2024 Spring — Geometric Representation Theory (1)

HOMEWORK PROBLEMS

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Homework 6 (Due on May 27)

Problem 6.1 (Lecture 12, Exercise 3.9)**.** Show that

$$\dim \mathscr{F}\ell_G^{=w} = \ell(w).$$

If you do not know the basics about reductive groups, prove this for semisimple G.

Solution. By [Lecture 12, Lemma 3.7] we have that

$$\mathscr{F}\ell_G^{=w} \simeq N/(\mathrm{Ad}_w(N) \cap N) \simeq \mathrm{Ad}_w(N) \cap N^-.$$

Recall that $\dim N$ is the number of positive roots, and the goal now is to also reformulate $\mathrm{Ad}_w(N) \cap N^-$ in terms of positive roots in Φ^+. For this, provided the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, let $X_\alpha \in \mathfrak{g}_\alpha$ be the root vector corresponding to α; this defines the root subgroup

$$U_\alpha \coloneqq \{\exp(sX_\alpha) \colon s \in k\}.$$

Observe that each U_α is a 1-dimensional algebraic group over k, and we can view the generator of U_α as the map $u_\alpha \colon \mathbb{G}_\mathrm{a} \to G,\ s \mapsto \exp(sX_\alpha)$. Also, to consider the W-adjoint action on U_α, we compute $u_{w(\alpha)}(s) = \exp(sX_{w(\alpha)}) = \exp(s\,\mathrm{Ad}_w(X_\alpha)) = (\mathrm{Ad}_w(u_\alpha))(s)$, and it follows that

$$\mathrm{Ad}_w(U_\alpha) = U_{w(\alpha)}.$$

On the other hand, there are isomorphisms of k-schemes $N \simeq \prod_{\alpha\in\Phi^+} U_\alpha$ and $N^- \simeq \prod_{\alpha\in\Phi^-} U_\alpha$, so in particular $\mathrm{Ad}_w(N) \simeq \prod_{\alpha\in\Phi^+} U_{w(\alpha)}$. To conclude, we obtain

$$\mathrm{Ad}_w(N) \cap N^- \simeq \prod_{\alpha\in\Phi^+} U_{w(\alpha)} \cap \prod_{\beta\in\Phi^+} U_{-\beta} \simeq \prod_{\substack{\alpha\in\Phi^+\\ w(\alpha)\in\Phi^-}} U_\alpha,$$

because $U_{w(\alpha)} \cap U_{-\beta} = \emptyset$ unless $w(\alpha) = -\beta \in \Phi^-$. Again, by $\dim U_\alpha = 1$ we see

$$\dim \mathscr{F}\ell_G^{=w} = \#\{\alpha \in \Phi^+ \colon w(\alpha) \in \Phi^-\} = \ell(w),$$

where the last equality is by definition of $\ell(w)$. □

Problem 6.2 (Lecture 12, Exercise 4.9)**.** Deduce the BGG theorem from the localization theorem (see [Lecture 12, Theorem 4.7])[1].

Solution. Given $\lambda, \mu \in \mathfrak{t}^*$, there are w and w' such that $\lambda = w \cdot (-2\rho)$ and $\mu = w' \cdot (-2\rho)$. To deduce the BGG theorem, assume $[M_\lambda : L_\mu] \neq 0$ and it suffices to prove $w' \leq w$.

By localization theorem, we have

$$M_\lambda \longleftrightarrow \Delta_w, \quad L_\mu \longleftrightarrow \mathrm{IC}_{w'}.$$

Here $\Delta_w \coloneqq i_{w,!}\mathcal{O}_{\mathscr{F}\ell_G^{=w}}$ with $i_w \colon \mathscr{F}\ell_G^{=w} \hookrightarrow \mathscr{F}\ell_G$. Since the Schubert cell $\mathscr{F}\ell_G^{=w}$ is a locally closed subset of $\mathscr{F}\ell_G$ and $\mathrm{supp}(\Delta_w) \subset \mathrm{supp}(\mathcal{O}_{\mathscr{F}\ell_G^{=w}})$ by property of !-pushforward, we see Δ_w is set-theoretically supported on the closure of $\mathscr{F}\ell_G^{=w}$, which is $\mathscr{F}\ell_G^{\leq w}$. In particular, any subquotient of Δ_w is set-theoretically supported on $\mathscr{F}\ell_G^{\leq w}$. From the assumption $[M_\lambda : L_\mu] \neq 0$, we know $\mathrm{IC}_{w'}$ is isomorphic to a subquotient of Δ_w, and hence supported on $\mathscr{F}\ell_G^{\leq w}$.

[1] Hint: Prove any subquotient of Δ_w is set-theoretically support on the Schubert variety $\mathscr{F}\ell_G^{\leq w}$.

Provided the above, we get $\mathrm{supp}(\Delta_{w'}) = \mathscr{F}\ell_G^{\leq w'}$, and it suffices to show the Bruhat cell $C_{w'} := \mathscr{F}\ell_G^{=w'}$ is contained in $\mathrm{supp}(\mathrm{IC}_{w'})$. Indeed, by definition we have $\mathrm{IC}_{w'} = \mathrm{im}(\Delta_{w'} \to \nabla_{w'})$; on the other hand $\Delta_{w'}|_{C_{w'}} = \nabla_{w'}|_{C_{w'}} = \mathcal{O}_{C_{w'}}$, implying that $\mathrm{IC}_{w'}|_{C_{w'}} = \mathcal{O}_{C_{w'}}$. This proves that $\mathrm{IC}_{w'}$ is non-zero when restricted to $\mathscr{F}\ell_G^{=w'}$, and hence $w' \leq w$. It follows that $\mu \preceq_{\subset} \lambda$ as desired. ☐

Problem 6.3 (Lecture 12, Exercise 4.10)**.** Prove that when $G = \mathrm{SL}_2$, the homomorphism $\alpha\colon U(\mathfrak{g}) \to \mathcal{D}(\mathscr{F}\ell_G)$ induces an isomorphism

$$U(\mathfrak{g})_{\chi_0} \simeq \mathcal{D}(\mathscr{F}\ell_G).$$

This is a special case of part (1) of localization theorem (see [Lecture 12, Theorem 4.7]).

Solution. When $G = \mathrm{SL}_2$, we have $\mathscr{F}\ell_G = \mathbb{P}^1$ and need to consider $\alpha\colon U(\mathfrak{g}) = U(\mathfrak{sl}_2) \longrightarrow \mathcal{D}(\mathbb{P}^1)$. Let e, f, h be a standard basis of $\mathfrak{g} = \mathfrak{sl}_2$. Recall that $Z(U(\mathfrak{g}))$ is generated by the Casimir element $\Omega = ef + fe + h^2/2 \in U(\mathfrak{g})$. Let $U = \{(x:1)\colon x \in k\}$ be a standard affine open in $\mathbb{P}^1$. For each coordinate parameter u of U, by computing

$$\alpha(\mathbf{g}) \cdot u = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\mathbf{g})(\exp(-t\mathbf{g})u), \quad \mathbf{g} \in \{e, f, h\},$$

we see, depending on the choice of u,

$$\alpha(e) = -\partial_u, \quad \alpha(f) = u^2\partial_u, \quad \alpha(h) = -2u\partial_u.$$

It follows that

$$\begin{aligned}\alpha(\Omega) &= \alpha(ef + fe + h^2/2) \\ &= (-\partial_u)u^2\partial_u + u^2\partial_u(-\partial_u) + (2u\partial_u)^2/2 \\ &= -2u\partial_u - u^2\partial_u^2 - u^2\partial_u^2 + 2u\partial_u + 2u^2\partial_u^2 \\ &= 0.\end{aligned}$$

Since the central character χ_0 corresponds to $Z(U(\mathfrak{g})) = k[\Omega]$ and $\alpha(\Omega) = 0$, the map α factors through $U(\mathfrak{g})_{\chi_0}$. So we get

$$\alpha_{\chi_0}\colon U(\mathfrak{g})_{\chi_0} \longrightarrow \mathcal{D}(\mathscr{F}\ell_G).$$

Note that both sides of α_{χ_0} admits graded structure, and we claim that

$$\mathrm{gr}^1(U(\mathfrak{g})_{\chi_0}) = \mathfrak{g} \twoheadrightarrow \mathcal{T}(\mathscr{F}\ell_G) = \mathrm{gr}^1\mathcal{D}(\mathscr{F}\ell_G)$$

is surjective. If this is true, then α_{χ_0} is surjective as well. Indeed, the claim follows from that $\mathcal{T}(\mathscr{F}\ell_G) = \mathcal{O}_{\mathscr{F}\ell_G}(2)$, whose global section is generated by $\alpha(e), \alpha(f), \alpha(h)$, meaning that α is surjective when restricted to $\mathfrak{g}$. Thus we have proved the claim as well as the surjectivity of α_{χ_0}. Now it remains to check the injectivity. For this, we have the embedding $\mathrm{gr}^\bullet\mathcal{D}(\mathscr{F}\ell_G) \hookrightarrow (\mathrm{Sym}^\bullet\,\mathcal{T})(\mathscr{F}\ell_G)$ that is an identity on degree 1 part. Also, the diagram below commutes:

$$\begin{array}{ccc} \mathrm{gr}^\bullet(U(\mathfrak{g})_{\chi_0}) & \xrightarrow{\alpha_{\chi_0}} & \mathrm{gr}^\bullet\mathcal{D}(\mathscr{F}\ell_G) \\ & \searrow\!\!\!\!\searrow & \downarrow \\ & & (\mathrm{Sym}^\bullet\,\mathcal{T})(\mathscr{F}\ell_G). \end{array}$$

Here the map $\mathrm{gr}^\bullet(U(\mathfrak{g})_{\chi_0}) \twoheadrightarrow (\mathrm{Sym}^\bullet\,\mathcal{T})(\mathscr{F}\ell_G)$ is surjective because we have shown α_{χ_0} is surjective on degree 1 part. To show this map is injective, we only need for all $n \geqslant 1$ that

$$\dim \mathrm{gr}^n(U(\mathfrak{g})_{\chi_0}) = \dim\,(\mathrm{Sym}^n\,\mathcal{T})(\mathscr{F}\ell_G).$$

Note that $\mathrm{Sym}^n\,\mathcal{T} = \mathrm{Sym}^n\,\mathcal{O}(2) = \mathcal{O}_{\mathbb{P}^1}(2n)$[2], so the right-hand side equals $\dim\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2n)) = 2n{+}1$. To compute the left-hand side, using PBW theorem, $\mathrm{gr}^n(U(\mathfrak{g})_{\chi_0})$ has a basis $\{e^i f^{n-i}\colon 0 \leqslant i \leqslant n\} \cup \{e^i f^{n-i-1}h\colon 0 \leqslant i \leqslant n-1\}$, so it also has dimension $2n+1$. This completes the proof that α_{χ_0} is injective, and hence an isomorphism. ☐

[2]See the proof of Problem 6.5 for details in this standard fact.

Problem 6.4 (Lecture 13, Exercise 1.5)**.** Let $e \in X \simeq G/B$ be the closed point corresponding to the chosen Borel subgroup B. We aim to prove that, as stated in the localization theorem, we can produce the (left) Verma module $M_{-2\rho}$ with highest weight -2ρ if using the left $\mathcal{D}$-module corresponding to δ_e. Let $\delta_e^{\mathrm{l}} \simeq \delta_e \otimes \omega_X^{-1}$ be the left $\mathcal{D}$-module corresponding to δ_e. Consider the left $U(\mathfrak{g})$-module $V := \Gamma(X, \delta_e^{\mathrm{l}})$.

(1) Prove: There is a *canonical* isomorphism

$$\delta_e^{\mathrm{l}} \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell,$$

where ℓ is the fiber of ω_X^{-1} at e, viewed as a skyscraper sheaf.

(2) Let $\ell \hookrightarrow V$ be the injection induced by taking global sections for the embedding $\mathcal{O}_X \otimes_{\mathcal{O}_X} \ell \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell$. Prove[3]: This line in V is a weight subspace of weight -2ρ.

(3) Prove[4]: The subalgebra $\mathfrak{b} \subset \mathfrak{g}$ stabilizes the line $\ell \subset V^{2\rho}$.

(4) Construct a $U(\mathfrak{g})$-linear map

$$M_{-2\rho} \longrightarrow V$$

and prove it is an isomorphism.

Solution. (1) By Problem 5.4, there is an isomorphism $\delta_e \simeq k_e \otimes_{\mathcal{O}_X} \mathcal{D}_X$ of right $\mathcal{D}_X$-modules. On the other hand, by definition we have the commutative diagram

$$\begin{array}{ccc} D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) & \xrightarrow{(-)\otimes_{\mathcal{O}_X}\omega_X^{-1}} & D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) \\ \downarrow{\scriptstyle \mathrm{ind}^{\mathrm{r}}} & & \downarrow{\scriptstyle \mathrm{ind}^{\mathrm{l}}} \\ D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) & \xrightarrow{(-)\otimes_{\mathcal{O}_X}\omega_X^{-1}} & D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathrm{qc}}) \end{array}$$

which leads to $k_e \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} k_e \otimes_{\mathcal{O}_X} \omega_X^{-1}$. If we write $i\colon e \hookrightarrow X$, then by construction we have an isomorphism $\ell \simeq i_* i^* \omega_X^{-1}$ of $\mathcal{O}_X$-modules. Thus,

$$\begin{aligned} \delta_e^{\mathrm{l}} &\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} i_* k \otimes_{\mathcal{O}_X} \omega_X^{-1} && \text{(by argument above)} \\ &\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} i_*(k \otimes_{\mathcal{O}_e} i^* \omega_X^{-1}) && \text{(by projection formula)} \\ &= \mathcal{D}_X \otimes_{\mathcal{O}_X} i_* i^* \omega_X^{-1} && \text{(by } \mathcal{O}_e = k\text{)} \\ &\simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell. && \text{(by prescribed description of } \ell\text{)} \end{aligned}$$

(2) By construction we have $\ell \simeq \wedge^d \mathcal{T}_{X,e} \simeq \wedge^d \mathfrak{n}^-$. Let v be a generator of this line ℓ, then v must be of form $\bigwedge_{\alpha\in\Phi^-} v_\alpha$. For each $h \in \mathfrak{h}$, we have by definition of $\mathfrak{h}$-action that

$$h(v) = \sum_{\beta\in\Phi^-} \bigwedge_{\alpha\in\Phi^- - \{\beta\}} v_\alpha \wedge h(v_\beta) = \sum_{\beta\in\Phi^-} \beta(h) \cdot \bigwedge_{\alpha\in\Phi^-} v_\alpha = \sum_{\beta\in\Phi^-} \beta(h)\cdot v.$$

Note that $\sum_{\beta\in\Phi^-} \beta(h) = -2\rho(h)$, so $\ell \subset V$ is a weight subspace of weight -2ρ.

(3) Recall that the PBW theorem for $\mathcal{D}_X$ dictates $\mathrm{gr}^\bullet \mathcal{D}_X \cong \mathrm{Sym}^\bullet_{\mathcal{O}_X} \mathcal{T}_X$, which further gives rise to a canonical filtration on V, where $\mathsf{F}^{\leqslant i} V$ is the image of ℓ under the action of $\mathsf{F}^{\leqslant i}\mathcal{D}_X$. Recall that the $\mathfrak{g}$-action is realized by $\mathcal{D}_X$ through the map $\mathfrak{g} \to \mathcal{T}_X \to \mathsf{F}^{\leqslant 1}\mathcal{D}_X \hookrightarrow \mathcal{D}_X$. Restricting this to $\mathfrak{b} \subset \mathfrak{g}$, the map $\mathfrak{b} \otimes \ell \to V$ factors through $\mathsf{F}^{\leqslant 1} V$ (as the canonical map $U(\mathfrak{g}) \to \mathcal{D}_X$ is compatible with the filtration). To show that ℓ is stabilized by $\mathfrak{b}$, it suffices to show that via the prescribed map $\mathfrak{b} \hookrightarrow \mathfrak{g} \to \mathcal{D}_X$, the image of $\mathfrak{b} \otimes \ell$ lies in $\mathcal{O}_X \otimes_{\mathcal{O}_X} \ell$. For this, we only need to show the composition

$$\mathfrak{b} \otimes \ell \longrightarrow \mathsf{F}^{\leqslant 1} V \longrightarrow \mathrm{gr}^1 V$$

[3]Hint: $\ell \simeq \wedge^d \mathcal{T}_{X,e}$ and $\mathcal{T}_{X,e} \simeq \mathfrak{n}^-$.

[4]Hint: Consider the PBW filtration of $\mathcal{D}_X$ and the induced filtration on V. Show that $\mathfrak{b} \otimes \ell \to V$ factors through $\mathsf{F}^{\leqslant 1} V$ and the composition $\mathfrak{b} \otimes \ell \to \mathsf{F}^{\leqslant 1} V \to \mathrm{gr}^1 V$ is zero.

is zero; but we observe $\mathrm{gr}^1 V = (\mathrm{gr}^1 \mathcal{D}_X)(\ell) = \mathcal{T}_X(\ell) \simeq \mathfrak{n}^- \otimes \ell$, which means the desired composition is induced by the map $\mathfrak{b} \to \mathfrak{n}^-$.

(4) Choose a set-theoretical map $M_{-2\rho} \to V$ such that the highest weight vector $v_{-2\rho} \in M_{-2\rho}$ is mapped to a generator $v \in \ell$ of $\ell \subset V$. By part (3), $\ell = kv$ is stabilized by $\mathfrak{b}$, so the map

$$M_{-2\rho} = k_{-2\rho} \otimes_{U(\mathfrak{b})} U(\mathfrak{g}) \longrightarrow V$$

is $U(\mathfrak{g})$-linear. To show this is an isomorphism, it suffices to show the bijectivity. The surjectivity follows from that V is generated by $\mathcal{D}_X$ from ℓ and $U(\mathfrak{g}) \twoheadrightarrow \mathcal{D}_X$ is surjective. As for the injectivity, suppose $t \in U(\mathfrak{n}^-)$ is such that $t \cdot v = 0$ in V, then we must have $t \cdot v_{-2\rho} = 0$ in $M_{-2\rho}$ because the map is $U(\mathfrak{g})$-linear. This completes the proof that $M_{-2\rho} \simeq V$. □

Problem 6.5 (Lecture 13, Exercise 3.12)**.** For $G = \mathrm{SL}_2$, prove $\mathfrak{p}\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is the blow-up of $\mathcal{N}$ at the point $0 \in \mathcal{N}$.

Solution. For $\mathfrak{g} = \mathfrak{sl}_2$, we have $\mathcal{N} = \{X \in \mathfrak{sl}_2\colon \det X = 0\}$. If we write $X = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathrm{M}_2(k)$ satisfying the condition $\det X = -a^2 - bc = 0$, we can realize the nilpotent cone $\mathcal{N}$ as the affine k-scheme

$$\mathcal{N} = \operatorname{Spec} k[a,b,c]/(a^2 + bc).$$

Recall that the blow-up of $\mathcal{N}$ at 0 is $\mathrm{Proj}(\mathcal{R}(\mathcal{I}))$, where $\mathcal{R}(\mathcal{I}) = \bigoplus_{n \geqslant 0} \mathcal{I}^n$ is the Rees algebra with $\mathcal{I}$ the defining ideal of $0 \in \mathcal{N}$, generated by images of a, b, c satisfying $a^2 + bc = 0$.

On the other hand, by definition of Springer resolution, at $0 \in \mathcal{N}$ we have

$$\widetilde{N} \simeq T^*(G/B) = T^*\mathbb{P}^1 \simeq \operatorname{Spec}_{\mathbb{P}^1}(\operatorname{Sym}^{\bullet}_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1}).$$

Here the last isomorphism is given by [Lecture 13, Construction 2.4]. To compute the relative spectrum on the right-hand side, we use $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$[5] to get

$$\operatorname{Sym}^{\bullet}_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{T}_{\mathbb{P}^1} = \operatorname{Sym}^{\bullet}_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(2) = \bigoplus_{n \geqslant 0} \operatorname{Sym}^n \mathcal{O}_{\mathbb{P}^1}(2) = \bigoplus_{n \geqslant 0} \mathcal{O}_{\mathbb{P}^1}(2n).$$

Combining these up, to show that $\widetilde{N}$ is the blow-up of $\mathcal{N}$ at 0, it remains to check $\mathrm{Proj}(\mathcal{R}(\mathcal{I})) \simeq \operatorname{Spec}_{\mathbb{P}^1}(\bigoplus_{n \geqslant 0} \mathcal{O}_{\mathbb{P}^1}(2n))$ as k-schemes. But this is true because each section of $\mathcal{R}(\mathcal{I})$ generates a regular function in a, b, c, and the degree of a is reduced by $2k \in \mathbb{Z}$ for some k via the relation $a^2 + bc = 0$. □

[5]For an explanation, notice that $\deg(\mathcal{T}_{\mathbb{P}^1}) + \deg(\mathcal{O}_{\mathbb{P}^1}) = \chi(\mathbb{P}^1) = 2$ with $\deg(\mathcal{O}_{\mathbb{P}^1}) = 0$, so $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$. Alternatively, we can construct the tangent bundle $\mathcal{T}_{\mathbb{P}^1}$ explicitly as follows. Cover $\mathbb{P}^1$ by standard affine opens $U_0 = \{(1 : x_1)\colon x_1 \in k\}$ and $U_1 = \{(x_0 : 1)\colon x_0 \in k\}$ and write down the transition function on $U_0 \cap U_1$ as $x_1 = x_0^{-1}$. It follows that $d(x_0^{-1})/dx = -1/x^2$, which also proves $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$.