2024 Spring — Geometric Representation Theory (1)

HOMEWORK PROBLEMS

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Homework 4 (Due on April 29)

Problem 4.1 (Lecture 8, Exercise 1.6)**.** Prove the following.

(1) For $w, w' \in W$ such that $\ell(w) + \ell(w') = \ell(ww')$, we have

$$\delta_w \delta_{w'} = \delta_{ww'}.$$

(2) For $w \in W$ and $s \in S$, we have

$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } w < ws, \\ (v^{-1} - v)\delta_w + \delta_{ws} & \text{if } w > ws. \end{cases}$$

and

$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } w < sw, \\ (v^{-1} - v)\delta_w + \delta_{sw} & \text{if } w > ws. \end{cases}$$

Solution. (1) Recall that if $w = s_{i_1} \cdots s_{i_k}$ with $k = \ell(w)$ is a reduced decomposition of w, then $\delta_w = \delta_{s_{i_1}} \cdots \delta_{s_{i_k}}$. Provided that $\ell(w) + \ell(w') = \ell(ww')$, we see a reduced decomposition of ww' can be constructed by concatenating those of w and w'. Then

$$\delta_w \delta_{w'} = \delta_{ww'}.$$

(2) We first compute $\delta_w \delta_s$. Suppose $w < ws$ then $\ell(ws) = \ell(w) + 1 = \ell(w) + \ell(s)$. Then by part (1) the result $\delta_w \delta_s = \delta_{ws}$ follows. Whenever $w > ws$ we may write $w = w's$ for some $w' \in W$; note that if this is the case then $ws = w's^2 = w'$, and hence $\delta_w = \delta_{w'}\delta_s$ with $\delta_{w'} = \delta_{ws}$. Recall the inverse formula

$$\delta_s^{-1} = \delta_s + (v - v^{-1})$$

which is equivalent to $\delta_s^2 = (v^{-1} - v)\delta_s + 1$. Combining the ingredients above we obtain that

$$\delta_w \delta_s = \delta_{w'} \delta_s^2 = \delta_{w'}((v^{-1} - v)\delta_s + 1) = (v^{-1} - v)\delta_{w'}\delta_s + \delta_{w'} = (v^{-1} - v)\delta_w + \delta_{ws}.$$

This proves the formula of $\delta_w \delta_s$, and the case of $\delta_s \delta_w$ follows from a similar argument. □

Problem 4.2 (Lecture 8, Exercise 4.8)**.** For any $\lambda \in \mathfrak{t}^*$, prove

(1) The surjection $P_\lambda \twoheadrightarrow L_\lambda$ factors as $P_\lambda \to M_\lambda \to L_\lambda$.
(2) The obtained map $P_\lambda \to M_\lambda$ is surjective and exhibits P_λ as a projective cover of M_λ.

Solution. (1) By definition P_λ is a projective cover of L_λ; in particular, it is a projective module. On the other hand, $M_\lambda \twoheadrightarrow L_\lambda$ is the canonical projection in Verma module, so $P_\lambda \twoheadrightarrow L_\lambda$ factors through M_λ by the universal property of projective module P_λ.

(2) Recall that M_λ is generated by a highest weight vector v_λ; so to show that $P_\lambda \to M_\lambda$ is surjective, it boils down to figuring out the preimage of v_λ in P_λ. Since $P_\lambda \twoheadrightarrow L_\lambda$ is surjective, if we write $\overline{v}_\lambda$ for the image of v_λ along $M_\lambda \twoheadrightarrow L_\lambda$, then there exists some $w_\lambda \in P_\lambda$ mapping to $\overline{v}_\lambda$. Note that w_λ is nonzero of weight λ, and hence mapped to a highest weight vector $v'_\lambda \in M_\lambda$; on the other hand, the subspace in M_λ spanned by highest weight vector is 1-dimensional, so $v'_\lambda = c \cdot v_\lambda$ for some $c \in k^\times$, which deduces that v_λ is the image of $c^{-1}w_\lambda$. Therefore, $P_\lambda \twoheadrightarrow M_\lambda$ is surjective.

To exhibit P_λ as a projective cover of M_λ, notice that the surjectivity result above can be reinterpreted as follows: P_λ surjects to M_λ if and only if it surjects to L_λ. We need to show that any proper submodule of P_λ cannot be mapped to M_λ by any surjection. For this, if $N \subset P_\lambda$ is a proper submodule with $N \twoheadrightarrow M_\lambda$, then we also have $N \twoheadrightarrow L_\lambda$, which contradicts to the fact P_λ is a projective cover of L_λ. $\square$

Problem 4.3 (Lecture 8, Exercise 4.12)**.** For $\lambda \in \mathfrak{t}^*$, let $P \twoheadrightarrow L_\lambda$ be the surjection constructed in the proof of [Lecture 8, Theorem 4.3], i.e., P represents the functor

$$\mathcal{O}_\chi \longrightarrow \mathsf{Vect}, \quad M \longmapsto M^{\mathrm{wt}=\lambda}.$$

Prove:

1. This map factors as $P \to P_\lambda \twoheadrightarrow L_\lambda$. Moreover, $P \to P_\lambda$ is surjective.
2. For $\mathfrak{g} = \mathfrak{sl}_2$, the obtained map $P \to P_\lambda$ happens to be an isomorphism[1].
3. In general, $P \to P_\lambda$ is not an isomorphism[2].

Solution. (1) Recall from proof of [Lecture 8, Theorem 4.3] that P is projective, so the factorization follows from the universal property, provided that $P_\lambda \twoheadrightarrow L_\lambda$ is a projective cover by Problem 4.2. Notice that the surjection $P \twoheadrightarrow L_\lambda$ induces another surjection $\mathrm{im}(P \to P_\lambda) \twoheadrightarrow L_\lambda$, and then $P_\lambda = \mathrm{im}(P \to P_\lambda)$ because P_λ is essential as a projective cover. This shows $P = \mathrm{im}(P \to P_\lambda)$, namely $P \twoheadrightarrow P_\lambda$ is surjective.

(2) As P is projective, by part (1) we get $P \cong P_\lambda \oplus K$ with $K \coloneqq \ker(P \twoheadrightarrow P_\lambda)$. Note that the injection $P_\lambda \hookrightarrow P$ induces $\mathrm{Hom}_{\mathcal{O}_\chi}(P_\lambda, M) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_\chi}(P, M)$ for all $M \in \mathcal{O}_\chi$. It then suffices to show

$$(*) \qquad \dim \mathrm{Hom}_{\mathcal{O}_\chi}(P, M) = \dim \mathrm{Hom}_{\mathcal{O}_\chi}(P_\lambda, M),$$

because if this is true then the two spaces are the same and Yoneda lemma deduces $P = P_\lambda$. To prove $(*)$, since P represents the functor $M \mapsto M^{\mathrm{wt}=\lambda}$, we have

$$\mathrm{Hom}_{\mathcal{O}_\chi}(P, M) = M^{\mathrm{wt}=\lambda}.$$

On the other hand, by [Lecture 8, Corollary 4.9], we have

$$\dim \mathrm{Hom}_{\mathcal{O}_\chi}(P_\lambda, M) = [M : L_\lambda].$$

So it remains to prove $\dim M^{\mathrm{wt}=\lambda} = [M : L_\lambda]$ for any simple objects $M \in \mathcal{O}_\chi$. But in the case of $\mathfrak{g} = \mathfrak{sl}_2$, this follows from the classification in [Lecture 8, Example 3.17], together with the fact that when $l, l' \in \mathbb{Z}$, the l'-weight space of M_l is 1-dimensional (so that for distinct weights $\mu \neq \lambda$ there is no vector in L_μ of weight λ).

(3) We give a counter-example for $\mathfrak{g} = \mathfrak{sl}_3$ to show the equality $\dim M^{\mathrm{wt}=\lambda} = [M : L_\lambda]$ in the proof of part (2) fails. In the context of Problem 3.2(3), with $\chi = \varpi(0)$, we have for $\lambda = -2\alpha_1 - \alpha_2$ and $M = M_0$ that $[M_0 : L_{-2\alpha_1-\alpha_2}] = 1$. This comes from combining part (1) and (2) of Problem 3.3 (which dictates that $[M_0 : L_{-\alpha_2}] = [M_{-\alpha_2} : M_{-2\alpha_1-\alpha_2}] = 1$). However, since $2\alpha_1 + \alpha_2$ admits two formulations into sum of positive roots, which are $\alpha_1 + \alpha_1 + \alpha_2$ and $\alpha_1 + (\alpha_1 + \alpha_2)$ respectively, we have $\dim M_0^{\mathrm{wt}=-2\alpha_1-\alpha_2} \geqslant 2$. Thus, $[M : L_\lambda] \neq \dim M^{\mathrm{wt}=\lambda}$. $\square$

Problem 4.4 (Lecture 9, Exercise 1.11)**.** Let λ be a weight. Prove:

1. If $M \in \mathcal{O}$ such that $\mathrm{wt}(M) \cap \{\mu \mid \mu \succeq \lambda\} = \emptyset$, then $\mathrm{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geqslant 0$[3].
2. $\mathrm{Ext}^i_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) = 0$ for $i > 0$.
3. Combining (1) and (2), deduce $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, L_\mu) = 0$ and $\mathrm{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$ for $i > 0$ and $\mu \not\succ \lambda$.

[1]Hint: Use [Lecture 8, Corollary 4.9].

[2]Hint: What we have learned so far can (at least) prove this for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$.

[3]Hint: **Step I.** Reduce to the case $M = L_\mu$ with $\varpi(\mu) = \varpi(\lambda)$ and $\mu \not\succeq \lambda$.
Step II. Consider $0 \to K \to M_\mu \to L_\mu \to 0$ and note that $\mathrm{wt}(K) \prec \mu$.

Solution. (1) For any $M \in \mathcal{O}$ satisfying $\mathrm{wt}(M) \cap \{\mu \mid \mu \succeq \lambda\} = \emptyset$, if L_μ is an irreducible subquotient of M then $\mu \not\succeq \lambda$ holds. Thus it reduces to considering $M = L_\mu$ with $\mu \not\succeq \lambda$. If $\varpi(\mu) \neq \varpi(\lambda)$, i.e. M and $M_\lambda^\vee$ come from different blocks in $\mathcal{O}$, we have $\mathrm{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geqslant 0$ by [Lecture 9, Lemma 1.10]; so we may assume $\varpi(\mu) = \varpi(\lambda)$. We then consider the short exact sequence

$$0 \longrightarrow K \longrightarrow M_\mu \longrightarrow L_\mu \longrightarrow 0$$

induced by the canonical projection $M_\mu \twoheadrightarrow L_\mu$ with kernel K. Recall [Lecture 9, Corollary 1.3] that we have $\mathrm{Ext}^i_{\mathcal{O}}(M_\mu, M_\lambda^\vee) = 0$ for $i > 0$. Moreover, by [Lecture 8, Lemma 3.14] and the assumption $\mu \not\succeq \lambda$, we get $\mathrm{Hom}_{\mathcal{O}}(M_\mu, M_\lambda^\vee) = \mathrm{Ext}^0_{\mathcal{O}}(M_\mu, M_\lambda^\vee) = 0$, and consequently

$$\mathrm{Ext}^0_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0.$$

Now it remains to show $\mathrm{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$ for $i > 0$. For this, applying the contravariant functor $\mathrm{Hom}_{\mathcal{O}}(-, M_\lambda^\vee)$ to the short exact sequence above, we have a long exact sequence:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \| & & \\ \cdots \longrightarrow & \mathrm{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) & \longrightarrow & \mathrm{Ext}^i_{\mathcal{O}}(M_\mu, M_\lambda^\vee) & \longrightarrow & \boxed{\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee)} & \\ \longrightarrow & \boxed{\mathrm{Ext}^{i+1}_{\mathcal{O}}(L_\mu, M_\lambda^\vee)} & \longrightarrow & \mathrm{Ext}^{i+1}_{\mathcal{O}}(M_\mu, M_\lambda^\vee) & \longrightarrow & \mathrm{Ext}^{i+1}_{\mathcal{O}}(K, M_\lambda^\vee) & \longrightarrow \cdots \\ & & & \| & & & \\ & & & 0 & & & \end{array}$$

It then follows that for $i \geqslant 0$,

$$\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) \cong \mathrm{Ext}^{i+1}_{\mathcal{O}}(L_\mu, M_\lambda^\vee).$$

So we only need to prove $\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) = 0$ for $i \geqslant 0$. Since there are only finitely many irreducible objects in $\mathcal{O}_{\varpi(\mu)}$, this can be done by induction on μ with respect to $\prec$:

- ⋄ If μ is dot-antidominant, then M_μ is irreducible and $K = 0$. In this case, the vanishing result obviously follows.
- ⋄ Suppose $\mathrm{Ext}^i_{\mathcal{O}}(L_{\mu'}, M_\lambda^\vee) = 0$ holds for all $i \geqslant 0$ and $\mu' \prec \mu$. Notice that $K \in \mathcal{O}$ is always an extension of $L_{\mu'}$'s with $\mu' \prec \mu$. So we get $\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) = 0$ for $i \geqslant 0$ as desired.

This completes the induction and we conclude that $\mathrm{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geqslant 0$.

(2) Suppose As in part (1), consider the short exact sequence

$$0 \longrightarrow K \longrightarrow M_\lambda \longrightarrow L_\lambda \longrightarrow 0.$$

Notice again that for each $\mu \in \mathrm{wt}(K)$ we have $\mu \prec \lambda$, so $\mathrm{wt}(K) \cap \{\mu \mid \mu \succeq \lambda\} = \emptyset$, and then by part (1) we get $\mathrm{Ext}^i_{\mathcal{O}}(K, M_\lambda^\vee) = 0$ for $i \geqslant 0$. On the other hand, as in part (1), we know $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) = 0$ for $i > 0$. Therefore, applying the contravariant functor $\mathrm{Hom}_{\mathcal{O}}(-, M_\lambda^\vee)$ to the short exact sequence above, we have a long exact sequence as follows:

$$\begin{array}{ccccccc} & & & & & 0 & \\ & & & & & \| & \\ 0 \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}(K, M_\lambda^\vee) & \\ \longrightarrow & \boxed{\mathrm{Ext}^1_{\mathcal{O}}(L_\lambda, M_\lambda^\vee)} & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}}(K, M_\lambda^\vee) & \longrightarrow \cdots \\ & & & \| & & \| & \\ & & & 0 & & 0 & \end{array}$$

Thus, $\mathrm{Ext}^i_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) = 0$ for $i > 0$ is implied by $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, M_\lambda^\vee) = \mathrm{Ext}^{i-1}_{\mathcal{O}}(K, M_\lambda^\vee) = 0$; but note that this may not vanish at $i = 0$.

(3) Using the isomorphism $L_\mu \cong L_\mu^\vee$, the vanishing result of $\mathrm{Ext}^i_{\mathcal{O}}(M_\lambda, L_\mu)$ follows from that of $\mathrm{Ext}^i_{\mathcal{O}}$ by duality. Thus it suffices to check $\mathrm{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$. If $\mu \prec \lambda$ then $M = L_\lambda$ satisfies

the condition of (1), otherwise $\mu = \lambda$ and we are in the case of (2), and in both cases we have the desired vanishing result. $\square$

Problem 4.5 (Lecture 9, Exercise 3.18)**.** Let μ be any dot-antidominant integral weight. Prove[4]:

(1) For any $w \in W$, $(\Xi_\mu : M_{w\cdot\mu}) = 1$ and there is a surjection $\Xi_\mu \twoheadrightarrow M_\mu$.
(2) For any $w \in W$, $(P_\mu : M_{w\cdot\mu}) \geqslant 1$ and there is a surjection $P_\mu \twoheadrightarrow M_\mu$.
(3) There exists an isomorphism $\Xi_\mu \simeq P_\mu$ compatible with the surjections to M_μ.
(4) For any $w \in W$, $[M_{w\cdot\mu} : L_\mu] = 1$[5].

Solution. (1) By [Lecture 9, Lemma 3.5], if ν is the unique dominant integral weight in $W(\mu - (-\rho)) = W(\mu+\rho)$, then $\Xi_\mu \coloneqq T^\mu_{-\rho}(L_{-\rho}) \cong T^\mu_{-\rho}(M_{-\rho}) \in \mathcal{O}_{\varpi(\mu)}$ admits a standard filtration and we have the multiplicity

$$(\Xi_\mu : M_{w\cdot\mu}) = (T^\mu_{-\rho}(M_{-\rho}) : M_{w\cdot\mu}) = \dim L_\nu^{\mathrm{wt}=w\cdot\mu-(-\rho)} = 1.$$

Here the last equality follows from $w \cdot \mu + \rho \in W(\nu)$ by our construction. In particular, taking $w = 1$, we deduce

$$\dim \mathrm{Hom}_{\mathcal{O}}(\Xi_\mu, M_\mu^\vee) = (\Xi_\mu : M_\mu) = 1.$$

Consequently, there exists a non-zero map $\phi\colon \Xi_\mu \to M_\mu^\vee$ in $\mathcal{O}$. As μ is dot-antidominant, we see $M_\mu \cong M_\mu^\vee$ is irreducible, and hence Ξ_μ must surjects onto M_μ through $\Xi_\mu \to M_\mu^\vee \simeq M_\mu$, which is the desired surjection.

(2) Using BGG reciprocity [Lecture 9, Theorem 2.2], we have

$$(P_\mu : M_{w\cdot\mu}) = [M_{w\cdot\mu} : L_\mu].$$

To compute the right hand side, by [Lecture 7, Corollary 4.15] and that μ is dot-antidominant integral, $L_\mu \cong M_\mu \subset M_{w\cdot\mu}$ as a submodule. So we get $[M_{w\cdot\mu} : L_\mu] \geqslant 1$ and hence $(P_\mu : M_{w\cdot\mu}) \geqslant 1$. Also, the surjection $P_\mu \twoheadrightarrow M_\mu$ is constructed as in Problem 4.2(2).

(3) Since we have the surjection $\Xi_\mu \twoheadrightarrow M_\mu$ from part (1) and Ξ_μ is by definition a projective object, the surjection $P_\mu \twoheadrightarrow M_\mu$ induces a factorization diagram

$$\begin{array}{ccc} \Xi_\mu & \overset{\eta}{\dashrightarrow} & P_\mu \\ & \searrow\!\!\!\!\searrow & \downarrow\!\!\downarrow \\ & & M_\mu \end{array}$$

and then we get a map $\eta\colon \Xi_\mu \to P_\mu$ compatible with the surjections to M_μ.

Now it remains to show that η is an isomorphism. Recall from Problem 4.2 that P_μ is a projective cover of M_μ, so η must be surjective by the same argument thereof. To show the injectivity, define the weight submodule $K \coloneqq \ker(\Xi_\mu \twoheadrightarrow P_\mu) \subset \Xi_\mu$. Then by (1) and (2), for any $w \in W$,

$$(\Xi_\mu : M_{w\cdot\mu}) = 1 \leqslant (P_\mu : M_{w\cdot\mu}) \implies (K : M_{w\cdot\mu}) = 0.$$

This makes sense because K admits a standard filtration by [Lecture 9, Lemma 1.8]. Therefore, we get $K = 0$, and $\eta\colon \Xi_\mu \xrightarrow{\sim} P_\mu$ is an isomorphism.

(4) It follows from part (3) directly that

$$[M_{w\cdot\mu} : L_\mu] = (P_\mu : M_{w\cdot\mu}) = (\Xi_\mu : M_{w\cdot\mu}) = 1.$$

$\square$

[4]Hint: [Lecture 9, Lemma 3.5] for (1); [Lecture 9, Theorem 2.2] and [Corollary 4.15, Lecture 7] for (2). For (3), first find a surjection $\Xi_\mu \twoheadrightarrow P_\mu$ and then use (1) and (2).

[5]See [Gai05, Proposition 4.20] for a different proof of this fact.

References

[Gai05] Dennis Gaitsgory. *Course Notes for Geometric Representation Theory*. 2005. available at https://people.mpim-bonn.mpg.de/gaitsgde/267y/cat0.pdf.