

HOMEWORK PROBLEMS

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HOMEWORK 1 (DUE ON MARCH 18)

Problem 1.1 (Lecture 2, Exercise 2.6). Prove the following.

(1) The map

$$U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b}) \xrightarrow{\text{mult}} U(\mathfrak{g})$$

is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$ -bimodules.

(2) As an \mathfrak{n}^- -module, M_λ is freely generated by v_λ , i.e.,

$$U(\mathfrak{n}^-) \longrightarrow M_\lambda, \quad x \longmapsto x \cdot v_\lambda$$

is an isomorphism.

Solution. (1) It is clear from the construction that mult is a homomorphism of left $U(\mathfrak{n}^-)$ -modules and right $U(\mathfrak{b})$ -modules. Note that the target $U(\mathfrak{g})$ is regarded as a $(U(\mathfrak{n}^-), U(\mathfrak{b}))$ -bimodule via restricting the $U(\mathfrak{g})$ -action on itself to $U(\mathfrak{b})$ and $U(\mathfrak{n}^-)$, respectively. It remains to show that mult is an isomorphism of vector spaces by PBW theorem [Lecture 2, Theorem 1.5]. For this, according to the context of [Lecture 2, Corollary 1.6], pick $\{x_1, \dots, x_n\}$ as a basis of \mathfrak{n}^- and $\{y_1, \dots, y_m\}$ as a basis of \mathfrak{b} . Then the bases of k -vector spaces $U(\mathfrak{n}^-)$, $U(\mathfrak{b})$, $U(\mathfrak{g})$ are $\{x_1^{k_1} \cdots x_n^{k_n}\}_{k_i \geq 0}$, $\{y_1^{l_1} \cdots y_m^{l_m}\}_{l_j \geq 0}$, $\{x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_m^{l_m}\}_{k_i, l_j \geq 0}$, respectively; it follows that $U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b})$ has a basis $\{x_1^{k_1} \cdots x_n^{k_n} \otimes y_1^{l_1} \cdots y_m^{l_m}\}_{k_i, l_j \geq 0}$. Moreover,

$$\text{mult}: x_1^{k_1} \cdots x_n^{k_n} \otimes y_1^{l_1} \cdots y_m^{l_m} \longmapsto x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_m^{l_m}.$$

This completes the proof that mult is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$ -bimodules.

(2) Using (1), we have that

$$U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} k_\lambda \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda = M_\lambda.$$

Here note that each $x \cdot v_\lambda \in M_\lambda$ can be identified with $x \otimes v_\lambda \in U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$. On the other hand, the left-hand side is isomorphic to $U(\mathfrak{n}^-) \otimes_k k_\lambda$, which is further isomorphic to $U(\mathfrak{n}^-)$. This completes the proof. \square

Problem 1.2 (Lecture 2, Exercise 2.17). In the case $\mathfrak{g} = \mathfrak{sl}_2$, show the Verma module M_l is irreducible unless $l \in \mathbb{Z}_{\geq 0}$. In the latter case, show there is a non-split short exact sequence

$$0 \longrightarrow M_{-l-2} \longrightarrow M_l \longrightarrow L_l \longrightarrow 0$$

such that L_l is a finite-dimensional irreducible \mathfrak{sl}_2 -module with highest weight l .

Solution. Suppose M_l is not irreducible. Then there is a nonzero proper submodule $N \subset M_l$ of highest weight l' ; denote by $v_{l'}$ the highest weight vector. In this case l' is also regarded as a weight of M_l . Recall that for $\mathfrak{g} = \mathfrak{sl}_2$, any weight of M_l is of form $l - 2n$ with $n \geq 0$. So we may assume $l' = l - 2n$. Since $v_{l'}$ generates N via \mathfrak{g} -action, the hypothesis $N \subsetneq M_l$ implies $l' \neq l$ (otherwise $M_l = N$), or equivalently $n > 0$. Let $e, f, g \in \mathfrak{sl}_2$ be the standard generators. The

weight of $e \cdot v_l$ is either 0 or $l' + 2$. Since l is the highest weight of M_l , we must be in the former case that $e \cdot v_l = 0$. Thus, we obtain from $[e, f] = h$ and $h \cdot f^j - f^j \cdot h = -2j f^j$ that¹

$$\begin{aligned} e \cdot f^n \cdot v_l &= \sum_{1 \leq i \leq n} f^{n-i} \cdot [e, f] \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l \\ &= \sum_{1 \leq i \leq n} f^{n-i} \cdot h \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l \\ &= \sum_{1 \leq i \leq n} (l - 2(i-1)) \cdot f^{n-1} \cdot v_l \\ &= n(l - (n-1)) \cdot f^{n-1} \cdot v_l. \end{aligned}$$

Since the above is 0 whereas $f^{n-1} \cdot v_l \neq 0$, it implies that $l = n - 1 \geq 0$. This shows the irreducibility of M_l for $l < 0$.

If $l = n - 1$ for $n > 0$, the same computation shows that $e \cdot f^n \cdot v_l = e \cdot v_{l-2n} = 0$, with $l - 2n = -l - 2$, i.e. the vector $v_{l-2n} \in M_l$ generates a submodule of M_l that is isomorphic to M_{-l-2} . By fixing an isomorphism $L_l \simeq M_l/M_{-l-2}$, we get a quotient module L_l of M_l as well as the desired short exact sequence $0 \rightarrow M_{-l-2} \rightarrow M_l \rightarrow L_l \rightarrow 0$. Such L_l is clearly finite-dimensional of highest weight l . If the sequence splits, then there is a nonzero section map $M_l \rightarrow M_{-l-2}$, and hence a nonzero vector in M_{-l-2} of weight l , which is impossible. So the sequence is non-split.

It remains to show the irreducibility of L_l . For this, note that whenever $l = n - 1$ the highest weight of a proper submodule N of M_l must be $-l - 2$, so there is no proper submodule of M_l containing M_{-l-2} . Correspondingly, the quotient L_l must be irreducible. \square

Problem 1.3 (Lecture 2, Exercise 3.9). Recall for any $V_1, V_2 \in \mathfrak{g}\text{-mod}$, the tensor product $V_1 \otimes V_2$ of the underlying vector spaces has a natural \mathfrak{g} -module structure defined by $x \cdot (v_1 \otimes v_2) := (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$.

- (1) Prove that if V_1 and V_2 are weight modules, so is $V_1 \otimes V_2$. Determine the weights and weight spaces of $V_1 \otimes V_2$ in terms of those for V_1 and V_2 .
- (2) Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Prove that the tensor product of two Verma modules is not contained in \mathcal{O} .

Solution. (1) If V_1, V_2 are weight modules, then we can respectively take $V_{1,\lambda} \subset V_1$ and $V_{2,\nu} \subset V_2$ to be λ - and ν -eigenspaces, where $\lambda, \nu \in \mathfrak{t}^*$. For all $t \in \mathfrak{t}$ together with $v_1 \in V_{1,\lambda}$ and $v_2 \in V_{2,\nu}$, we have $t \cdot (v_1 \otimes v_2) = (t \cdot v_1) \otimes v_2 + v_1 \otimes (t \cdot v_2) = \lambda(t)v_1 \otimes v_2 + \nu(t)v_1 \otimes v_2 = (\lambda(t) + \nu(t)) \cdot v_1 \otimes v_2$. It follows that $v_1 \otimes v_2 \in V_1 \otimes V_2$ is of weight $\lambda + \nu$. This proves that $V_1 \otimes V_2$ is a weight module; each weight space of $V_1 \otimes V_2$ is of form $\bigoplus_{\lambda+\nu=\mu} V_{1,\lambda} \otimes V_{2,\nu}$ for some fixed $\mu \in \mathfrak{t}^*$.

(2) Let M_λ, M_ν be two Verma modules. We prove $M_\lambda \otimes M_\nu \notin \mathcal{O}$ by showing that it is not finitely generated. Suppose for the sake of contradiction that $M_\lambda \otimes M_\nu$ is generated by m_1, \dots, m_n for some $n \in \mathbb{Z}$. Indeed, $M_\lambda \otimes M_\nu$ is isomorphic to a quotient module of an extension of finitely many Verma modules. When $\mathfrak{g} = \mathfrak{sl}_2$, the weight spaces of each of these Verma modules are all 1-dimensional. (Note that this is not true for general \mathfrak{g} .) After taking the quotient towards $M_\lambda \otimes M_\nu$, it follows that the dimension of any weight space is at most n . Also, since $\mathfrak{g} = \mathfrak{sl}_2$, there turns out to be a weight space of $M_\lambda \otimes M_\nu$ of weight $\lambda + \nu - 2n$ and dimension $n + 1$. This is impossible, and thus $M_\lambda \otimes M_\nu$ cannot be finitely generated. \square

Problem 1.4 (Lecture 3, Exercise 5.6).

- (1) Find all maps between k -schemes $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \setminus \{0\}$.
- (2) Find all 1-dimensional representations of the additive group \mathbb{G}_a .
- (3) Find all maps between k -schemes $\mathbb{A}^1 \setminus \{0\} \rightarrow \mathbb{A}^1 \setminus \{0\}$.
- (4) Find all 1-dimensional representations of the multiplicative group \mathbb{G}_m .

¹There is a typo in the proof of [Gai05, Proposition 1.9], c.f. the third line of the computation.

Solution. (1) It suffices to recognize all homomorphisms $k[x]_x \rightarrow k[x]$ between k -algebras; here $k[x]_x$ is the localization of $k[x]$ at the point with coordinate x . Since x is invertible in $k[x]_x$, its image in $k[x]$ must be invertible as well, which implies that the image of x must be an element of k^\times . It follows that all $k[x]_x \rightarrow k[x]$ (with $1 \mapsto 1$ and $x \mapsto t \in k^\times$) are parameterized by k^\times . Therefore, all maps between k -schemes $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \setminus 0$ are exactly parametrized by k^\times , which are constant maps sending all points on \mathbb{A}^1 to some point on $\mathbb{A}^1 \setminus 0$.

(2) Any 1-dimensional representation of \mathbb{G}_a is given by the map $\mathbb{G}_a \rightarrow \mathbb{G}_m$ of group schemes over some (algebraically closed) field, say k . At the level of k -schemes, the data of $\mathbb{G}_a \rightarrow \mathbb{G}_m$ can be specialized to the data of $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \setminus 0$. But the result of (1) forces $\mathbb{G}_a \rightarrow \mathbb{G}_m$ to be a constant map, which can only be the trivial representation of \mathbb{G}_a . In other words, the 1-dimensional representation of \mathbb{G}_a can only be trivial.

(3) As in (1), we aim to figure out all homomorphisms $k[x]_x \rightarrow k[x]_x$ between k -algebras. It suffices to determine the image of x . Note that all invertible elements of $k[x]_x$ are of form tx^n with $t \in k^\times$ and $n \in \mathbb{Z}$. We thus conclude that each map $\mathbb{A}^1 \setminus 0 \rightarrow \mathbb{A}^1 \setminus 0$ is parametrized by some $tx^n \in k^\times x^{\mathbb{Z}}$, sending $P \in \mathbb{A}^1 \setminus 0$ to tP^n .

(4) As in (2), consider the map $\mathbb{G}_m \rightarrow \mathbb{G}_m$ of group schemes over k . Note that this is determined by $\mathbb{A}^1 \setminus 0 \rightarrow \mathbb{A}^1 \setminus 0$, which must be of form $P \mapsto tP^n$ by (3). Moreover, as a group homomorphism, we must have $\mathbf{1} \mapsto \mathbf{1}$ where $\mathbf{1}$ is the identity element of \mathbb{G}_m ; it hence implies $t = 1 \in k^\times$. Therefore, the desired 1-dimensional representation of \mathbb{G}_m must be $\mathbb{G}_m \rightarrow \mathbb{G}_m, g \mapsto g^k$ for some $k \in \mathbb{Z}$. \square

Problem 1.5 (Lecture 3, Exercise 6.6). Let G be any semisimple algebraic group with Lie algebra \mathfrak{g} . Prove any Verma module of \mathfrak{g} is not G -integrable.

Solution. Suppose M_λ is a G -integrable Verma module of \mathfrak{g} , i.e. M_λ is a \mathfrak{g} -module coming from a representation of G through the functor $\text{Rep}(G) \rightarrow \mathfrak{g}\text{-mod}$. By [Lecture 3, Proposition 5.7], if this is the case, then the G -action on M_λ is locally finite, i.e. M_λ is a union of finite-dimensional subrepresentations.

Let v_λ be the highest weight vector in M_λ . Note that v_λ generates M_λ through the orbit of \mathfrak{g} -action. Since M_λ is locally finite, there exists a finite-dimensional \mathfrak{g} -submodule containing v_λ that is also a G -invariant subspace; it must equal to M_λ for the prescribed reason. In particular, in this case M_λ is finite-dimensional, which is a contradiction. \square

REFERENCES

[Gai05] Dennis Gaitsgory. *Course Notes for Geometric Representation Theory*. 2005. available at <https://people.mpim-bonn.mpg.de/gaitsgde/267y/cat0.pdf>.