2024 Spring — Geometric Representation Theory (1)

HOMEWORK PROBLEMS

LECTURER: LIN CHEN
TA: WENHAN DAI

Homework 3 (Due on April 15)

Problem 3.1 (Lecture 6, Exercise 1.18)**.** Let $\mathfrak{g} = \mathfrak{sl}_n$ and let $\sigma_i \in \mathrm{Fun}(\mathfrak{t})^W$ be the basic symmetric polynomial of degree i (so that $\prod_{j=1}^n (x + x_j) = x^n + \sigma_1 x^{n-1} + \cdots + \sigma_n$). Consider the homomorphisms (c.f. [Lecture 6, (1.2)–(1.3)])

$$\mathrm{Fun}(\mathfrak{n}) \otimes \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \otimes \mathrm{Fun}(\mathfrak{n}^-) \longrightarrow \mathrm{Fun}(\mathfrak{g}), \tag{1.2}$$

$$\mathrm{Fun}(\mathfrak{n}) \otimes \mathrm{Fun}(\mathfrak{t})^W \otimes \mathrm{Fun}(\mathfrak{n}^-) \longrightarrow \mathrm{Fun}(\mathfrak{g}), \tag{1.3}$$

which are induced by the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$.

(1) For each $1 < i \leqslant n$, find the unique element $\tilde{\sigma}_i \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$ corresponding to σ_i via the Chevalley isomorphism $\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{Fun}(\mathfrak{t})^W$. In other words, find the unique extension of σ_i to an adjoint invariant polynomial function on $\mathfrak{g}$.

(2) For $\mathfrak{g} = \mathfrak{sl}_2$, prove that (1.2) and (1.3) are both injective and have the same image.

(3) For $\mathfrak{g} = \mathfrak{sl}_3$, prove that $\tilde{\sigma}_3$ is contained in the image of (1.2) but not in the image of (1.3).

Solution. (1) We need to construct $(\tilde{\sigma}_i \colon \mathfrak{g} \to k) \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$. For this, given $X \in \mathfrak{g}$, the construction is given by its characteristic polynomial

$$\det(\lambda \cdot \mathrm{id} - X) = \lambda^n + \sum_{i=1}^n (-1)^i \tilde{\sigma}_i(X) \cdot \lambda^{n-i} \in k[\lambda].$$

It is clear that coefficients $\tilde{\sigma}_i \in \mathrm{Fun}(\mathfrak{g})$, and $\tilde{\sigma}_i|_{\mathfrak{t}} = \sigma_i \colon \mathfrak{t} \to k$. Then it remains to check the $\mathfrak{g}$-adjoint invariance. But note that the $\mathfrak{g}$-invariance is the same as G-invariance; then this can be proved by noticing that the characteristic polynomial is invariant under adjoint action, i.e. $\det(\lambda \cdot \mathrm{id} - UXU^{-1}) = \det(\lambda \cdot \mathrm{id} - X)$ for $U \in \mathfrak{g}$, and so also is $\tilde{\sigma}_i(X)$.

(2) Fix an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ that induces $\mathfrak{t} \simeq \mathfrak{t}^*$. For $\mathfrak{g} = \mathfrak{sl}_2$ with the standard basis e, f, h, recall from [Lecture 6, Example 1.16] that

$$\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \simeq \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} = k[\Omega], \quad \mathrm{Fun}(\mathfrak{t})^W \simeq \mathrm{Sym}(\mathfrak{t})^W = k[h^2].$$

Here $\Omega = h^2 + 4ef$ is the image of the Casimir element. Then we can respectively identify (1.2) and (1.3) with

$$k[e] \otimes k[\Omega] \otimes k[f] \longrightarrow k[e, f, h], \quad f_1(e) \otimes f_2(\Omega) \otimes f_3(f) \longmapsto f_1(e) f_2(\Omega) f_3(f)$$

and

$$k[e] \otimes k[h^2] \otimes k[f] \longrightarrow k[e, f, h], \quad f_1(e) \otimes f_2(h^2) \otimes f_3(f) \longmapsto f_1(e) f_2(\Omega) f_3(f).$$

Then using the property of tensor product both maps are clearly injective. Moreover, the images are respectively given by $k[e, \Omega, f]$ and $k[e, h^2, f]$, which are the same in $k[e, f, h]$.

(3) We know from part (1) that $\tilde{\sigma}_3 \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$, and is thus contained in the image of (1.2). If we write $X = (x_{ij}) \in \mathfrak{sl}_3$ as a 3×3-matrix then $\tilde{\sigma}_3(X)$ is a polynomial in all entries x_{ij}. If $\tilde{\sigma}_3$ lies in the image of (1.3) then $\tilde{\sigma}_3(X) = \det(X)$ must be a symmetric polynomial in x_{11}, x_{22}, x_{33} (with regarding other x_{ij} with $i \neq j$ as constant coefficients), which is impossible by elementary computation of $\det(\lambda I - X)$. □

Problem 3.2 (Lecture 6, Exercise 3.1)**.** Consider $\mathfrak{g} = \mathfrak{sl}_3$ and its standard Borel $\mathfrak{b}$ and Cartan subalgebras $\mathfrak{t}$. Let α_1 and α_2 be the two simple positive roots.

(1) Prove: the elements in $W \cdot 0$ are given by

0	$s_1 \cdot 0$	$s_2 \cdot 0$	$s_1 s_2 \cdot 0$	$s_2 s_1 \cdot 0$	$w_0 \cdot 0$
0	$-\alpha_1$	$-\alpha_2$	$-2\alpha_1 - \alpha_2$	$-\alpha_1 - 2\alpha_2$	$-2\alpha_1 - 2\alpha_2$

Here $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is the *longest element* in W.

(2) Prove: $M_{-2\alpha_1-2\alpha_2}$ is irreducible.

(3) Prove: $M_{-\alpha_1-2\alpha_2}$ contains $M_{-2\alpha_1-2\alpha_2}$ as a submodule[1] and the quotient is irreducible[2]. Deduce $\mathrm{length}(M_{-\alpha_1-2\alpha_2}) = 2$ and

$$[M_{-\alpha_1-2\alpha_2} : L_{-\alpha_1-2\alpha_2}] = [M_{-\alpha_1-2\alpha_2} : L_{-2\alpha_1-2\alpha_2}] = 1.$$

Solution. (1) Since the positive roots of $\mathfrak{sl}_3$ are exactly $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$, we have the half-sum $\rho = \frac{1}{2}(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2)) = \alpha_1 + \alpha_2$. Recall that the W-dotted action is given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \in W.$$

Thus to compute $w \cdot 0$ we first notice that

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2,$$
$$s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad s_2(\alpha_2) = -\alpha_2.$$

Using these we immediately get $s_1 \cdot 0 = -\alpha_1$ and $s_2 \cdot 0 = -\alpha_2$. Also, $s_1 s_2 \cdot 0 = s_1(s_2(\rho)) - \rho = s_1(\alpha_1) - (\alpha_1 + \alpha_2) = -2\alpha_1 - \alpha_2$, and $s_2 s_1 \cdot 0 = \alpha_1 - 2\alpha_2$ by a similar computation. It then follows that $w_0 \cdot 0 = s_2(s_1 s_2(\rho)) - \rho = s_2(-2\alpha_1 - \alpha_2) - (\alpha_1 + \alpha_2) = -2\alpha_1 - 2\alpha_2$.

(2) Recall from [Lecture 6, Corollary 2.3] that if $M_{-2\alpha_1-2\alpha_2}$ contains an irreducible submodule of form L_μ, then

$$\mu = w \cdot (-2\alpha_1 - 2\alpha_2) \preceq -2\alpha_1 - 2\alpha_2.$$

Note that this implies $\mu \in W \cdot 0$. On the other hand, since part (1) has given all elements in $W \cdot 0$, we deduce $\mu = -2\alpha_1 - 2\alpha_2$ and thus $M_{-2\alpha_1-2\alpha_2} = L_{-2\alpha_1-2\alpha_2}$, which proves the irreducibility.

(3) From the proof of part (1) we see $-2\alpha_1 - 2\alpha_2 = s_1(-\alpha_1 - 2\alpha_2)$, and s_1 is defined by the simple root $\alpha_1 \in \Delta$. Also, it is clear that $\langle (-\alpha_1 - 2\alpha_2) + \rho, \check{\alpha}_1 \rangle = \langle -\alpha_2, \alpha_1 \rangle \in \mathbb{Z}^{\geqslant 0}$. Then $M_{-\alpha_1-2\alpha_2}$ contains $M_{-2\alpha_1-2\alpha_2}$ as a submodule by [Lecture 5, Lemma 1.3].

To show the irreducibility of $M_{-\alpha_1-2\alpha_2}/M_{-2\alpha_1-2\alpha_2}$, suppose N is a proper submodule of $M_{-\alpha_1-2\alpha_2}$ containing $M_{-2\alpha_1-2\alpha_2}$. We claim that

$$\dim M_{-\alpha_1-2\alpha_2}^{\mathrm{wt}=-2\alpha_1-2\alpha_2} = 1.$$

Indeed, if we write $f_1, f_2, f_3 \in \mathfrak{n}^-$ for the root vectors corresponding to $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$, respectively, then they generate a PBW basis of $U(\mathfrak{n}^-)$. Also, if v_μ is a weight vector of weight μ, then $f_i \cdot v_\mu = v_{\mu-\alpha_i}$ for $i = 1, 2$. Known this, we see as in our case there is no more weights between $-\alpha_1 - 2\alpha_2$ and $-2\alpha_1 - 2\alpha_2$ with respect to the order $\preceq$, the vectors in $M_{-\alpha_1-2\alpha_2}$ of weight $-2\alpha_1 - 2\alpha_2$ can appear exactly at once in $M_{-2\alpha_1-2\alpha_2}$ through the action of $f_1 \in U(\mathfrak{n}^-)$. As $M_{-2\alpha_1-2\alpha_2}$ is free of rank one over $U(\mathfrak{n}^-)$, we have proved claim. Granting the claim, note that the highest weight of N is at least $-2\alpha_1 - 2\alpha_2$, which by our assumption on N implies $N = M_{-2\alpha_1-2\alpha_2}$. Therefore, we conclude that

$$M_{-\alpha_1-2\alpha_2}/M_{-2\alpha_1-2\alpha_2} \simeq L_{-\alpha_1-2\alpha_2}$$

is irreducible.

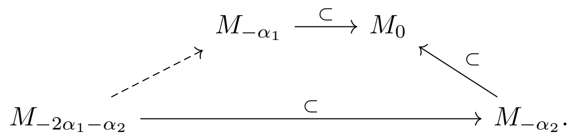
Combining this with part (2), we see $M_{-\alpha_1-2\alpha_2}$ contains an irreducible submodule whose quotient is also irreducible, and hence $\mathrm{length}(M_{-\alpha_1-2\alpha_2}) = 2$; here both $[M_{-\alpha_1-2\alpha_2} : L_{-\alpha_1-2\alpha_2}]$ and $[M_{-\alpha_1-2\alpha_2} : L_{-2\alpha_1-2\alpha_2}]$ are forced to be 1. □

[1]Hint: [Lecture 5, Lemma 1.3].

[2]Hint: Count the dimension of the $(-2\alpha_1 - 2\alpha_2)$-weight subspace of $M_{-\alpha_1-2\alpha_2}$.

Problem 3.3 (Lecture 6, Exercise 3.3)**.** We continue with the case $\mathfrak{g} = \mathfrak{sl}_3$.

(1) Prove: M_0 contains $M_{-\alpha_1}$ and $M_{-\alpha_2}$ as submodules and $[M_0 : L_{-\alpha_1}] = [M_0 : L_{-\alpha_2}] = 1$.
(2) Prove: $M_{-\alpha_2}$ contains $M_{-2\alpha_1-\alpha_2}$ as a submodule and $[M_{-\alpha_2} : L_{-2\alpha_1-\alpha_2}] = 1$.
(3) Prove: there exists a (unique) dotted arrow making the following diagram commutes[3]



Solution. (1) The argument is the same as that in Problem 3.2(3). For the first assertion, the result follows from $-\alpha_i = s_i \cdot 0$ and $\langle 0 + \rho, \check{\alpha}_i \rangle = 1 \in \mathbb{Z}^{\geqslant 0}$ for $i = 1, 2$. Further, since there is no more weights between 0 and $-\alpha_i$, the quotient $M_0/M_{-\alpha_i} \simeq L_0$ is irreducible. Thus, for the second assertion, as $L_{-\alpha_i}$ and L_0 are distinct, we get $[M_0 : L_{-\alpha_i}] = [M_{-\alpha_i} : L_{-\alpha_i}] = 1$ for $i = 1, 2$.

(2) As before, since $s_1 \cdot (-\alpha_2) = -2\alpha_1 - \alpha_2$ and $\langle -\alpha_2 + \rho, \check{\alpha}_1 \rangle = \langle \alpha_1, \alpha_1 \rangle = 2 \in \mathbb{Z}^{\geqslant 0}$, we know that $M_{-\alpha_2}$ contains $M_{-2\alpha_1-\alpha_2}$ as a submodule. Similar to Problem 3.2(3), we have that

$$\dim M_{-\alpha_1}^{\mathrm{wt}=-2\alpha_1-\alpha_2} = 1.$$

This proves $[M_{-\alpha_2} : L_{-2\alpha_1-\alpha_2}] = 1$.

(3) For each $\lambda \in W \cdot 0$, let v_λ be the highest weight vector of M_λ. Following the hint, it suffices to show $f_1^2 f_2 \cdot v_0 = u \cdot v_{-\alpha_1} = u f_1 \cdot v_0$ for some $u \in U(\mathfrak{n}^-)$. For this, notice that

$$f_1^2 f_2 = f_1([f_1, f_2] + f_2 f_1) = f_1[f_1, f_2] + f_1 f_2 f_1.$$

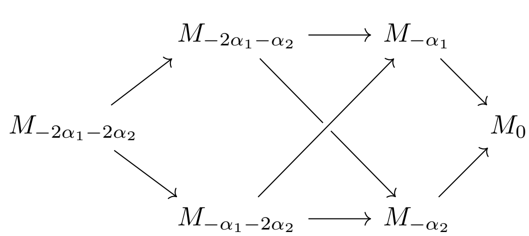
Also notice that both $f_1[f_1, f_2] \cdot v_0$ and $[f_1, f_2] f_1 \cdot v_0$ are highest weight vectors of $M_{-2\alpha_1-\alpha_2}$, because $[f_1, f_2]$ sends a vector of weight v_μ to that of weight $\mu - (\alpha_1 + \alpha_2)$. So there exists $\lambda \in k^\times$ such that $f_1[f_1, f_2] \cdot v_0 = \lambda [f_1, f_2] f_1 \cdot v_0$. It then leads to

$$f_1^2 f_2 \cdot v_0 = (\lambda [f_1, f_2] f_1 + f_1 f_2 f_1) \cdot v_0,$$

and taking $u = \lambda [f_1, f_2] + f_1 f_2$ completes the proof.[4] □

Problem 3.4 (Lecture 6, Exercise 3.4)**.** We continue with the case $\mathfrak{g} = \mathfrak{sl}_3$. Prove: for $\lambda, \mu \in W \cdot 0$, $[M_\lambda : L_\mu] \neq 0$ if and only if $\lambda \succeq \mu$.

Solution. It is clear that $[M_\lambda : L_\mu] \neq 0$ implies $\lambda \succeq \mu$. Conversely, we need to apply an enumeration and use the known results in Problems 3.2 and 3.3. We need to show that each map in the following diagram is an inclusion of modules, because these maps are exactly all maps of the form $M_\mu \to M_\lambda$ with $\mu \preceq \lambda$.



For this, it is known that

- $M_{-\alpha_i} \hookrightarrow M_0$ for $i = 1, 2$ by Problem 3.3(1);

[3] Hint: Show $f_1^2 f_2 \cdot v_0 = u \cdot v_{-\alpha_1}$ for some u. Here v_0 is the highest weight of M_0, $v_{-\alpha_1} = f_1 \cdot v_0$ is the highest weight of $M_{-\alpha_1}$, and $f_i \in \mathfrak{n}^-$ is the root vector corresponding to α_i.

[4] In fact, one can even deduce the formula of u in a more explicit sense. As in Problem 3.2(3), let f_1, f_2, f_3 be (standard) generators of PBW basis of $U(\mathfrak{n}^-)$ corresponding to simple roots $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$, respectively. Then there are relations turning out to be $[f_1, f_2] = f_3$ and $[f_1, f_3] = 0$. Using these together with some elementary computation, we get $u = f_1 f_2 + f_3$, and hence $\lambda = 1$ in the original proof.

- $M_{-2\alpha_1-\alpha_2} \hookrightarrow M_{-\alpha_2}$ by Problem 3.3(2);
- $M_{-2\alpha_1-\alpha_2} \hookrightarrow M_{-\alpha_1}$ by Problem 3.3(3);
- $M_{-2\alpha_1-2\alpha_2} \hookrightarrow M_{-\alpha_1-2\alpha_2}$ by Problem 3.2(3).

So it remains to show $M_{-\alpha_1-2\alpha_2} \hookrightarrow M_{-\alpha_1}$, $M_{-\alpha_1-2\alpha_2} \hookrightarrow M_{-\alpha_2}$, and $M_{-2\alpha_1-2\alpha_2} \hookrightarrow M_{-2\alpha_1-\alpha_2}$. But these follow from similar arguments as in 3.3(2), 3.3(3), and 3.2(3), respectively. □

Problem 3.5 (Lecture 7, Exercise 4.20)**.** For any weight $\lambda \in \mathfrak{t}^*$, let $\mathcal{O}_\lambda \subset \mathcal{O}$ be the full subcategory containing those objects M whose composition factors are of the form L_μ for $\mu \in W_{[\lambda]} \cdot \lambda$. Prove:

(1) Each $\mathcal{O}_\lambda$ is indecomposable.
(2) For $M \in \mathcal{O}$, suppose we have a decomposition $M \simeq M_1 \oplus M_2$ as $\mathfrak{t}$-modules such that the set $\mathrm{wt}(M_1) - \mathrm{wt}(M_2) := \{\lambda_1 - \lambda_2 \mid \lambda_i \in \mathrm{wt}(M_i)\}$ has empty intersection with Λ_r, then this is also a decomposition of $\mathfrak{g}$-modules.
(3) For any central character χ of $Z(\mathfrak{g})$, we have a direct sum decomposition
$$\mathcal{O}_\chi \simeq \bigoplus_{\substack{\lambda \in \varpi^{-1}(\chi) \\ \text{dot-antidominant}}} \mathcal{O}_\lambda.$$
(4) Conclude that
$$\mathcal{O} \simeq \bigoplus_{\lambda \text{ dot-antidominant}} \mathcal{O}_\lambda.$$

Solution. (1) The proof uses the same idea as [Lecture 7, Proposition 4.19]. We may assume λ is dot-antidominant and suppose $\mathcal{O}_\lambda \simeq \mathcal{O}_1 \oplus \mathcal{O}_2$. Recall any Verma module M_μ is indecomposable because it has a unique irreducible quotient L_μ. Moreover, if $\mu \in W_{[\lambda]} \cdot \lambda$, then $M_\lambda \subset M_\mu$ as a submodule by [Lecture 7, Corollary 4.15(2)]. Note that M_λ lies in either $\mathcal{O}_1$ or $\mathcal{O}_2$, so all M_μ are simultaneously contained in one of $\mathcal{O}_1$ and $\mathcal{O}_2$, and thus either $\mathcal{O}_1 \simeq 0$ or $\mathcal{O}_2 \simeq 0$ holds.

(2) It suffices to show that M_1 is a $\mathfrak{g}$-module, and the same argument applies to M_2. For this, taking $v \in M_1$, we need that $x \cdot v \in M_1$ for all root vectors $x \in \mathfrak{g}$. If this is not true then $x_0 \cdot v \in M_2$ for some root vector $x_0 \in \mathfrak{g}$ and then
$$(\text{weight of } v) - (\text{weight of } x_0 \cdot v) \in (\mathrm{wt}(M_1) - \mathrm{wt}(M_2)) \cap \Lambda_r,$$
which is impossible by assumption that $(\mathrm{wt}(M_1) - \mathrm{wt}(M_2)) \cap \Lambda_r = \emptyset$. Thus both M_1 and M_2 can be regarded as $\mathfrak{g}$-modules.

(3) Considering W-dotted action on a fixed element $\mu \in \varpi^{-1}(\chi) \subset \mathfrak{t}^*$, we have a decomposition
$$W \cdot \mu = \bigsqcup_{\lambda \text{ dot-antidominant}} W_{[\lambda]} \cdot \lambda.$$
Recall that $\mathcal{O}_\chi$ is the full subcategory of $\mathcal{O}$ supported on χ. Then for each object $M \in \mathcal{O}_\chi$, the decomposition of $W \cdot \mu$ above leads to a decomposition of $\mathfrak{t}$-modules
$$M = \bigoplus_{\substack{\lambda \in \varpi^{-1}(\chi) \\ \text{dot-antidominant}}} M_{[\lambda]}.$$
Notice that for two distinct dot-antidominant weights $\lambda, \lambda' \in \varpi^{-1}(\chi)$, we obtain $(\mathrm{wt}(M_{[\lambda]}) - \mathrm{wt}(M_{[\lambda']})) \cap \Lambda_r = \emptyset$ by decomposition of $W \cdot \mu$ together with the definition of $W_{[\lambda]}$; in particular, $\mathrm{wt}(M_{[\lambda]}) \cap \mathrm{wt}(M_{[\lambda']}) = \emptyset$ as $0 \in \Lambda_r$. Then by part (2) the decomposition of M above upgrades to a decomposition of $\mathfrak{g}$-modules, with the property that $\mathrm{Hom}_{\mathcal{O}_\chi}(M_{[\lambda]}, M_{[\lambda']}) = 0$. So the desired decomposition of categories follows.

(4) From the block decomposition we have that
$$\mathcal{O} \simeq \bigoplus_{\chi \in \mathrm{Spec}\, Z(\mathfrak{g})} \mathcal{O}_\chi.$$

Note that for each $\chi \in \operatorname{Spec} Z(\mathfrak{g})$ the set of dot-antidominant weights in $\varpi^{-1}(\chi)$ is nonempty, because $\varpi\colon \mathfrak{t}^* \to \operatorname{Spec} Z(\mathfrak{g}) \simeq \mathfrak{t}^* /\!\!/ W$ is surjective. So it makes sense to apply part (3) to deduce

$$\mathcal{O} \simeq \bigoplus_{\chi \in \operatorname{Spec} Z(\mathfrak{g})} \bigoplus_{\substack{\lambda \in \varpi^{-1}(\chi) \\ \text{dot-antidominant}}} \mathcal{O}_\lambda \simeq \bigoplus_{\lambda \text{ dot-antidominant}} \mathcal{O}_\lambda.$$

□