PROBLEM SET 6

Due: Dec 26, noon

100 credits + 50 bonus

Problem 1 (10 credits)**.** Use the valuative criterion to show that $\mathbb{P}^n_{\mathbb{Z}} \to \mathsf{Spec}(\mathbb{Z})$ is proper.

Problem 2 (10 bonus credits)**.** Let k be a field. Show that the direct image functor along the closed immersion $0 \to \mathbb{A}^\infty_k$ does not preserve coherent modules.

Problem 3 (10 credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring and $f \in S_+$ be a homogeneous element. Show that the composition

$$U_+(f) \subseteq U(f) \simeq \mathsf{Spec}(S_f) \to \mathsf{Spec}(S_{(f)})$$

is a homeomorphism.

Problem 4 (40 bonus credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring and

$$\mathsf{Spec}(S)^\circ := \mathsf{Spec}(S) \smallsetminus \mathsf{Spec}(S_0).$$

(1) (10 bonus credits) Show that the map

$$\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S),\ \mathfrak{p} \mapsto \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (\mathfrak{p} \cap S_n)$$

is well-defined and continuous. Moreover, for any homogeneous $f \in S_+$,

$$\pi^{-1}(U_+(f)) = U(f).$$

(2) (10 bonus credits) Show that π is an epimorphism of schemes.

(3) (10 bonus credits) Show that $\mathsf{Spec}(S)^\circ \xrightarrow{\subseteq} \mathsf{Spec}(S) \to \mathsf{Spec}(S_0)$ uniquely factors through $\mathsf{Proj}(S)$.

(4) (10 bonus credits) Suppose S_+ is generated by S_1. Show that for any $\mathbb{Z}$-graded S-module M,

$$\pi^*(\widetilde{M}) \simeq \widetilde{M}|_{\mathsf{Spec}(S)^\circ}$$

as objects in $\mathsf{QCoh}(\mathsf{Spec}(S)^\circ)$.

Problem 5 (10 credits)**.** Let R be a commutative ring and $n \geq 0$. Let $S = R[x_0, \cdots, x_n]$ be the $\mathbb{Z}_{\geq 0}$-graded ring such that $S_0 = R$ and $x_i \in S_1$. Show that $\mathsf{Proj}(S) \simeq \mathbb{P}^n_R$.

Problem 6 (20 credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring and d be a positive integer.

(1) (10 credits) Let $S' = \bigoplus_{n \geq 0} S'_{nd}$ be the $\mathbb{Z}_{\geq 0}$-graded ring such that $S'_{nd} := S_n$. Show that $\mathsf{Proj}(S') \simeq \mathsf{Proj}(S)$.

(2) (10 credits) Let $S'' \subseteq S$ be the subring generated by elements in S_{nd}, $n \geq 0$. Show that $\mathsf{Proj}(S'') \simeq \mathsf{Proj}(S)$.

Problem 7 (20 credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring such that S_+ is generated by S_1. Let M be a $\mathbb{Z}$-graded S-module that is of finite presentation. Show that

(1) (10 credits) The $\mathcal{O}$-module $\widetilde{M}$ is of finite presentation.
(2) (10 credits) For any $\mathbb{Z}$-graded S-module N, we have

$$\underline{\mathsf{Hom}}_{\mathcal{O}}(\widetilde{M}, \widetilde{N}) \simeq \widetilde{\mathsf{Hom}^{\mathsf{gr}}_S(M, N)},$$

where $\mathsf{Hom}^{\mathsf{gr}}_S(M, N)$ is the $\mathbb{Z}$-graded S-module such that

$$\mathsf{Hom}^{\mathsf{gr}}_S(M, N)_d := \mathsf{Hom}_{S-\mathsf{mod}_{\mathsf{gr}}}(M, N(d)).$$

Problem 8 (10 credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring such that S_+ is generated by finitely many elements in S_1. Let M be a $\mathbb{Z}$-graded S-module that is eventually of finite type. Show that $\widetilde{M} = 0$ iff M is eventually zero.

Problem 9 (10 credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring such that S_+ is generated by S_1. Show that for any $n \geq 0$, the $\mathcal{O}_{\mathsf{Proj}(S)}$-module $\mathcal{O}_{\mathsf{Proj}(S)}(n)$ is generated by its global sections that belong to the image of the canonical map

$$S_n \to \Gamma(\mathsf{Proj}(S), \mathcal{O}(n)).$$

Problem 10 (10 credits)**.** Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring such that S_+ is generated by S_1. Let $I \subseteq S$ be a graded ideal and $i : \mathsf{Proj}(S/I) \to \mathsf{Proj}(S)$ be the corresponding closed immersion. Show that

$$i^*(\mathcal{O}_{\mathsf{Proj}(S)}(n)) \simeq \mathcal{O}_{\mathsf{Proj}(S/I)}(n).$$