PROBLEM SET 1

Due: Oct 29, noon

100 credits + 50 bonus

Problem 1 (10 credits)**.** Let k be an algebraically closed field. Describe the underlying topological space of $\mathsf{Spec}(k[x,y]/(xy))$.

Problem 2 (10 bonus credits)**.** Give an example of $R, R' \in \mathsf{CRing}$ such that there exists a morphism $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ between *ringed spaces* which is not a morphism between *locally ringed spaces*.

Problem 3 (10 credits)**.** Let X be a scheme over $\mathsf{Spec}(\mathbb{F}_q)$.

(1) (5 credits) For any open subset $U \subseteq X$, show that the map $\beta_U : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$, $f \mapsto f^q$ is a homomorphism, and these maps give an endomorphism $\beta : \mathcal{O}_X \to \mathcal{O}_X$ of the structure sheaf.

(2) (5 credits) Show that $\mathsf{Frob}_{X,q} := (\mathsf{id}_X, \beta)$ is an endomorphism of the scheme X defined over $\mathsf{Spec}(\mathbb{F}_q)$.

Problem 4 (20+10 credits)**.** Let X be a topological space. For any open subset $U \subseteq X$, let $\mathcal{C}_X(U)$ be the commutative ring of $\mathbb{R}$-valued[1] continuous functions on U. Note that $U \mapsto \mathcal{C}_X(U)$ defines a sheaf of commutative rings on X.

(1) (10 credits) Show that $(X, \mathcal{C}_X)$ is a locally ringed space.

(2) (10 credits) Show that a continuous map $X \to X'$ induces a morphism $(X, \mathcal{C}_X) \to (X', \mathcal{C}_{X'})$ between locally ringed spaces.

(3) (10 bonus credits) Show that $(\mathbb{R}, \mathcal{C}_{\mathbb{R}})$ is not a scheme.

Problem 5 (10 credits)**.** Let k be a field and $R = k[x,y]$. Consider the point $(0,0) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x,y). Let $U := \mathsf{Spec}(R) \smallsetminus \{(0,0)\}$ be the complementary open subset.

(1) (5 credits) Find the commutative ring $\mathcal{O}(U)$.

(2) (5 credits) Show that U is not an affine scheme.

Problem 6 (20 bonus credits)**.** Let k be a field of characteristic 0 and $R := k[x,y]/(y^2 - x^3)$. Consider the point $(1,1) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal $(x-1, y-1)$. $U := \mathsf{Spec}(R) \smallsetminus \{(1,1)\}$ be the complementary open subset.

(1) (10 bonus credits) Show that U is not a standard open subset of $\mathsf{Spec}(R)$.

(2) (10 bonus credits) Show that U is an affine scheme.

Problem 7 (20 points)**.** Let R be a commutative ring and k be an algebraically closed field.

(1) (10 points) Find $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R)$. Deduce that $\mathbb{P}^n_R$ is not affine for $n \geq 1$.

(2) (10 points) Show that the closed points of $\mathbb{P}^n_k$ can be canonically identified with elements in $(k^{n+1} \smallsetminus 0)/k^\times$, where $k^\times$ acts on the vector space k^{n+1} via scaler multiplication.

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[1]We equip $\mathbb{R}$ with the usual topology.

Problem 8 (10 points)**.** Let R be any commutative ring and $I = \{1, 2\}$. Let

$$X_1 = X_2 := \mathbb{A}^1_R := \mathsf{Spec}(R[t])$$

and

$$U_{12} = U_{21} := U(t),\, U_{11} := X_1,\, U_{22} := X_2.$$

Let ϕ_{ij} be the identity morphisms. Consider the scheme X glued from the above gluing data. Show that X is not affine.

Problem 9 (10 bonus points)**.** Show that the embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ does not admit a right adjoint.

Problem 10 (10 points)**.** Show that the functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to the set $\mathsf{GL}_n(R)$ of $n \times n$ invertible matrices over R is represented by an affine scheme.

Problem 11 (10 points)**.** Let I be a set.

(1) (5 points) Show that the constant functor $\mathsf{CRing} \to \mathsf{Set}$, $R \mapsto I$ is not represented by a scheme unless $I \simeq \{*\}$.
(2) (5 points) What is the functor represented by the disjoint union $\bigsqcup_{i \in I} \mathsf{Spec}(\mathbb{Z})$?