2024 Spring — Geometric Representation Theory (1)

HOMEWORK PROBLEMS

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Homework 5 (Due on May 13)

Problem 5.1 (Lecture 10, Exercise 3.3)**.** Let X be an affine smooth k-scheme. Prove: Any k-derivation $\mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order 1, and the obtained map

$$\mathcal{O}(X) \oplus \mathcal{T}(X) \longrightarrow \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$$

is an isomorphism.

Solution. Let $\partial\colon \mathcal{O}(X) \to \mathcal{O}(X)$ be any k-derivation, regarded as an element of $\mathcal{T}(X)$. Then for any $f, g \in \mathcal{O}(X)$, we have

$$[\partial, f](g) = \partial(fg) - f \cdot \partial(g) = \partial(f) \cdot g.$$

Here the first equality above is by definition of Lie bracket, and the second is because ∂ satisfies Leibniz rule. Since multiplying by $\partial(f) \in \mathcal{O}(X)$ is known to be a differential operator of order 0, we see ∂ is a differential operator of order 1.

To prove the provided map is an isomorphism, we need to show that each element of $\mathsf{F}^{\leqslant 1}\mathcal{D}(X)$ can be uniquely written as a sum $f + \partial$ with $f \in \mathcal{O}(X)$ and $\partial \in \mathcal{T}(X)$. From the argument above we know $f \in \mathsf{F}^{\leqslant 0}\mathcal{D}(X)$ and $\partial \in \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$, so that $f + \partial \in \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$. Thus the following k-linear map makes sense:

$$\Phi\colon \mathcal{O}(X) \oplus \mathcal{T}(X) \longrightarrow \mathsf{F}^{\leqslant 1}\mathcal{D}(X), \quad (f, \partial) \longmapsto f + \partial.$$

Now it suffices to check the bijectivity of Φ.

For the injectivity, let $f + \partial = 0$ in $\operatorname{im}\Phi$, then $f \cdot g + \partial(g) = 0$ for all $g \in \mathcal{O}(X)$. In particular, when $g = \mathbb{1}$ we obtain

$$f + \partial(\mathbb{1}) = 0.$$

Here $\partial \in \mathcal{T}(X)$ satisfies Leibniz rule, so $\partial(\mathbb{1}) = \partial(\mathbb{1} \cdot \mathbb{1}) = \mathbb{1} \cdot \partial(\mathbb{1}) + \mathbb{1} \cdot \partial(\mathbb{1}) = 2\partial(\mathbb{1})$, which implies $\partial(\mathbb{1}) = 0$. It follows that $f = 0$, and thus $\partial = 0$ as well, which proves the injectivity.

For the surjectivity, let $D \in \mathsf{F}^{\leqslant 1}\mathcal{D}(X)$. Caution that D does not need to satisfy Leibniz rule. For any $f \in \mathcal{O}(X)$, the operator

$$[D, f]\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(X), \quad s \longmapsto D(fs) - f \cdot D(s)$$

is of order 0, so we get $[D, f](\mathbb{1}) \in \mathcal{O}(X)$, and $[D, f](s) = [D, f](\mathbb{1}) \cdot s$ holds for all $s \in \mathcal{O}(X)$. In particular, we take $s = \mathbb{1} \in \mathcal{O}(X)$ to deduce

$$[D, f](\mathbb{1}) = D(f) - f \cdot D(\mathbb{1}) \in \mathcal{O}(X).$$

Using this formula of $[D, f]$, for any $g \in \mathcal{O}(X)$, we deduce

$$D(fg) - f \cdot D(g) = [D, f](g) = [D, f](\mathbb{1}) \cdot g = D(f) \cdot g - D(\mathbb{1}) \cdot fg.$$

This then gives us a formula of $D(fg)$, written as

$$D(fg) = D(f) \cdot g + f \cdot D(g) - D(\mathbb{1}) \cdot fg.$$

On the other hand, the formula of $[D, f]$ also suggests to consider the map

$$D - D(\mathbb{1}) = [D, \text{–}](\mathbb{1})\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(X), \quad f \longmapsto [D, f].$$

We claim that it is an element of $\mathcal{T}(X) = \mathrm{Der}(\mathcal{O}(X), \mathcal{O}(X))$. Indeed, since the k-linearity of $[D, -](\mathbb{1})$ in f is clear, it is clearly an endomorphism on $\mathcal{O}(X)$, and we only need to verify Leibniz rule. For this, we compute for any $f, g \in \mathcal{O}(X)$ that

$$\begin{aligned} [D, fg](\mathbb{1}) &= D(fg) - D(\mathbb{1}) \cdot fg \\ &= D(f) \cdot g + f \cdot D(g) - 2D(\mathbb{1}) \cdot fg \\ &= (D(f) - D(\mathbb{1}) \cdot f) \cdot g + f \cdot (D(g) - D(\mathbb{1}) \cdot g) \\ &= [D, f](\mathbb{1}) \cdot g + f \cdot [D, g](\mathbb{1}). \end{aligned}$$

Here the first and last equalities above are due to the identity $D - D(\mathbb{1}) = [D, -](\mathbb{1})$, and the second is by the prescribed formula of $D(fg)$. This finishes the proof of surjectivity because

$$D = D(\mathbb{1}) + [D, -],$$

with $D(\mathbb{1}) \in \mathcal{O}(X)$ and $[D, -] \in \mathcal{T}(X)$. □

Problem 5.2 (Lecture 10, Exercise 7.3)**.** Let X be a smooth k-scheme dimension n. Prove: There is a unique right $\mathcal{D}_X$-module structure on Ω_X^n such that for local sections $f \in \mathcal{O}(U)$, $\partial \in \mathcal{T}(U)$, and $\omega \in \Omega_X^n(U)$, the right action is given by

$$\omega \cdot f = f\omega, \quad \omega \cdot \partial = -\mathcal{L}_\partial(\omega).$$

Solution. Since X is a smooth k-scheme, it suffices to check the unique $\mathcal{D}_X(U)$-module structure on $\Omega_X^n(U)$ for each open affine $U \subset X$. For this, we only need to concern about the action of $\mathsf{F}^{\leqslant 1}\mathcal{D}_X(U)$. By Problem 5.1, it suffices to consider the actions of $\mathcal{O}(U)$ and $\mathcal{T}(U)$.

Let $\{x_i\}_{i=1}^n$ be an étale coordinate system locally on U. Then, according to [Lecture 10, Proposition–Definition 1.21], $\{\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_m} : 1 \leqslant i_1 < \cdots < i_m \leqslant n\}$ form a free basis of $\Omega_X^n(U)$ as an $\mathcal{O}(U)$-module, and $\{\partial_{i_1} \cdots \partial_{i_m} : 1 \leqslant i_1 < \cdots < i_m \leqslant n\}$ is the dual basis of $\mathcal{T}(U)$. Note that for $l \neq k$ we have $[\partial_l, \partial_k] = 0$. To recognize the action of $\mathcal{T}(U)$, we compute for $\omega \in \mathcal{O}_X^n(U)$ that

$$\omega \cdot (\partial_{i_1} \cdots \partial_{i_m}) = (-1)^m (\mathcal{L}_{\partial_{i_1}} \circ \cdots \circ \mathcal{L}_{\partial_{i_m}})(\omega).$$

Also, to check the compatibility of actions by $\mathcal{O}(U)$ and $\mathcal{T}(U)$, we need to check:

- The definition of Lie bracket $[\partial, f]$, i.e. $(\omega \cdot \partial) \cdot f - (f\omega) \cdot \partial = \omega \cdot \partial(f)$, and
- The composition equality $\omega \cdot (f\partial) = (\omega \cdot f) \cdot \partial$.

For the former, the given action means it is equivalent to $-f\mathcal{L}_\partial(\omega) + \mathcal{L}_\partial(f\omega) = \partial(f)\omega$. As for the latter, it suffices to check $-\mathcal{L}_{f\partial}(\omega) = -\mathcal{L}_\partial(f\omega)$. We see both equalities above are clear by definition of $\mathcal{L}_\partial$[1], so we get the right $\mathcal{D}_X$-module structure. The uniqueness also follows from the argument. □

Problem 5.3 (Lecture 10, Exercise 7.6)**.** In [Lecture 10, Example 7.5], prove $\mathcal{M}_{x^\lambda}$ is isomorphic to $\mathcal{O}_X$ as left $\mathcal{D}_X$-modules if and only if $\lambda \in \mathbb{Z}$.

Solution. Recall the construction of $\mathcal{M}_{x^\lambda}$ as follows. On $X = \mathbb{A}^1 - 0 = \operatorname{Spec} k[x^{\pm 1}]$, with $\lambda \in k$, the $\mathcal{T}_X$-action on the left $\mathcal{D}_X$-module $\mathcal{M}_{x^\lambda}$ is given by $\partial_x \cdot x^\lambda - \lambda x^{-1} \cdot x^\lambda = 0$, and then we have an $\mathcal{D}_X$-module isomorphism

$$\mathcal{M}_{x^\lambda} \simeq \mathcal{D}_X / \mathcal{D}_X \cdot (\partial_x - \lambda x^{-1}).$$

Note that we have an isomorphism between underlying $\mathcal{O}_X$-module structures; note that any isomorphism $\mathcal{M}_{x^\lambda} \simeq \mathcal{O}_X$ of $\mathcal{O}_X$-modules must be of form

$$\iota \colon \mathcal{O}_X \longrightarrow \mathcal{M}_{x^\lambda}, \quad f \longmapsto uf$$

[1]By definition, when U is affine with étale coordinate system $\{x_i\}_{i=1}^n$, if we write $\omega = \omega(x_1, \ldots, x_n)$, then

$$\mathcal{L}_\partial(\omega) \coloneqq \partial(\omega) - \sum_{i=1}^n \omega^{(i)},$$

where $\omega^{(i)} = \omega(x_1, \ldots, [\partial, x_i], \ldots, x_n)$.

for some $u \in \mathcal{O}_X^\times$. Fix such an ι in the following, then by PBW theorem for $\mathcal{D}_X$, this ι upgrades to an isomorphism of $\mathcal{D}_X$-modules if and only if u is such that

$$\partial_x(f) = \partial_x \cdot (uf),$$

where the right-hand side means the left action image of $\partial_x \in \mathcal{D}_X$. Since $\partial_x \in \mathcal{T}_X$, it satisfies Leibniz rule and we get

$$\partial_x \cdot (uf) = \partial_x(uf) - u \cdot \partial_x(f) = \partial_x(u) \cdot f = \lambda x^{-1} \cdot (uf).$$

As this holds for all $f \in \mathcal{O}_X$, we see $u = x^\lambda \in \mathcal{O}_X^\times$. Also, $x^\lambda \in \mathcal{O}_X^\times = k[x^{\pm 1}]^\times$ if and only if $\lambda \in \mathbb{Z}$. So we see ι is an isomorphism of $\mathcal{D}_X$-modules if and only if $\lambda \in \mathbb{Z}$. □

Problem 5.4 (Lecture 11, Exercise 6.7)**.** Let $x\colon \mathrm{pt} \to X$ be a closed point of X. We write $\delta_x := x_{*,\mathrm{dR}}(k)$. Prove:

(1) $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ as a right $\mathcal{D}_X$-module.
(2) δ_x is set-theoretically supported at x, i.e., for the complement open $U := X - x$, we have $\delta_x|_U = 0$.
(3) There exists a unique section Dirac_x of δ_x such that $\mathsf{Dirac}_x \cdot f = f(x)\mathsf{Dirac}_x$ for any local section f of $\mathcal{O}_X$ defined near x.
(4) δ_x is generated by Dirac_x as a right $\mathcal{D}_X$-module.

Solution. (1) Note that x is an affine morphism (implying that $x_{*,\mathrm{dR}}$ is t-exact) and $\mathcal{D}_{\mathrm{pt}\to X}$ is locally free as an $\mathcal{O}_X$-module. Then by definition we have that

$$\delta_x = x_{*,\mathrm{dR}}(k) = x_*(k \otimes_{\mathcal{D}_{\mathrm{pt}}} \mathcal{D}_{\mathrm{pt}\to X}),$$

where $\mathcal{D}_{\mathrm{pt}} = \mathcal{O}_{\mathrm{pt}} = k$ and $\mathcal{D}_{\mathrm{pt}\to X} = x^*\mathcal{D}_X$. Combining this with derived projection formula, we get an isomorphism of $\mathcal{D}_X$-modules

$$\delta_x = x_*(k \otimes_{\mathcal{O}_{\mathrm{pt}}} x^*\mathcal{D}_X) \simeq x_*k \otimes_{\mathcal{O}_X} \mathcal{D}_X = k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

(2) Consider the open embedding $j\colon U \hookrightarrow X$. Using the isomorphism of part (1), we have

$$j^*\delta_x \simeq j^*(k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq j^*k_x \otimes_{\mathcal{O}_X} j^*\mathcal{D}_X.$$

But as we know $k_x = x_*k$, it is clear that $j^*k_x = 0$. Here we regard k_x as the skyscraper sheaf at x and k the constant sheaf on X. So we have $\delta_x|_U = j^*\delta_x = 0$ as desired.

(3) We see from part (1) that $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$. We claim that $\mathsf{Dirac}_x \in \delta_x$ is the isomorphic image of the canonical element $1 \otimes \mathbb{1} \in k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$, and hence the existence and uniqueness follows. To check $1 \otimes \mathbb{1}$ satisfies the desired property, we let U_x be any open neighborhood of x in X and compute for any local section $f \in \mathcal{O}_X(U_x)$ that

$$(1 \otimes \mathbb{1}) \cdot f = 1 \otimes (\mathbb{1} \cdot f) = 1 \otimes f|_{\{x\}} = f(x)(1 \otimes \mathbb{1}).$$

This completes the proof of $\mathsf{Dirac}_x \cdot f = f(x)\mathsf{Dirac}_x$.

(4) By argument in part (3), we can identify Dirac_x with $1 \otimes \mathbb{1} \in k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Since $1 \otimes \mathbb{1}$ clearly generates $k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ under the right $\mathcal{D}_X$-action, the conclusion follows. □

Problem 5.5 (Lecture 11, Exercise 8.3)**.** Let $x\colon \mathrm{pt} \to X$ be a closed point of X. Prove

$$\delta_x \otimes^! \delta_x \simeq \delta_x.$$

(A formal proof exists, but you are encouraged to do some direct calculations to see $\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = 0$ unless $i = 0$, and $\mathcal{H}^0(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = \delta_x$.)

Solution. We present a formal proof. Apply base change theorem to the following Cartesian diagram

$$\begin{array}{ccc} \mathrm{pt} & \xrightarrow{\mathrm{id}} & \mathrm{pt} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle x} \\ \mathrm{pt} & \xrightarrow{x} & X \end{array}$$

we see that $x^! \circ x_{*,\mathrm{dR}} \cong \mathrm{id}_{*,\mathrm{dR}} \circ \mathrm{id}^!$. Thus we are able to compute

$$\begin{aligned} \delta_x \otimes^! \delta_x &= x_{*,\mathrm{dR}} k \otimes^! \delta_x && \text{(by definition)} \\ &\simeq x_{*,\mathrm{dR}}(k \otimes^! x^! \delta_x) && \text{(by projection formula)} \\ &= x_{*,\mathrm{dR}}(k \otimes^! x^! x_{*,\mathrm{dR}} k) && \text{(by definition)} \\ &\simeq x_{*,\mathrm{dR}}(k \otimes^! \mathrm{id}_{*,\mathrm{dR}} \mathrm{id}^! k) && \text{(by base change)} \\ &\simeq x_{*,\mathrm{dR}}(k \otimes^! k). \end{aligned}$$

But it is clear that $k \otimes^! k \simeq k$, and hence we get $\delta_x \otimes^! \delta_x \simeq x_{*,\mathrm{dR}} k = \delta_x$. $\square$