LECTURE 10

From this lecture on, we turn to the geometric side of the localization theory. The main player will be (algebraic) D-modules on the flag variety G/B. In this lecture, we define D-modules. There are many good references for this theory. For example, [HTT] is a thorough textbook, while [B] is a short notes.

1. Recollection: (co)tangent sheaves

Recall the following definitions in algebraic geometry.

Definition 1.1. Let A be a k-algebra and M be an A-module. A **k-derivation** of A into M is a k-linear map $D : A \to M$ satisfying the **Lebniz rule**

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

Let $\mathsf{Der}(A, M)$ be the set of such k-derivations. This is naturally an A-module.

Definition 1.2. Let $(X, \mathcal{O}_X)$ be any k-ringed space and $\mathcal{M}$ be an $\mathcal{O}_X$-module. A **k-derivation** of $\mathcal{O}_X$ into $\mathcal{M}$ is a k-linear morphism $D : \mathcal{O}_X \to \mathcal{M}$ such that for any open subscheme $U \subset X$, $D(U) : \mathcal{O}(U) \to \mathcal{M}(U)$ is a k-derivation. Let $\mathsf{Der}(\mathcal{O}_X, \mathcal{M})$ be the space of k-derivations.

Proposition-Definition 1.3. *Let X be a k-scheme. The functor $\mathcal{O}_X-\mathsf{mod} \to \mathsf{Vect}$, $\mathcal{M} \mapsto \mathsf{Der}(\mathcal{O}_X, \mathcal{M})$ is represented by a quasi-coherent $\mathcal{O}_X$-module Ω^1_X, i.e.*

$$\mathsf{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{M}) \simeq \mathsf{Der}(\mathcal{O}_X, \mathcal{M}).$$

*We call Ω^1_X the **sheaf of differentials**, or the **cotangent sheaf**, of X over k.*

The identity map on Ω^1_X corresponds to a k-derivation

$$d : \mathcal{O}_X \to \Omega^1_X,$$

*which is called the **universal k-derivation** of $\mathcal{O}_X$.*

*When $X = \mathsf{Spec}(A)$ is affine, let $\Omega^1_A \in A-\mathsf{mod}$ be such that $\Omega^1_X \simeq \widetilde{\Omega^1_A}$. We call Ω^1_A the **module of differentials** of A over k.*

Example 1.4. If $A = \mathsf{Sym}(V)$ for a k-vector space $V \in \mathsf{Vect}$, then $\Omega^1_A \simeq A \otimes_k V$ and $d : A \to \Omega^1_A$ sends $v \in V \subset \mathsf{Sym}(V)$ to $dv = 1 \otimes v \in A \otimes_k V$.

Construction 1.5. *Let $f : X \to Y$ be a morphism between k-schemes. For any $\mathcal{O}_X$-module $\mathcal{M}$, consider the composition*

$$\mathsf{Der}(\mathcal{O}_X, \mathcal{M}) \to \mathsf{Der}(f_*\mathcal{O}_X, f_*\mathcal{M}) \to \mathsf{Der}(\mathcal{O}_Y, f_*\mathcal{M})$$

that sends a k-derivation $D : \mathcal{O}_X \to \mathcal{M}$ to $\mathcal{O}_Y \to f_*\mathcal{O}_X \xrightarrow{f_*(D)} f_*\mathcal{M}$. By definition, we obtain maps

$$\mathsf{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{M}) \to \mathsf{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, f_*\mathcal{M}) \simeq \mathsf{Hom}_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{M})$$

that are functorial in $\mathcal{M}$. This gives an $\mathcal{O}_X$-linear morphism

$$f^*\Omega^1_Y \to \Omega^1_X. \tag{1.1}$$

Date: Apr 29, 2024.

Lemma 1.6. *If $f : X \to Y$ is an open embedding, or more generally an étale morphism, then $f^*\Omega^1_Y \to \Omega^1_X$ is an isomorphism.*

Lemma 1.7. *Let $f : X \to Y$ be a closed embedding corresponding to the ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. Then we have*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} f^*\Omega^1_Y \to \Omega^1_X \to 0,$$

where the map $\mathcal{I}/\mathcal{I}^2 \to f^*\Omega^1_Y \simeq \Omega^1_Y/\mathcal{I}\Omega^1_Y$ is induced by $\mathcal{I} \hookrightarrow \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y$.

Remark 1.8. Since any affine scheme is a closed subscheme of $\mathsf{Spec}(\mathsf{Sym}(V))$ for some $V \in \mathsf{Vect}$, the above lemmas, together with Example 1.4, allow us to calculate Ω^1_X for any k-scheme X.

Corollary 1.9. *If X is (locally) of finite type over k, then Ω^1_X is (locally) coherent.*

Corollary 1.10. *If X is a smooth k-scheme of dimension n, then Ω^1_X is locally free of rank n.*

From now on, we always assume X is a smooth k-scheme.

Construction 1.11. *Let X be a smooth k-scheme. Define $\Omega^n_X := \bigwedge^n_{\mathcal{O}_X}(\Omega^1_X)$, i.e., the anti-symmetric quotient of $\Omega^1_X \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \Omega^1_X$. As in the study of differential geometry, there is a unique complex*

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots$$

*such that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^m(\alpha \wedge d\beta)$, where $\alpha \in \Omega^m_X(U)$ is a m-form. This is the **de Rham complex** of X.*

Construction 1.12. *Let X be a smooth k-scheme. We define the **tangent sheaf** $\mathcal{T}_X$ of X over k to be the dual of Ω^1_X, i.e.,*

$$\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

Note that $\mathcal{T}_X$ is quasi-coherent, and for any open subset U, we have

$$\mathcal{T}(U) := \mathsf{Der}(\mathcal{O}_U, \mathcal{O}_U).$$

By definition, for $\partial \in \mathcal{T}(U)$ and $f \in \mathcal{O}(U)$, we have $\partial(f) = \langle \partial, df \rangle$.

Corollary 1.13. *We have equivalences between functors $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathsf{Vect}$:*

$$\mathsf{Der}_k(\mathcal{O}_X, -) \simeq \Gamma(X, \mathcal{T}_X \underset{\mathcal{O}_X}{\otimes} -).$$

Remark 1.14. The **tangent space** $T_{X,x}$ introduced in [Section 3, Lecture 3] can be identified with the stalk of $\mathcal{T}_X$ at x.

Corollary 1.15. *If X is a smooth k-scheme of dimension n, then $\mathcal{T}_X$ is locally free of rank n.*

Construction 1.16. *Let $f : X \to Y$ be a morphism between smooth k-schemes. The morphism* (1.1) *induces an $\mathcal{O}_X$-linear morphism*

$$df : \mathcal{T}_X \to f^*\mathcal{T}_Y.$$

Lemma 1.17. *Let $X = X_1 \times X_2$ be a smooth k-scheme. Then the $\mathcal{O}_X$-linear morphism*

$$(d\mathsf{pr}_1, d\mathsf{pr}_2) : \mathcal{T}_X \to \mathsf{pr}_1^*\mathcal{T}_{X_1} \oplus \mathsf{pr}_2^*\mathcal{T}_{X_2}$$

is an isomorphism.

Construction 1.18. *Let X be a smooth k-scheme. For any open subset U of X, $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ defines a Lie bracket on $\mathsf{Der}(\mathcal{O}_U, \mathcal{O}_U)$. By defition, $\mathcal{O}_U$ is a representation of the obtained Lie algebra.*

It follows that $\mathcal{T}_X$ is a sheaf of Lie algebras *on X, and $\mathcal{O}_X$ is a $\mathcal{T}_X$-module.*

Warning 1.19. *The category* $\mathcal{O}_X\text{–}\mathsf{mod}$ *has a natural symmetric monoidal structure but* $\mathcal{T}_X$ *is* not *a Lie algebra object in this symmetric monoidal category. In other words, the Lie bracket* $[-,-]:\mathcal{T}_X\otimes_k\mathcal{T}_X\to\mathcal{T}_X$ *does not factor through* $\mathcal{T}_X\otimes_{\mathcal{O}_X}\mathcal{T}_X$.

Construction 1.20. *As in the study of differential geometry, for any local section* $\partial\in\mathcal{T}(U)$, *we can define the* ***contraction*** *operator*

$$i_\partial:\Omega_X^{n+1}(U)\to\Omega_X^n(U)$$

and the ***Lie derivative***

$$\mathcal{L}_\partial:\Omega_X^n(U)\to\Omega_X^n(U).$$

They satisfy all the identities in differential geometry. In particular, we have the ***Cartan's magic formula****:*

$$\mathcal{L}_\partial(\omega)=i_\partial(d\omega)+d(i_\partial\omega).$$

We have the following useful result, which allows us to apply the techniques in differential geometry to the study of $\mathcal{T}_X$ and Ω_X. For a proof, see [HTT, Theorem A.5.1].

Proposition-Definition 1.21. *Let* X *be any* n*-dimensional smooth* k*-scheme and* $p\in X$ *be a closed point. Then there exists an affine open neighborhood* U *of* p *and functions* $x_i\in\mathcal{O}(U)$, $i=1,\cdots,n$, *such that*

(i) $\{dx_i\}$ *is a free basis of* $\Omega^1(U)$ *as an* $\mathcal{O}(U)$*-module;*
(ii) *Let* $\{\partial_i\}$ *be the dual basis of* $\mathcal{T}(U)$, *then* $[\partial_i,\partial_j]=0$ *and* $\partial_i(x_j)=\delta_{ij}$.
(iii) *The images of* x_i *in the local ring* $\mathcal{O}_{X,p}$ *generate the maximal ideal* $\mathfrak{m}_{X,p}$.

We call such a system $\{x_i\}$ *an* ***étale coordinate system*** *of* X *near* p.

Remark 1.22. (ii) actually follows from (i).

Remark 1.23. The functions x_i define an étale map $X\to\mathbb{A}^n$ sending $p\in X$ to the origin $0\in\mathbb{A}^n$.

2. Tangent sheaf vs. Lie algebra

Construction 2.1. *Let* X *be a finite type* k*-scheme equipped with a* right *action of an algebraic group* G. *Let* $\mathfrak{g}:=\mathsf{Lie}(G)$ *be the Lie algebra of* G. *We construct a Lie algebra homomorphism*

$$a:\mathfrak{g}\to\mathcal{T}(X)$$

as follows. Consider the action map $\mathsf{act}:X\times G\to X$ *and the* $\mathcal{O}_{X\times G}$*-linear morphisms*

$$\mathcal{O}_X\otimes\mathcal{T}_G\simeq\mathsf{pr}_2^*\mathcal{T}_G\hookrightarrow\mathcal{T}_{X\times G}\xrightarrow{d\mathsf{act}}\mathsf{act}^*\mathcal{T}_X.$$

By restricting along $X\xrightarrow{\mathsf{Id}\times e}X\times G$, *we otain* $\mathcal{O}_X$*-linear morphisms*

$$\mathcal{O}_X\otimes\mathfrak{g}\to\mathcal{T}_X,$$

which induces the desired k*-linear map* $a:\mathfrak{g}\to\mathcal{T}(X)$.

Remark 2.2. Using the language of differential geometry, for $x\in\mathfrak{g}$, $a(x)$ is the vector field of the flow $\mathsf{act}(-,\exp(tx)):X\to X$.

Lemma 2.3. *The above map* $a:\mathfrak{g}\to\mathcal{T}(X)$ *is a Lie algebra homomorphism.*

Proof. Unwinding the definitions, the corresponding map $\mathsf{Der}(\mathcal{O}_G,k_e)\to\mathsf{Der}(\mathcal{O}_X,\mathcal{O}_X)$ sends a k-derivation $\partial:\mathcal{O}_G\to k_e$ to the composition[1]

$$a(\partial_1):\mathcal{O}_X\to\mathcal{O}_X\otimes\mathcal{O}_G\xrightarrow{\mathsf{Id}\otimes\partial}\mathcal{O}_X\otimes k_e\simeq\mathcal{O}_X.$$

[1]For non-affine G, we use the morphism between $\mathcal{O}_X$-modules: $\mathcal{O}_X\to\mathsf{act}_*(\mathcal{O}_G\otimes\mathcal{O}_X)\xrightarrow{\partial\otimes\mathsf{Id}}\mathsf{act}_*(k_e\otimes\mathcal{O}_X)\simeq\mathcal{O}_X$.

Using the axioms of group actions, we can identify $a(\partial_1)\circ a(\partial_2)$ with

$$\mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_G \xrightarrow{\mathsf{Id}\otimes\Delta} \mathcal{O}_X \otimes \mathcal{O}_G \otimes \mathcal{O}_G \xrightarrow{\mathsf{Id}\otimes\partial_1\otimes\partial_2} \mathcal{O}_X \otimes k_e \otimes k_e \simeq \mathcal{O}_X.$$

This implies

$$[a(\partial_1), a(\partial_2)] = a(\partial_1)\circ a(\partial_2) - a(\partial_2)\circ a(\partial_1) = a([\partial_1,\partial_2]),$$

where the last equation is due to [Remark 4.5, Lecture 3].

□

Remark 2.4. Let X be a finite type k-scheme equipped with a *left* action of an algebraic group G. We can obtain a right G-action by precomposing with $g \mapsto g^{-1}$. It follows that we also have a Lie algebra homomorphism $a : \mathfrak{g} \to \mathcal{T}(X)$. Using the language of differential geometry, for $x \in \mathfrak{g}$, $a(x)$ is the vector field of the flow $\mathsf{act}(\exp(-tx), -) : X \to X$.

Example 2.5. Consider the left and right multplication actions of G on itself. We obtain Lie algebra homomorphisms

$$\mathfrak{g} \xrightarrow{a_l} \mathcal{T}(G) \xleftarrow{a_r} \mathfrak{g}.$$

By construction, the image of a_r consists of *left* invariant vector fields on G, while the image of a_l consists of *right* invariant ones.

It is easy to see that the images of $a_l : \mathfrak{g} \to \mathcal{T}(G)$ and $a_r : \mathfrak{g} \to \mathcal{T}(G)$ commute with respect to the Lie bracket, i.e., $[a_l(x), a_r(y)] = 0$ for $x, y \in \mathfrak{g}$. Indeed, this follows by considering the $(G\times G)$-action on G given by $(g_1, g_2)\cdot x := g_1^{-1} x g_2$.

It is easy to show the obtained $\mathcal{O}_G$-linear maps, which we denote by the same symbols,

$$\mathcal{O}_G \otimes \mathfrak{g} \xrightarrow{a_l} \mathcal{T}_G \xleftarrow{a_r} \mathcal{O}_G \otimes \mathfrak{g}$$

are isomorphisms. Note that the stalks of the above maps at $e \in G$ are given by

$$\mathfrak{g} \xrightarrow{-\mathsf{Id}} \mathfrak{g} \xleftarrow{\mathsf{Id}} \mathfrak{g}.$$

The following terminology is not standard[2].

Definition 2.6. Let X be a smooth k-scheme. A **quasi-coherent $\mathcal{T}_X$-module** is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ equipped with a $\mathcal{T}_X$-module structure such that

(i) The map $\mathcal{T}_X \otimes_k \mathcal{M} \to \mathcal{M}$ is $\mathcal{O}_X$-linear, where $\mathcal{T}_X \otimes_k \mathcal{M}$ is viewed as an $\mathcal{O}_X$-module via the first factor.

(ii) The map $\mathcal{O}_X \otimes_k \mathcal{M} \to \mathcal{M}$ is $\mathcal{T}_X$-linear, where $\mathcal{O}_X \otimes_k \mathcal{M}$ is viewed as an $\mathcal{T}_X$-module via the diagonal action.

Let $\mathcal{T}_X\text{-}\mathsf{mod}_{\mathsf{qc}}$ be the category of quasi-coherent $\mathcal{T}_X$-module, where morphisms are defined in the obvious way.

Remark 2.7. Unwinding the definitions, (ii) means the action of $\mathcal{T}_X$ on $\mathcal{M}$ satisfies the **Lebniz rule**: for any local sections $\partial \in \mathcal{T}(U)$, $m \in \mathcal{M}(U)$ and $f \in \mathcal{O}(U)$, we have

$$\partial(f\cdot m) = f\cdot\partial(m) + \partial(f)\cdot m.$$

Example 2.8. The structure sheaf $\mathcal{O}_X$, equipped with the standard $\mathcal{T}_X$-action, is a quasi-coherent $\mathcal{T}_X$-module.

Remark 2.9. It is easy to show $\mathcal{T}_X\text{-}\mathsf{mod}_{\mathsf{qc}}$ is an abelian category and the forgetful functor $\mathcal{T}_X\text{-}\mathsf{mod}_{\mathsf{qc}} \to \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{qc}}$ is exact.

[2]The standard terminology would be: $\mathcal{M}$ is a module for the *Lie algebroid* $\mathcal{T}_X$.

Construction 2.10. *Let X be a finite type k-scheme equipped with an action of an algebraic group G. We have a functor*

$$\Gamma : \mathcal{T}_X\text{–}\mathsf{mod}_{\mathsf{qc}} \to \mathfrak{g}\text{–}\mathsf{mod},\ \mathcal{M} \mapsto \mathcal{M}(X),$$

where $\mathcal{M}(X)$ is viewed as a $\mathfrak{g}$-module via the Lie algebra homomorphism $a : \mathfrak{g} \to \mathcal{T}(X)$.

Remark 2.11. One version of the localizaiton theory says for $X = G/B$, the above functor induces an equivalence

$$\Gamma : \mathcal{T}_X\text{–}\mathsf{mod}_{\mathsf{qc}} \xrightarrow{\simeq} \mathfrak{g}\text{–}\mathsf{mod}_{\chi_0},$$

where $\chi_0 = \varpi(0)$ is the central character of M_0.

3. Differential operators

Just like the associative algebra $U(\mathfrak{g})$ plays a significant role in the study of $\mathfrak{g}$-modules, there is a sheaf of associative algebras, known as the sheaf of differential operators $\mathcal{D}_X$, that plays a similar role in the study of $\mathcal{T}_X$-modules[3].

Definition 3.1. Let X be an affine smooth k-scheme. We define the notion of differential operator on X inductively.

A k-linear map $D : \mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order $-n$ $(n > 0)$ iff $D = 0$.

A k-linear map $D : \mathcal{O}(X) \to \mathcal{O}(X)$ is a **differential operator of order** n $(n \geq 0)$ if for any function $f \in \mathcal{O}(X)$, the k-linear map $[D, f] := D \circ f - f \circ D$, i.e.,

$$g \mapsto D(fg) - fD(g)$$

is a differential operator of order $n - 1$.

Let $\mathsf{F}^{\leq n}\mathcal{D}(X)$ be the space of differential operators of order n on X, and $\mathcal{D}(X) := \cup_n \mathsf{F}^{\leq n}\mathcal{D}(X)$ be the space of all differential operators on X.

Example 3.2. Multplication by any $f \in \mathcal{O}(X)$ is a differential operator of order 0 and the map $\mathcal{O}(X) \to \mathsf{F}^{\leq 0}\mathcal{D}(X) \simeq \mathsf{gr}^0\mathcal{D}(X)$ is an isomorphism.

Exercise 3.3. This is Homework 5, Problem 1. Let X be an affine smooth k-scheme. Prove: any k-derivation $\mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order 1, and the obtained map $\mathcal{O}(X) \oplus \mathcal{T}(X) \to \mathsf{F}^{\leq 1}\mathcal{D}(X)$ is an isomorphism.

Lemma 3.4. *Let X be an affine smooth k-scheme, and $D_1 \in \mathsf{F}^{\leq m}\mathcal{D}(X)$, $D_2 \in \mathsf{F}^{\leq n}\mathcal{D}(X)$ be differential operators. Prove:*

(1) *The composition $D_1 \circ D_2$ is a differential operator of order $m + n$.*
(2) *The commutator $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ is a differential operator of order $m + n - 1$.*

In other words, $\mathsf{F}^\bullet\mathcal{D}(X)$ is a filtered associative algebra such that $\mathsf{gr}^\bullet\mathcal{D}(X)$ is commutative.

Proof. Follow from the above exercise and induction. □

Remark 3.5. For *any* k-linear map $D : \mathcal{O}(X) \to \mathcal{O}(X)$ and functions $u, v \in \mathcal{O}(X)$, we have

$$[D, uv] = [D, u]v + u[D, v]. \tag{3.1}$$

It follows that in Definition 3.1, we can replace "for any function $f \in \mathcal{O}(X)$" by "for generators f of $\mathcal{O}(X)$".

Note that we also have

$$D(uv) = [D, u](v) + uD(v). \tag{3.2}$$

[3]The sheaf $\mathcal{D}_X$ is so useful that $\mathcal{D}_X$-modules were introduced much earlier than $\mathcal{T}_X$-modules. The latter point of view was ignored until the study of general Lie algebroids.

Remark 3.6. For a disjoint union $V_1 \sqcup V_2$ of affine smooth k-schemes, we have $\mathsf{F}^\bullet\mathcal{D}(V_1 \sqcup V_2) \simeq \mathsf{F}^\bullet\mathcal{D}(V_1) \oplus \mathsf{F}^\bullet\mathcal{D}(V_2)$.

Warning 3.7. *Definition 3.1 makes sense even when X is singular, but the obtained algebra $\mathcal{D}(X)$ is ill-behaved and is not the correct algebra to consider. In modern point of view, for singular X, the "correct" $\mathcal{D}(X)$ should be a DG algebra.*

4. Sheaf of differential operators

In this section, for any smooth k-scheme, we construct a sheaf $\mathcal{D}_X$ of filtered associative algebras such that for any affine open subscheme U, we have $\mathcal{D}_X(U) \simeq \mathcal{D}(U)$. The proofs in this section are technical and can be treated as blackboxes.

Throughout this section, let V be a *connected* affine smooth k-scheme and $f \in \mathcal{O}(V)$ be a nonzero function. Let $U \subset V$ be the affine open subscheme where $f \neq 0$, i.e. $\mathcal{O}(U) \simeq \mathcal{O}(V)_f$. Note that U is also connected. We identify $\mathcal{O}(V)$ as a subalgebra of $\mathcal{O}(U)$.

Lemma 4.1. *Any differential operator $D \in \mathcal{D}(U)$ is determined by its restriction $D|_{\mathcal{O}(V)} : \mathcal{O}(V) \to \mathcal{O}(U)$.*

Proof. Let $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$. We prove by induction in n. When $n < 0$, there is nothing to prove. For $n \geq 0$, for any $g \in \mathcal{O}(V)$ and $m \geq 0$, we have

$$D(g) = D(f^m \cdot \frac{g}{f^m}) = f^m \cdot D(\frac{g}{f^m}) + [D, f^m](\frac{g}{f^m}). \tag{4.1}$$

It follows that

$$D(\frac{g}{f^m}) = \frac{D(g)}{f^m} - \frac{[D, f^m](\frac{g}{f^m})}{f^m}. \tag{4.2}$$

Note that $[D, f^m] \in \mathsf{F}^{\leq n-1}\mathcal{D}(U)$ hence it is determined by $[D, f^m]|_{\mathcal{O}(V)} = [D|_{\mathcal{O}(V)}, f^m]$, which is determined by $D|_{\mathcal{O}(V)}$. Now (4.2) implies D is also determined by $D|_{\mathcal{O}(V)}$. □

Lemma 4.2. *For any differential operator $D_0 \in \mathsf{F}^{\leq n}\mathcal{D}(V)$, there is a unique differential operator $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{O}(V) & \xrightarrow{D_0} & \mathcal{O}(V) \\ \downarrow & & \downarrow \\ \mathcal{O}(U) & \xrightarrow{D} & \mathcal{O}(U). \end{array}$$

Proof. The uniqueness follows from Lemma 4.1. It remains to show the existence. We prove by induction in n. When $n < 0$, there is nothing to prove. For $n \geq 0$ and any $m \geq 0$, by induction hypothesis, there is a unique differential operator $[D_0, f^m]^\sharp \in \mathsf{F}^{\leq n-1}\mathcal{D}(U)$ that extends $[D_0, f^m] \in \mathsf{F}^{\leq n-1}\mathcal{D}(V)$. Motivated by (4.2), for any $g \in \mathcal{O}(V)$, we define

$$D(\frac{g}{f^m}) := \frac{D_0(g)}{f^m} - \frac{[D_0, f^m]^\sharp(\frac{g}{f^m})}{f^m}.$$

We need to show the map D is well-defined, i.e., $D(\frac{g}{f^m}) = D(\frac{f^l g}{f^{l+m}})$ for any $l \geq 0$. Note that we have

$$D_0(f^l g) = f^l D_0(g) + [D_0, f^l](g).$$

We also have $[D_0, f^{l+m}] = [D_0, f^l]f^m + f^m[D_0, f^l]$. This implies

$$[D_0, f^{l+m}]^\sharp = [D_0, f^l]^\sharp f^m + f^m[D_0, f^l]^\sharp.$$

Combining the above two equations, a direct calculation shows $D(\frac{g}{f^m}) = D(\frac{f^l g}{f^{l+m}})$ as desired.

It remains to show D is a differential operator of order n. By Remark 3.5, we only need to show $[D, f^{-1}]$ and $[D, h]$, $h \in \mathcal{O}(V)$ are differential operators of order $n-1$. By (3.1), we have $0 = [D, ff^{-1}] = f[D, f^{-1}] + [D, f]f^{-1}$. Hence $[D, f^{-1}] = -f^{-1}[D, f]f^{-1}$. Therefore we only need to show $D' := [D, h] \in \mathsf{F}^{\leq n-1}\mathcal{D}(U)$. Write $D'_0 := [D_0, h] \in \mathsf{F}^{\leq n-1}\mathcal{D}(V)$. A direct calculations shows D' can be obtained from D'_0 using the same formula that defines D from D_0, i.e.,

$$D'(\frac{g}{f^m}) := \frac{D'_0(g)}{f^m} - \frac{[D'_0, f^m]^\sharp(\frac{g}{f^m})}{f^m}.$$

Hence we win by induction in n (again).

□

Lemma 4.3. *For any differential operator $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$, there exists an integer $N \geq 0$ such that $f^N D$ and Df^N send $\mathcal{O}(V)$ into $\mathcal{O}(V)$.*

Proof. We prove by induction in n. When $n < 0$, there is nothing to prove. For $n \geq 0$,let $g_i \in \mathcal{O}(V)$, $i \in I$ be a *finite* set of generators. By induction hypothesis, there exists an integer $N \geq 0$ such that $f^N[D, g_i]$ and $[D, g_i]f^N$ preserve $\mathcal{O}(V)$ for any $i \in I$. By (3.1), $f^N[D, g]$ and $[D, g]f^N$ preserve $\mathcal{O}(V)$ for any $g \in \mathcal{O}(V)$. By enlarging N, we can assume $f^N D(1) \in \mathcal{O}(V)$. Hence by (3.2), $f^N D(g) = f^N[D, g](1) + f^N g D(1) \in \mathcal{O}(V)$ and $D(f^{2N}g) = [D, f^N](f^N g) + f^N D(f^N g) \in \mathcal{O}(V)$.

□

Construction 4.4. *Let $\mathsf{F}^\bullet\mathcal{D}(V) \to \mathsf{F}^\bullet\mathcal{D}(U)$ be the map defined by Lemma 4.2. Recall $\mathcal{D}(U)$ is an $\mathcal{O}(U)$-bimodule and $\mathcal{D}(V)$ is an $\mathcal{O}(V)$-bimodule. It is easy to see the above map is compatible with the homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$. It follows that we have maps*

$$\mathcal{O}(U) \underset{\mathcal{O}(V)}{\otimes} \mathsf{F}^\bullet\mathcal{D}(V) \to \mathsf{F}^\bullet\mathcal{D}(U) \leftarrow \mathsf{F}^\bullet\mathcal{D}(V) \underset{\mathcal{O}(V)}{\otimes} \mathcal{O}(U), \tag{4.3}$$

such that the rightward map is left $\mathcal{O}(U)$-linear while the leftward map is right $\mathcal{O}(U)$-linear.

Lemma 4.5. *The maps* (4.3) *are isomorphisms.*

Proof. We first show the maps are injective. Note that any element in $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathsf{F}^\bullet\mathcal{D}(V)$ is a pure tensor $f^{-m} \otimes D_0$ for some $m \geq 0$ and $D_0 \in \mathsf{F}^n\mathcal{D}(V)$. The rightward map sends it to $f^{-m}D_0 \in \mathsf{F}^n\mathcal{D}(U)$. If $f^{-m}D_0 = 0$, then $D_0 = 0$ because $\mathcal{O}(V)$ is integral. This proves the rightward map is injective. A similar argument shows the leftward map is injective.

To prove the maps are surjective, let $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$. By Lemma 4.3, there exists $N \geq 0$ such that $D_0 := f^N D$ and $D'_0 := Df^N$ preserve $\mathcal{O}(V)$. We view D_0 and D'_0 as elements in $\mathsf{F}^{\leq n}\mathcal{D}(V)$. By Lemma 4.1, the rightward (resp. leftward) map sends $f^{-N} \otimes D_0$ (resp. $D'_0 \otimes f^{-N}$) to D.

□

Construction 4.6. *Let X be a smooth k-scheme. By Lemma 4.5 and Remark 3.6, for any n, there exists a unique sheaf $\mathsf{F}^{\leq n}\mathcal{D}_X$ such that:*

- *For any affine open subscheme $U \subset X$,*

$$\mathsf{F}^{\leq n}\mathcal{D}_X(U) := \mathsf{F}^{\leq n}\mathcal{D}(U);$$

- *For $U \subset V \subset X$ as before, the connecting map $\mathsf{F}^{\leq n}\mathcal{D}_X(V) \to \mathsf{F}^{\leq n}\mathcal{D}_X(U)$ is given by the map $\mathsf{F}^{\leq n}\mathcal{D}(U) \to \mathsf{F}^{\leq n}\mathcal{D}(V)$ in Construction 4.4.*

*We call $\mathsf{F}^{\leq n}\mathcal{D}_X$ the **sheaf of differential operators of order** n on X.*

*Define the **sheaf of differential operators** on X to be*

$$\mathcal{D}_X := \cup_n \mathsf{F}^{\leq n}\mathcal{D}_X.$$

Note that $\mathsf{F}^\bullet\mathcal{D}_X$ *is naturally a sheaf of filtered associtive* k*-algebras such that* $\mathsf{gr}^\bullet\mathcal{D}_X$ *is commutative.*

Remark 4.7. Unwinding the definitions, for any open subscheme $V \subset X$, $\mathcal{D}_X(V)$ can be identified with the algebra of k-linear morphisms $\mathcal{O}_V \to \mathcal{O}_V$ such that for any affine open subscheme $U \subset V$, the obtained map $\mathcal{O}(U) \to \mathcal{O}(U)$ is a differential operator.

Corollary 4.8. *Let* X *be a smooth* k*-scheme. We have natural isomorphisms* $\mathcal{O}_X \simeq \mathsf{F}^{\le 0}\mathcal{D}_X$ *and* $\mathcal{O}_X \oplus \mathcal{T}_X \simeq \mathsf{F}^{\le 1}\mathcal{D}_X$.

Corollary 4.9. *Let* X *be a smooth* k*-scheme. Then* $\mathcal{D}_X$ *is naturally an* $\mathcal{O}_X$*-bimodule, and it is quasi-coherent for both the left and right* $\mathcal{O}_X$*-module structures.*

5. The PBW theorem for $\mathcal{D}_X$

Construction 5.1. *Let* X *be a smooth* k*-scheme. By Corollary 4.8, we have a homomorphism* $\mathcal{O}_X \to \mathsf{gr}^0\mathcal{D}_X$ *and a morphism* $\mathcal{T}_X \to \mathsf{gr}^1\mathcal{D}_X$ *between their modules. By the universal property of symmetric algebras, we obtain a graded homomorphism*

$$\mathsf{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathsf{gr}^\bullet\mathcal{D}_X. \tag{5.1}$$

The proof of the following theorem can be treated as a blackbox[4].

Theorem 5.2 (PBW for $\mathcal{D}_X$)**.** *The above graded homomorphism is an isomorphism.*

Proof. We can assume X is affine and connected. Hence we only need to show

$$\mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)) \to \mathsf{gr}^n\mathcal{D}(X)$$

is an isomorphism. We prove by induction in n. For $n < 0$, there is nothing to prove. For $n \ge 0$ and $D \in \mathsf{F}^{\le n}\mathcal{D}(X)$, consider the composition

$$\mathcal{O}(X) \xrightarrow{[D,-]} \mathsf{F}^{\le n-1}\mathcal{D}(X) \twoheadrightarrow \mathsf{gr}^{n-1}\mathcal{D}(X).$$

By (3.1), this is a k-derivation of $\mathcal{O}(X)$ into $\mathsf{gr}^{n-1}\mathcal{D}(X)$, where the latter is viewed as an $\mathcal{O}(X)$-module via the homomorphism $\mathcal{O}(X) \to \mathsf{gr}^0\mathcal{D}(X)$. Moreover, this k-derivation only depends on the image of D in $\mathsf{gr}^n\mathcal{D}(X)$. In other words, we have a k-linear map

$$\mathsf{gr}^n\mathcal{D}(X) \to \mathsf{Der}_k(\mathcal{O}(X), \mathsf{gr}^{n-1}\mathcal{D}(X)),\ D \mapsto [D,-].$$

Note that $[gD, f] = g[D, f]$ for any $f, g \in \mathcal{O}(X)$. It follows that the above map is $\mathcal{O}(X)$-linear. By induction hypothesis and Corollary 1.13, we have

$$\mathsf{Der}_k(\mathcal{O}(X), \mathsf{gr}^{n-1}\mathcal{D}(X)) \simeq \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{gr}^{n-1}\mathcal{D}(X) \simeq \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{Sym}^{n-1}_{\mathcal{O}(X)}(\mathcal{T}(X)).$$

Composing with the multplication map

$$\mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{Sym}^{n-1}_{\mathcal{O}(X)}(\mathcal{T}(X)) \to \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)),$$

we obtain an $\mathcal{O}(X)$-linear map

$$\mathsf{gr}^n\mathcal{D}(X) \to \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)).$$

By considering affine open subschemes, we obtain an $\mathcal{O}_X$-linear morphism

$$\mathsf{gr}^n\mathcal{D}_X \to \mathsf{Sym}^m_{\mathcal{O}_X}(\mathcal{T}_X).$$

[4]In fact, for the purpose of *this* course, we can define $\mathcal{D}_X$ to be the subsheaf of $\mathcal{H}om_k(\mathcal{O}_X, \mathcal{O}_X)$ generated by the images $\mathcal{O}_X$ and $\mathcal{T}_X$ under compositions. Then the PBW theorem becomes obvious. However, it is not obvious to show this definition coincides with ours.

We claim it is inverse to (5.1) up to multplication by n.

A direct calculation shows

$$\mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)) \to \mathsf{gr}^n\mathcal{D}(X) \to \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X))$$

is given by multplication by n. Indeed, for $\partial_1, \cdots, \partial_n \in \mathcal{T}(X)$, we have

$$[\partial_1 \cdots \partial_n, f] \equiv \sum_{i=1}^{n} \partial_i(f)\partial_1 \cdots \widehat{\partial_i} \cdots \partial_n \quad \in \quad \mathsf{gr}^{\leq n-1}\mathcal{D}(X).$$

Hence $\partial_1 \cdots \partial_n$ is sent to the element

$$\sum_{i=1}^{n} \partial_i \otimes \partial_1 \cdots \widehat{\partial_i} \cdots \partial_n \in \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{Sym}^{n-1}_{\mathcal{O}(X)}(\mathcal{T}(X)),$$

which is sent to $n\partial_1 \cdots \partial_n \in \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X))$ as desired.

It remains to show

$$\mathsf{gr}^n\mathcal{D}_X \to \mathsf{Sym}^m_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathsf{gr}^n\mathcal{D}_X \tag{5.2}$$

is given by multplication by n. Note that the problem is Zariski local in X, hence we can assume X admits an étale coordinate system $x_1, \cdots, x_d$. Now let $D \in \mathsf{F}^{\leq n}\mathcal{D}_X$ and

$$\sum_{i=1}^{d} \partial_i \otimes D_i \in \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{gr}^{n-1}\mathcal{D}(X)$$

be its image. By definition, for any function $f \in \mathcal{O}(X)$, we have

$$[D, f] \equiv \sum_{i=1}^{d} \partial_i(f) D_i \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X)$$

Taking $f = x_i$, we obtain $D_i \equiv [D, x_i]$. Therefore for any $f \in \mathcal{O}(X)$,

$$[D, f] \equiv \sum_{i=1}^{d} \partial_i(f)[D, x_i] \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X) \tag{5.3}$$

and the composition (5.2) sends D to $\sum_{i=1}^{d} \partial_i[D, x_i]$. It remains to show

$$nD \equiv \sum_{i=1}^{d} \partial_i[D, x_i] \quad \in \quad \mathsf{gr}^n\mathcal{D}(X)$$

To prove this, we use induction in n (again). For $n \leq 0$, the claim is obvious. For $n > 0$, we only need to show

$$n[D, f] \equiv \sum_{i=1}^{d} [\partial_i[D, x_i], f] \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X) \tag{5.4}$$

for any $f \in \mathcal{O}(X)$.

A direct calculation shows

$$[\partial_i[D, x_i], f] = \partial_i[[D, f], x_i] + \partial_i(f)[D, x_i]. \tag{5.5}$$

By induction hypothesis, we have

$$(n-1)[D, f] \equiv \sum_{i=1}^{d} \partial_i[[D, f], x_i] \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X). \tag{5.6}$$

Now (5.3)+(5.6) implies (5.4) by (5.5).

□

Corollary 5.3. *Let X be a affine smooth k-scheme, then the associative algebra $\mathcal{D}(X)$ is generated by the images of $\mathcal{O}(X)$ and $\mathcal{T}(X)$ subject to the following relations:*

$$f_1 \star f_2 = f_1 f_2,\ \partial_1 \star \partial_2 - \partial_2 \star \partial_1 = [\partial_1, \partial_2],\ f \star \partial = f\partial,\ \partial \star f - f \star \partial = \partial(f)$$

for $f, f_1, f_2 \in \mathcal{O}(X)$ and $\partial, \partial_1, \partial_2 \in \mathcal{T}(X)$. Here $\star$ (temporarily) denotes the multplication in $\mathcal{D}(X)$.

Example 5.4. For $X = \mathbb{A}^d$, we have $\mathcal{D}(X) = k[x_1, \cdots, x_d, \partial_1, \cdots, \partial_d]$ such that $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{i,j}$. This is known as the **Weyl algebra**.

6. Definition of D-modules

Definition 6.1. Let X be a smooth k-scheme. A **left (resp. right) $\mathcal{D}_X$-module** is a sheaf $\mathcal{M}$ of k-vector spaces equipped with a left (resp. right) action by $\mathcal{D}_X$.

Let $\mathcal{D}_X\text{-}\mathsf{mod}^l$ (resp. $\mathcal{D}_X\text{-}\mathsf{mod}^r$) be the category of left (resp. right) $\mathcal{D}_X$-modules, where morphisms are defined in the obvious way.

Construction 6.2. *Let X be a smooth k-scheme. Restricting along the homomorphism $\mathcal{O}_X \to \mathcal{D}_X$, we obtain forgetful functors*

$$\mathsf{oblv}^l : \mathcal{D}_X\text{-}\mathsf{mod}^l \to \mathcal{O}_X\text{-}\mathsf{mod},\ \mathsf{oblv}^r : \mathcal{D}_X\text{-}\mathsf{mod}^r \to \mathcal{O}_X\text{-}\mathsf{mod}.$$

*We say a left (resp. right) $\mathcal{D}_X$-module $\mathcal{M}$ is **quasi-coherent** if the underlying $\mathcal{O}_X$-module is quasi-coherent.*

Let $\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ (resp. $\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$) be the category of left (resp. right) $\mathcal{D}_X$-modules.

Remark 6.3. It is easy to see the above categories are abelian categories and the forgetful functors are exact.

The following result follows from the PBW theorem:

Proposition 6.4. *Restricting along $\mathcal{T}_X \to \mathcal{D}_X$ defines an equivalence*

$$\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \simeq \mathcal{T}_X\text{-}\mathsf{mod}_{\mathsf{qc}}.$$

Remark 6.5. In fact, we could have defined a notion of *quasi-coherent right $\mathcal{T}_X$-modules* such that they form an abelian category equivalent to $\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$. We did not introduce this notion because people are less familiar with right modules of Lie algebras.

7. Examples of D-modules

Example 7.1. The sheaf $\mathcal{D}_X$ itself is a left and a right $\mathcal{D}_X$-module.

Example 7.2. The sheaf $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module with the action given by $D \cdot f := D(f)$.

Exercise 7.3. This is Homework 5, Problem 2. Let X be a smooth k-scheme of dimension n. Prove: there is a unique right $\mathcal{D}_X$-module structure on Ω^n_X such that for local sections $f \in \mathcal{O}(U)$, $\partial \in \mathcal{T}(U)$ and $\omega \in \Omega^n(U)$, the right action is given by

$$\omega \cdot f = f\omega,\ \omega \cdot \partial = -\mathcal{L}_\partial(\omega)$$

Example 7.4. For $X = \mathbb{A}^1 = \mathsf{Spec}(k[x])$, we define a left D-module $\mathcal{M}_{e^x}$ whose underlying $\mathcal{O}_X$-module is isomorphic to $\mathcal{O}_X$, with a generator denoted by "e^x". The $\mathcal{T}_X$-action is determined by the formular $\partial_x \cdot e^x = e^x$. It is easy to see $\mathcal{M}_{e^x} \simeq \mathcal{D}_X / \mathcal{D}_X \cdot (\partial_x - 1)$, where $\mathcal{D}_X \cdot (\partial_x - 1)$ is the left ideal of $\mathcal{D}_X$ generated by the section $\partial_x - 1$.

Example 7.5. For $X = \mathbb{A}^1 - 0 = \mathsf{Spec}(k[x^{\pm}])$ and $\lambda \in k$, we define a left D-module $\mathcal{M}_{x^\lambda}$ whose underlying $\mathcal{O}_X$-module is isomorphic to $\mathcal{O}_X$, with a generator denoted by "x^λ". The $\mathcal{T}_X$-action is determined by the formular $\partial_x \cdot x^\lambda = \lambda x^{-1} \cdot x^\lambda$. It is easy to see $\mathcal{M}_{x^\lambda} \simeq \mathcal{D}_X / \mathcal{D}_X \cdot (\partial_x - \lambda x^{-1})$, where $\mathcal{D}_X \cdot (\partial_x - \lambda x^{-1})$ is the left ideal of $\mathcal{D}_X$ generated by the section $\partial_x - \lambda x^{-1}$.

Exercise 7.6. This is Homework 5, Problem 3. In Example 7.5, prove $\mathcal{M}_{x^\lambda}$ is isomorphic to $\mathcal{O}_X$ as left $\mathcal{D}_X$-modules iff $\lambda \in \mathbb{Z}$.

8. $\mathcal{D}_X$ vs. $U(\mathfrak{g})$

Construction 8.1. *Let X be a smooth k-scheme equipped with an action by an algebraic group G. Consider the Lie algebra homomorphisms*

$$\mathfrak{g} \xrightarrow{a} \mathcal{T}(X) \to \mathcal{D}(X)$$

where $\mathcal{D}(X)$ is viewed as a Lie algebra via the forgetful functor $\mathsf{Alg} \to \mathsf{Lie}$. *By the universal property of $U(\mathfrak{g})$, we obtain a homomorphism*

$$U(\mathfrak{g}) \xrightarrow{a} \mathcal{D}(X).$$

By construction, this homomorphism is compatible with the PBW filtrations on both sides.

This induces a functor

$$\Gamma : \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \to U(\mathfrak{g})\text{-}\mathsf{mod},\ \mathcal{M} \mapsto \mathcal{M}(X).$$

By construction, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} & \xrightarrow{\simeq} & \mathcal{T}_X\text{-}\mathsf{mod}_{\mathsf{qc}} \\ \downarrow \Gamma & & \downarrow \Gamma \\ U(\mathfrak{g})\text{-}\mathsf{mod} & \xrightarrow{\simeq} & \mathfrak{g}\text{-}\mathsf{mod}. \end{array}$$

Remark 8.2. The localizaiton theory says for $X = G/B$, the homomorphism $U(\mathfrak{g}) \xrightarrow{a} \mathcal{D}(X)$ induces an isomorphism

$$a : U(\mathfrak{g})_{\chi_0} := U(\mathfrak{g}) \underset{Z(\mathfrak{g})}{\otimes} k_{\chi_0} \xrightarrow{\simeq} \mathcal{D}(X).$$

and the functor Γ induces an equivalence

$$\Gamma : \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \xrightarrow{\simeq} U(\mathfrak{g})\text{-}\mathsf{mod}_{\chi_0}.$$

References

[B] Bernstein, Joseph. Algebraic theory of D-modules, 1984, abailable at `https://gauss.math.yale.edu/~il282/Bernstein_D_mod.pdf`.

[G] Gaitsgory, Dennis. Course Notes for Geometric Representation Theory, 2005, available at `https://people.mpim-bonn.mpg.de/gaitsgde/267y/cat0.pdf`.

[H] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[HTT] Hotta, Ryoshi, and Toshiyuki Tanisaki. D-modules, perverse sheaves, and representation theory. Vol. 236. Springer Science & Business Media, 2007.