}, \mathbb{C})$ is the category of simplicial enriched functors from \mathbb{J} to \mathbb{C} , equipped with a suitable model structure induced by the model structure on \mathbb{C} .

Dually, the *homotopy colimit* of a diagram $u : \mathbb{J} \rightarrow \mathbb{C}$ is the value of the left derived functor of the naive colimit functor, i.e.,

$$\mathrm{hocolim} u := \mathbb{L}\mathrm{colim}(u),$$

1.5. Recall that Set_Δ is Cartesian closed and therefore has a natural simplicial enrichment given by $\mathrm{Fun}(-, -)$. This enrichment is compatible with the Kan–Quillen model structure. We denote the obtained simplicial model category by $\mathrm{Set}_\Delta^{\mathrm{KQ}}$. In Theorem 1.2, we can take $\mathbb{C} := \mathrm{Set}_\Delta^{\mathrm{KQ}}$. Note that $\mathbb{C}^\circ = \mathbb{K}\mathrm{on}$ is the simplicial category defined in [Lecture 5, §7]. We obtain:

Corollary 1.6. *Let \mathbb{J} be a small fibrant simplicial category. Then the homotopy (co)limit of a \mathbb{J} -indexed diagram in*

$$\mathbb{K}\mathrm{on} \subset \mathrm{Set}_\Delta^{\mathrm{KQ}}$$

calculates the (co)limit of the corresponding $\mathfrak{N}_\bullet(\mathbb{J})$ -indexed diagram in the quasi-categories

$$\mathcal{K}\mathrm{an} := \mathfrak{N}_\bullet(\mathbb{K}\mathrm{on}).$$

Remark 1.7. *Note that the latter models (co)limits in the ∞ -category Grpd_∞ , which essentially control (co)limits in any ∞ -category ([Lecture 7, Theorem 2.11]).*

Remark 1.8. *The above corollary provides a model-categorical algorithm to calculate small (co)limits of ∞ -groupoids. Let us take the second case as an example. Let $u : K \rightarrow \mathrm{Grpd}_\infty$ be a small diagram with $K \in \mathrm{Set}_\Delta$.*

- (i) *Choose a weak equivalence $\mathfrak{C}(K) \rightarrow \mathbb{J}$ in Cat_Δ .*
- (ii) *Find a functor $w : \mathbb{J} \rightarrow \mathbb{K}\mathrm{on}$ such that the composition*

$$K \rightarrow \mathfrak{N}_\bullet(\mathbb{J}) \rightarrow \mathfrak{N}_\bullet(\mathbb{K}\mathrm{on}) =: \mathcal{K}\mathrm{an}$$

represents u . By HTT.4.2.4.4, such w always exists.

- (iii) *View w as an object in the model category $\mathrm{Fun}(\mathbb{J}, \mathrm{Set}_\Delta^{\mathrm{KQ}})$. Calculate the derived (co)limits*

$$\mathbb{R}\mathrm{lim}(w), \mathbb{L}\mathrm{colim}(w) \in \mathrm{Set}_\Delta^{\mathrm{KQ}}[W^{-1}]$$

by finding a (co)fibrant replacement of w .

By Corollary 1.6, the obtained objects in $\mathrm{Set}_\Delta^{\mathrm{KQ}}[W^{-1}] \simeq \mathrm{hGrpd}_\infty$ are canonically isomorphic to $\mathrm{lim} u$ and $\mathrm{colim} u$.

Exercise 1.9. *Prove the functor w in Step (ii) exists in the case when $\mathfrak{C}(K) \rightarrow \mathbb{J}$ is a cofibration.*

Remark 1.10. *In Remark 1.8, when $K = \mathfrak{N}_\bullet(\mathbb{J})$ is the nerve of an ordinary category, we can take $\mathbb{J} := \mathbb{J}$, viewed as a simplicial category with discrete enrichment. Then $\mathrm{Fun}(\mathbb{J}, \mathrm{Set}_\Delta^{\mathrm{KQ}})$ is just the category of functors between the ordinary categories $\mathbb{J} \rightarrow \mathrm{Set}_\Delta$, and the underived functor lim is the limit functor for ordinary categories.*

In fact, the above essentially covers all the cases because for any simplicial set K , there exists an initial morphism $\mathfrak{N}_\bullet(\mathbb{J}) \rightarrow K$ such that \mathbb{J} is an ordinary category or even a partially ordered set. See HTT.4.2.3.14.

Remark 1.11. In Remark 1.8, the obtained objects are contained in the homotopy category \mathbf{hGrpd}_∞ rather than in \mathbf{Grpd}_∞ . In other words, the above algorithm only calculates $\lim u$ up to homotopy. In particular, it cannot produce the entire limit diagram, nor the canonical lifting of $\lim u$ in $(\mathbf{Grpd}_\infty)_{/u}$.

To remedy this, one can try to find an extended diagram $\bar{w} : \mathbb{J}^\triangleleft \rightarrow \mathbf{Kan}$ that **exhibits $\bar{w}(\ast)$ as the homotopy limit** of w (see Definition 2.18 below). Then Theorem 1.2 says the corresponding diagram

$$\mathfrak{N}_\bullet(\mathbb{J})^\triangleleft \rightarrow \mathfrak{N}(\mathbf{Kan}) =: \mathbf{Kan}$$

is also a limit diagram. Now restriction along the categorical equivalence $K^\triangleleft \rightarrow \mathfrak{N}_\bullet(\mathbb{J})^\triangleleft$ produces a limit diagram extending $v : K \rightarrow \mathbf{Kan}$.

For general w , the above extension \bar{w} may not exist¹. Nevertheless, we can replace w by any fibrant replacement of it because they represent the same diagram in \mathbf{Grpd}_∞ . Under this additional assumption, such extension \bar{w} exists and is unique up to unique equivalence, because it has to be the limit diagram extending w .

Remark 1.12. In future lectures, we will see that any presentable ∞ -category can be realized as $\mathfrak{N}_\bullet(\mathbb{C}^\circ)$ for some combinatorial simplicial model category \mathbb{C} . This provides a model-categorical algorithm to calculate small (co)limits in any presentable ∞ -category.

2. DEFINITION OF HOMOTOPY LIMITS

2.1. In this section, we give the precise definitions for the notions used in Theorem 1.2.

Definition 2.2. Let \mathbb{C} be a simplicial category.

- (1) We say \mathbb{C} is **tensoried over \mathbf{Set}_Δ** if for any $S \in \mathbf{Set}_\Delta$ and $X \in \mathbb{C}$, the functor

$$\mathrm{Fun}(S, \mathrm{Hom}_{\mathbb{C}}(X, -)) : \mathbb{C} \rightarrow \mathbf{Set}_\Delta$$

is represented by an object in \mathbb{C} , which we denote by $S \otimes X$.

- (2) We say \mathbb{C} is **cotensoried over \mathbf{Set}_Δ** if for any $S \in \mathbf{Set}_\Delta$ and $Y \in \mathbb{C}$, the functor

$$\mathrm{Fun}(S, \mathrm{Hom}_{\mathbb{C}}(-, Y)) : \mathbb{C}^{\mathrm{op}} \rightarrow \mathbf{Set}_\Delta$$

is represented by an object in \mathbb{C} , which we denote by $\mathrm{Fun}(S, Y)$.

Proposition-Definition 2.3 (HTT.A.3.1.5, A.3.1.6). Let \mathbb{C} be a model category equipped with a simplicial enrichment such that it is both tensoried and cotensoried over \mathbf{Set}_Δ . Then the following conditions are equivalent:

- (i) Given any cofibration $j : X \rightarrow X'$ and any fibration $k : Y \rightarrow Y'$ in \mathbb{C} , the morphism

$$\mathrm{Hom}_{\mathbb{C}}(X', Y) \rightarrow \mathrm{Hom}_{\mathbb{C}}(X', Y') \times_{\mathrm{Hom}_{\mathbb{C}}(X, Y')} \mathrm{Hom}_{\mathbb{C}}(X, Y)$$

is a fibration in $\mathbf{Set}_\Delta^{\mathbf{KQ}}$, which is a weak equivalence if either j or k is so.

¹For example, consider $\mathrm{Sing}_\bullet(\{0\} \rightarrow [0, 1] \leftarrow \{1\})$.

- (ii) Given any cofibrations $i : S \rightarrow S'$ and $j : X \rightarrow X'$ respectively in $\text{Set}_\Delta^{\text{KQ}}$ and \mathbb{C} , the morphism

$$(S' \otimes X) \bigsqcup_{S \otimes X} (S \otimes X') \rightarrow S' \otimes X'$$

is a cofibration in \mathbb{C} , which is a weak equivalence if either i or j is so.

- (iii) Given any cofibration $i : S \rightarrow S'$ in $\text{Set}_\Delta^{\text{KQ}}$ and any fibration $k : Y \rightarrow Y'$ in \mathbb{C} , the morphism

$$\text{Fun}(S', Y) \rightarrow \text{Fun}(S', Y') \times_{\text{Fun}(S, Y')} \text{Fun}(S, Y)$$

is a fibration in \mathbb{C} , which is a weak equivalence if either i or k is so.

We say \mathbb{C} is a **simplicial model category** if the above conditions are satisfied.

Exercise 2.4. Let \mathbb{C} be a simplicial model category. Show that $\text{Hom}_{\mathbb{C}}(X, Y)$ is a Kan complex if X is cofibrant and Y is fibrant.

Exercise 2.5. Can you make $\text{Set}_\Delta^{\text{Joyal}}$ into a simplicial model category?

Exercise 2.6. Let \mathbb{C} be a simplicial model category. Construct a canonical equivalence $\text{h}\mathbb{C} \simeq \pi_0 \mathbb{C}$.

2.7. We also need the following technical size conditions.

Definition 2.8 (HTT.A.2.6.1). Let \mathbb{C} be a model category. We say \mathbb{A} is **combinatorial** if the following conditions are satisfied:

- (a) The category \mathbb{C} is presentable.
- (b) As a weakly saturated class of morphisms, (C) is generated by a set.
- (c) As a weakly saturated class of morphisms, $(C \cap W)$ is generated by a set.

Example 2.9. The model category $\text{Set}_\Delta^{\text{KQ}}$ equipped with the simplicial enrichment $\text{Fun}(-, -)$ is a combinatorial simplicial model category.

Proposition-Definition 2.10 (HTT.A.3.3.2). Let \mathbb{C} be a combinatorial simplicial model category and \mathbb{J} be a small simplicial category. Then there exists two combinatorial model structures on $\text{Fun}(\mathbb{J}, \mathbb{C})$:

- (1) The **projective model structure**, denoted by $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{proj}}$, where
 - (W) A **weak equivalence** is a natural transformation that is a pointwise weak equivalence.
 - (F) A **projective fibration** is a natural transformation that is a pointwise fibration.
 - (C) The collection of **projective cofibrations** is determined by $(C \cap W)$.
- (2) The **injective model structure**, denoted by $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}}$, where
 - (W) A **weak equivalence** is a natural transformation that is a pointwise weak equivalence.
 - (C) A **injective cofibration** is a natural transformation that is a pointwise cofibration.
 - (F) The collection of **injective cofibrations** is determined by $(F \cap W)$.

Example 2.11. When $\mathbb{J} = [0]$ is the singleton, both model structures coincide with the given model structure on \mathbb{C} .

Proposition 2.12 (HTT.A.3.3.6). *Let*

$$F : \mathbb{A} \rightleftarrows \mathbb{B} : G$$

be a Quillen adjunction between combinatorial simplicial model categories. Then for any small simplicial category \mathbb{J} , it induces Quillen adjunctions

$$F \circ - : \text{Fun}(\mathbb{J}, \mathbb{A})? \rightleftarrows \text{Fun}(\mathbb{J}, \mathbb{B})? : G \circ -,$$

*where ? can be either **proj** or **inj**. They are Quillen equivalences if $F : \mathbb{A} \rightleftarrows \mathbb{B} : G$ is so.*

Proposition 2.13 (HTT.A.3.3.7, A.3.3.8). *Let $\iota : \mathbb{J} \rightarrow \mathbb{J}'$ be a functor between small simplicial enriched categories. For a combinatorial simplicial model category \mathbb{C} , we have Quillen adjunctions*

$$\text{LKE}_\iota : \text{Fun}(\mathbb{J}, \mathbb{C})_{\text{proj}} \rightleftarrows \text{Fun}(\mathbb{J}', \mathbb{C})_{\text{proj}} : \iota \circ -;$$

$$\iota \circ - : \text{Fun}(\mathbb{J}', \mathbb{C})_{\text{inj}} \rightleftarrows \text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}} : \text{RKE}_\iota;$$

which are Quillen equivalences if $\iota : \mathbb{J} \rightarrow \mathbb{J}'$ is a weak equivalence in the model category Cat_Δ (see [Lecture 5, Definition 5.4]).

Definition 2.14. *We call the right derived functor of*

$$\text{RKE}_\iota : \text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}} \rightarrow \text{Fun}(\mathbb{J}', \mathbb{C})_{\text{inj}}$$

*the **homotopy right Kan extension** functor, and denote it by*

$$\text{hoRKE}_\iota : \text{Fun}(\mathbb{J}, \mathbb{C})[W^{-1}] \rightarrow \text{Fun}(\mathbb{J}', \mathbb{C})[W^{-1}].$$

*When $\mathbb{J}' = [0]$ is the singleton, we obtain **homotopy limit** functor*

$$\text{holim} : \text{Fun}(\mathbb{J}, \mathbb{C})[W^{-1}] \rightarrow \mathbb{C}[W^{-1}],$$

which is the right derived functor of

$$\lim : \text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}} \rightarrow \mathbb{C}.$$

*Dually, we define the **homotopy left Kan extension** functor and the **homotopy colimit** functor.*

Warning 2.15. *Note that*

$$\text{Fun}(\mathbb{J}, \mathbb{C})[W^{-1}] \neq \text{Fun}(\mathbb{J}, \mathbb{C}[W^{-1}]).$$

Hence homotopy limit is not a functorial construction about diagrams in $\mathbb{C}[W^{-1}]$.

Remark 2.16. *In fact, Theorem 1.2 implies the theory of ∞ -limits serves as a remedy for the non-functoriality of the classical theory of homotopy limits. Namely, instead of considering diagrams into the ordinary homotopy category $\mathbb{C}[W^{-1}]$, one should consider diagrams into the ∞ -category modelled by the quasi-category $\mathfrak{N}_\bullet(\mathbb{C}^\circ)$. In fact, the latter can be canonically identified with the quasi-categorical localization $\mathbf{N}_\bullet(\mathbb{C})[W^{-1}]$. See [Lecture 5, A.5] for more information.*

Construction 2.17. *Let $\bar{w} : \mathbb{J}^\natural \rightarrow \mathbb{C}$ be a functor and $w : \mathbb{J} \rightarrow \mathbb{C}$ be its restriction. There is an obvious morphism in $\text{Fun}(\mathbb{J}, \mathbb{C})$ from the constant functor $\bar{w}(\ast)$ to w . By adjunction, we obtain a canonical morphism*

$$\bar{w}(\ast) \rightarrow \lim w.$$

Definition 2.18. Let $\bar{w} : \mathbb{J}^\triangleleft \rightarrow \mathbb{C}$ be a functor and $w : \mathbb{J} \rightarrow \mathbb{C}$ be its restriction. We say \bar{w} *exhibits $\bar{w}(\ast)$ as the homotopy limit* of w if for any/all fibrant replacement $w \rightarrow w'$, the composition

$$\bar{w}(\ast) \rightarrow \lim w \rightarrow \lim w'$$

is an isomorphism.

3. EXAMPLES OF HOMOTOPY (CO)LIMITS

3.1. Throughout this section, \mathbb{C} is a combinatorial simplicial model category.

Exercise 3.2. Let \mathbb{J} be a set. Show that:

- (1) A functor $w : \mathbb{J} \rightarrow \mathbb{C}$ is fibrant in $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}}$ iff $w(j) \in \mathbb{C}$ is fibrant for any $j \in \mathbb{J}$.
- (2) A functor $w : \mathbb{J} \rightarrow \mathbb{C}$ is cofibrant in $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{proj}}$ iff $w(j) \in \mathbb{C}$ is cofibrant for any $j \in \mathbb{J}$.

Deduce that the homotopy (co)products of (co)fibrant objects can be calculated by the naive (co)products.

Exercise 3.3. Let $\mathbb{J} := \{a_0 \xrightarrow{f_0} b \xleftarrow{f_1} a_1\}$ be the index category of pullbacks. Show that a functor $w : \mathbb{J} \rightarrow \mathbb{C}$ is fibrant in $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}}$ iff $w(b) \in \mathbb{C}$ is fibrant and $w(f_i)$ are fibrations. Deduce a sufficient condition for a homotopy pullback diagram in \mathbb{C} .

Remark 3.4. Note that the above condition is stronger than those in [Lecture 1, Exercise A.1]. The reason is: the latter conditions are obtained by using another model structure on $\text{Fun}(\mathbb{J}, \mathbb{C})$, known as the Reedy model structure. For more information, see HTT.A.2.9.

Exercise 3.5. Let $\mathbb{J} := \{a \rightrightarrows b\}$ be the index category of equalizers. Show that a functor $w : \mathbb{J} \rightarrow \mathbb{C}$ is fibrant in $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}}$ iff $w(b) \in \mathbb{C}$ is fibrant and $w(a) \rightarrow w(b \times b)$ is a fibration. Deduce a sufficient condition for a homotopy equalizer diagram in \mathbb{C} .

Exercise 3.6. Let $\mathbb{J} := \{\dots < -2 < -1 < 0\}$ be the index category of sequential limits. Show that a functor $w : \mathbb{J} \rightarrow \mathbb{C}$ is fibrant in $\text{Fun}(\mathbb{J}, \mathbb{C})_{\text{inj}}$ iff $w(0) \in \mathbb{C}$ is fibrant and $w(-n) \rightarrow w(-n+1)$ is a fibration. Deduce a sufficient condition for a homotopy sequential limit diagram in \mathbb{C} .

Exercise 3.7. Can you find a weaker necessary condition for a homotopy sequential limit diagram in \mathbb{C} ?

Exercise 3.8. Let \mathbb{J} be a fibrant object in Cat_Δ and $\underline{X} : \mathbb{J} \rightarrow \mathbb{C}$ a constant functor with value X . Use the results in [Lecture 7, Appendix B] to show that

$$\begin{aligned} \text{holim } \underline{X} &\simeq \text{Fun}(\mathfrak{N}_\bullet(\mathbb{J}), X), \\ \text{hocolim } \underline{X} &\simeq \mathfrak{N}_\bullet(\mathbb{J}) \otimes X. \end{aligned}$$

APPENDIX A. MARKED SIMPLICIAL SETS

Definition A.1. Let Set_Δ^+ be the ordinary category defined by:

- Objects are pairs (X, E) , where X is a simplicial set and $E \subset X_1$ is a subset of 1-simplices in X , called the set of **marked 1-simplices**.
- A morphism from (X, E) to (X', E') is a morphism $X \rightarrow X'$ in Set_Δ^+ such that E is sent into E' .

We call it the category of **marked simplicial set**.

Construction A.2. There is a functor

$$(-)^{\flat} : \mathbf{Set}_{\Delta} \rightarrow \mathbf{Set}_{\Delta}^{+}$$

such that the marked 1-simplexes in X^{\flat} are given by degenerate 1-simplexes in X .

There is a functor

$$(-)^{\sharp} : \mathbf{Set}_{\Delta} \rightarrow \mathbf{Set}_{\Delta}^{+}$$

such that any 1-simplex in X is a marked 1-simplex in X^{\sharp} .

There is a functor

$$(-)^{\natural} : \mathbf{QCat} \rightarrow \mathbf{Set}_{\Delta}^{+}$$

such that the marked 1-simplexes in X^{\natural} are given by isomorphisms in X .

Exercise A.3. Let X and Y be objects in $\mathbf{Set}_{\Delta}^{+}$. Show that the functor

$$\mathrm{Hom}_{\mathbf{Set}_{\Delta}^{+}}((-)^{\flat} \times X, Y) : \mathbf{Set}_{\Delta}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

is represented by an object in \mathbf{Set}_{Δ} , which we denote by $\mathrm{Fun}^{\flat}(X, Y)$.

Similarly, show that the functor

$$\mathrm{Hom}_{\mathbf{Set}_{\Delta}^{+}}((-)^{\sharp} \times X, Y) : \mathbf{Set}_{\Delta}^{\mathrm{op}} \rightarrow \mathbf{Set}$$

is represented by an object in \mathbf{Set}_{Δ} , which we denote by $\mathrm{Fun}^{\sharp}(X, Y)$.

Exercise A.4. Construct a canonical model structure on $\mathbf{Set}_{\Delta}^{+}$ such that

- Any morphism is a cofibration;
- Bifibrant objects are given by X^{\natural} for $X \in \mathbf{QCat}$;
- Weak equivalence between bifibrant objects are given by categorical equivalences between quasi-categories;
- The model structure is compatible with the simplicial enrichment given by $\mathrm{Fun}^{\sharp}(-, -)$.

A.5. **Suggested readings.** HTT.3.1.