PROBLEM SET 1

Due: Oct 10, noon

100 credits + 50 bonus

Problem 1 (10 credits)**.** Let k be an algebraic closed field and $\mathbb{A}^1_k = \mathsf{Spm} k[x]$.

(1) (5 credits) Show that any bijection $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is continuous for the Zariski topology.
(2) (5 credits) Find those bijections coming from a homomorphism $k[x] \to k[x]$.

Problem 2 (10+10 credits)**.** Let X be a topological space and $\mathfrak{B}$ be a base of open subsets of X.

(1) (10 credits) Let $\mathcal{F}$ and $\mathcal{F}'$ be sheaves on X and $\alpha : \mathcal{F}|_{\mathfrak{B}} \to \mathcal{F}'|_{\mathfrak{B}}$ be a natural transformation between their restrictions on the full subcategory $\mathfrak{B}^{\mathsf{op}} \subseteq \mathfrak{U}(X)^{\mathsf{op}}$. Show that α can be uniquely extended to a morphism $\phi : \mathcal{F} \to \mathcal{F}'$.
(2) (10 bonus credits) Show that for presheaves, similar claims about existence and uniqueness are both false in general.

Problem 3 (10+10 credits)**.** Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X.

(1) (10 credits) Show that $\mathcal{F}$ is a sheaf iff for any open covering $U = \bigcup_{i\in I} U_i$, the sequence
$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$
is exact. Here the second map is
$$s \mapsto (s|_{U_i}),$$
and the third map is
$$(s_i) \mapsto (s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j}).$$
(2) (10 bonus credits) Suppose $\mathcal{F}$ is a sheaf, can you further extend this exact sequence to the right?

Problem 4. (10 credits) Let X be a topological space and $x \in X$ be a point. For any set A, show that the skyscraper $\delta_{x,A}$ is a sheaf of sets.

Problem 5 (10+10 credits)**.** Let X be a topological space and $U \subseteq X$ be an open subset.

(1) (10 credits) Show that the functor
$$\mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{Set},\ \mathcal{F} \to \mathcal{F}(U)$$

admits a left adjoint. In other words, for any set A, there exists a presheaf $\underline{A}_U \in \mathsf{PShv}(X, \mathsf{Set})$ equipped with a map $f : A \to \underline{A}_U(U)$ such that for any presheaf $\mathcal{F}$, the following composition is a bijection

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\underline{A}_U, \mathcal{F}) \xrightarrow{(-)(U)} \mathsf{Hom}_{\mathsf{Set}}(\underline{A}_U(U), \mathcal{F}(U)) \xrightarrow{-\circ f} \mathsf{Hom}_{\mathsf{Set}}(A, \mathcal{F}(U)).$$

(2) (10 bonus credits) Show that this functor also admits a right adjoint.

Problem 6. (20+10 credits) Let X be a topological space and $E \to X$ and $E' \to X$ be two covering spaces. Consider the map

$$\mathsf{Hom}_X(E, E') \to \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathsf{Sect}_E, \mathsf{Sect}_{E'})$$

sending a continuous map $f : E \to E'$ defined over X to the morphism $\phi : \mathsf{Sect}_E \to \mathsf{Sect}_{E'}$ given by

$$\mathsf{Sect}_E(U) \to \mathsf{Sect}_{E'}(U),\ s \mapsto f \circ s.$$

(1) (10 credits) Show that the above map is a bijection.
(2) (10 credits) Show that f is bijective iff ϕ is an isomorphism in $\mathsf{Shv}(X, \mathsf{Set})$.
(3) (10 bonus credits) Show that (1) and the "if" claims in (2) can fail for general spaces E and E' over X.

Problem 7. (10 credits) Let $f : X \to X'$ be a continuous map between topological spaces. Show that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{PShv}(X', \mathsf{Set}) & \xrightarrow{f^{-1}_{\mathsf{PShv}}} & \mathsf{PShv}(X, \mathsf{Set}) \\ \downarrow{\scriptstyle (-)^\sharp} & & \downarrow{\scriptstyle (-)^\sharp} \\ \mathsf{Shv}(X', \mathsf{Set}) & \xrightarrow{f^{-1}} & \mathsf{Shv}(X, \mathsf{Set}). \end{array}$$

Problem 8. (10 credits) Let $f : X \to X'$ be a continuous map between topological spaces. Show that f^{-1} sends a constant sheaf to the constant sheaf associated to the same set.

Problem 9. (10 credits) Let $X' = \{s, b\}$ be the topological space with two points whose open subsets are exactly given by $\varnothing, \{b\}$ and X'. Consider the following diagram

$$\begin{array}{ccc} \varnothing & \xrightarrow{j} & \{s\} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \{b\} & \xrightarrow{j'} & X'. \end{array}$$

Show that the base-change natural transformations $f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$ and $f^{-1} \circ j'_* \to j_* \circ g^{-1}$ are not invertible.

Problem 10. (10 bonus credits) Let X be a topological space and $U \subseteq X$ be an open subset. Write $j : U \to X$ for the embedding. Show that both

$$j^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{PShv}(U, \mathsf{Set})$$

and

$$j^{-1} : \mathsf{Shv}(X, \mathsf{Set}) \to \mathsf{Shv}(U, \mathsf{Set})$$

admit a left adjoint, and give an explicit construction of these left adjoints.