PROBLEM SET 3

Due: Nov 12, noon

100 credits + 50 bonus

Problem 1 (10 credits)**.** Show that pushouts exist in CRing and we have

$$A \coprod_C B \simeq A \otimes_C B.$$

Problem 2 (10+10 credits)**.** Let $k \to k'$ be a finite extension of degree d and $\overline{k}$ be an algebraic closure of k.

(1) (10 credits) Suppose $k \to k'$ is a separable extension. Show that

$$\mathsf{Spec}(k') \underset{\mathsf{Spec}(k)}{\times} \mathsf{Spec}(\overline{k}) \simeq \coprod_d \mathsf{Spec}(\overline{k}).$$

(2) (10 bonus credits) For general finite extension $k \to k'$, how many points does $\mathsf{Spec}(k') \times_{\mathsf{Spec}(k)} \mathsf{Spec}(\overline{k})$ have? What are the residue fields of these points?

Problem 3 (10+10 credits)**.** Let X be a scheme.

(1) (10 credits) Suppose X is affine. Show that the intersection of two affine open subsets in X is still affine.
(2) (10 bonus credits) Show that (1) may fail for general X.

Problem 4 (10 credits)**.** Show that the base-change of an open immersion is still an open immersion.

Problem 5 (10 credits)**.** Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram of schemes. Show that the morphism

$$\coprod_{(x,y,s)} \mathsf{Spec}(\kappa_x) \underset{\mathsf{Spec}(\kappa_s)}{\times} \mathsf{Spec}(\kappa_y) \to X \underset{S}{\times} Y$$

induces a bijection between topological points. Here $x \in X$, $y \in Y$ and $s \in S$ are topological points such that $f(x) = g(y) = s$.

Problem 6 (10 bonus credits)**.** Show that the functor $\mathsf{Aff} \to \mathsf{Sch}$ sends monomorphisms to monomorphisms, but may fail to send epimorphisms to epimorphisms.

Problem 7 (10 credits)**.** Let k be a field. Consider $R = k[x,y]/(y^2 - x^3)$, $R' = k[t]$ and the homomorphism $R \to R'$, $x \mapsto t^3$, $y \mapsto t^2$. Show that $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ induces a homeomorphism between the underlying topological spaces, but it is not a closed immersion.

Problem 8 (10 credits)**.** Let k be a field. Consider the closed immersion $i : \mathbb{A}^1_k \to \mathbb{A}^2_k$ corresponding to the surjection $k[x,y] \to k[x]$. Find an open subset $U \subseteq \mathbb{A}^2_k$ such that $\mathcal{O}_{\mathbb{A}^2_k}(U) \to (i_* \mathcal{O}_{\mathbb{A}^1_k})(U)$ is not surjective.

Problem 9 (10 bonus credits)**.** Let $i : Y \to X$ be an open immersion between affine schemes. Show that $\mathcal{O}_X(U) \to i_*\mathcal{O}_Y(U)$ is surjective for any affine open subset $U \subseteq X$.

Problem 10 (10+10 credits)**.** Let $i : Y \to X$ be a closed immersion and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition.

(1) (10 credits) Show that $\mathcal{O}_X/\mathcal{I}_Y$ is the sheafification of the presheaf
$$U \mapsto \mathcal{O}_X(U)/\mathcal{I}_Y(U).$$
(2) (10 bonus credits) Give an example where this sheafification step is necessary.

Problem 11 (20 credits)**.** Let k be a field and $X := \mathbb{A}^1_k$ and $0 \in X$ be the zero point.

(1) (10 credits) Show that
$$\mathcal{I}(U) := \begin{cases} \mathcal{O}(U) & \text{for} \quad 0 \notin U \\ 0 & \text{for} \quad 0 \in U \end{cases}$$
defines an sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$.

(2) (10 credits) Show that $\mathcal{I}$ is not an ideal of definition for any closed immersions into X.