LECTURE 12

In this lecture, we discuss about the localization theorem.

1. Flag variety

In this section, G can be any connected affine algebraic group over an algebraic closed field k. Let B be a Borel subgroup of G, which is defined to be a maximal connected solvable subgroup. Consider the right multiplication action of B on G on itself. We will review some basic facts about the flag variety G/B.

Theorem 1.1. *The quotient G/B exists and is a projective k-scheme.*

Remark 1.2. The proof of this theorem in my prefered reference [M] is more involved than those in the other literatures because Milne uses scheme theory and allows nilpotency in the definition of algebraic groups. Below is a guideline to this proof.

- (Existence of quotient) In general, for any subgroup K of an algebraic group H over any field k, the quotient H/K exists (see [M, Section 5.c]). Such k-scheme H/K is called a **homogeneous space** under H. More precisely, the fppf sheafification of the naïve functor $\mathsf{CAlg}_k \to \mathsf{Set}$, $R \mapsto H(R)/K(R)$ is represented by a k-scheme H/K, and the map $H \to H/K$ exhibits H as a fppf K-torsor on H/K. When $\mathsf{char}(k) = 0$, "fppf" can be replaced by "étale"[1].
- (Quasi-projective) In general, any homogeneous space H/K under H is quasi-projective (see [M, Section 8.k]).
- The above two parts are more about scheme theory rather than representation theory.
- (Completeness) The quotient G/B is shown to be complete in [M, Section 17]. There are several important representation theoretic ingredients in this proof: Chevalley's theorem[2], Lie–Kolchin theorem[3]

Example 1.3. For $G = \mathsf{SL}_2$ or GL_2 and the standard Borel subgroup B, G/B is isomorphic to $\mathbb{P}^1$ such that the k-point $\mathsf{pt} \simeq B/B \to G/B$ corresponds to the k-point $\infty \in \mathbb{P}^1$.

Indeed, consider the standard 2-dimensional representation V of G. We view V as an affine k-scheme and $\mathbb{P}(V)$ as a projective k-scheme. We obtain a transitive action of G on $\mathbb{P}(V) \simeq \mathbb{P}^1$. Consider the k-point $\infty \in \mathbb{P}(V)$ given by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V$. The stablizer subgroup at this point is the standard Borel subgroup B. It follows that we have an isomorphism $G/B \simeq \mathbb{P}(V)$ that sends gB to $g \cdot \infty$.

Date: May 13, 2024.

[1] The contents of the above words are: for any $R \in \mathsf{CAlg}_k$, $H(R) \to (H/K)(R)$ factors as $H(R) \twoheadrightarrow H(R)/K(R) \hookrightarrow (H/K)(R)$. And for any element in $x \in (H/K)(R)$, there exists a finite presented faithfully flat R-algebra R' such that the image x' of x under $(H/K)(R) \to (H/K)(R')$ is contained in the image of $H(R') \to (H/K)(R')$. In zero characteristic, we can require R' to be an étale R-algebra.

[2] Every subgroup K of an affine algebraic group H can be realized as the stablizer of a 1-dimensional subspace L in a finite-dimensional representation V. See [M, Section 4.h].

[3] Irreducible representations of a smooth connected solvable algebraic group B are 1-dimensional. See [M, Section 16.d].

Example 1.4. For $G = \mathsf{SL}_n$ or GL_n and the standard Borel subgroup B, G/B classifes complete flags in $k^{\oplus n}$, i.e., subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = k^{\oplus n}$ such that $\dim(V_k) = k$. For this reason, G/B is called the **flag variety** of G.

In this case, it is easy to write down homogenous coordinates to show G/B is projective. See [M, Section 7.g].

Theorem 1.1, together with the Borel fixed point theorem[4], imply the following results, which were used in our previous lectures. See [M, Section 17].

Theorem 1.5. *Any two Borel subgroups of G are conjugate by an element of $G(k)$.*

Theorem 1.6. *The normalizer subgroup of B in G is equal to B, i.e., $N_G(B) = B$.*

Construction 1.7. *We have $(G/B)(k) \simeq G(k)/B(k)$. Then Theorem 1.5 and Theorem 1.6 imply there is an isomorphism*

$$(G/B)(k) \xrightarrow{\sim} \{\text{Borel subgroups of } G\},\ gB \mapsto \mathsf{Ad}_g(B)$$

and therefore

$$(G/B)(k) \xrightarrow{\sim} \{\text{Borel subalgebras of } \mathfrak{g}\},\ gB \mapsto \mathsf{Ad}_g(\mathfrak{b}).$$

Corollary 1.8. *For two Borel subgroups B and B' of G, there is a unique G-equivariant isomorphism $G/B \to G/B'$.*

Notation 1.9. *The above corollary says that as a k-scheme equipped with a G-action, G/B does not depend on the choice of G. We write Fl_G for it.*

Remark 1.10. In fact, one can *define* the moduli problem classifying Borel subalgebras of $\mathfrak{g}$ and show that it is representable by a k-scheme Fl_G which is isomorphic to G/B for any Borel subgroup B. Namely, let $n = \dim(G)$ and d be the common dimension of all the Borel subgroups. Then one can show the above moduli problem is a closed subfunctor of the Grassmannian $\mathsf{Gr}(d, \mathfrak{g})$ that classifies d-dimensional subspaces of $\mathfrak{g}$.

It is a well-known fact that the obvious map $\mathsf{Gr}(d, V) \to \mathbb{P}(\wedge^d V)$ is a closed embedding, known as the **Plücker embedding**. Hence the composition $\mathsf{Fl}_G \to \mathsf{Gr}(d, \mathfrak{g}) \to \mathbb{P}(\wedge^d \mathfrak{g})$ gives a closed embedding of Fl_G into a projective space.

2. Line bundles on flag variety

From now on, G is assumed to be connected and *reductive*. Recall this means the unipotent radical of G is trivial. We can classify the line bundles on the flag variety G/B using the characters of B (or equivalently, of $T \simeq B/[B, B]$). To describe this classfication, we need the following construction.

Definition 2.1. Let K be an algebraic group and Y be a k-scheme equipped with an action of H. Consider the maps $\mathsf{act}, \mathsf{pr} : K \times Y \rightrightarrows Y$. A **$K$-equivariant quasi-coherent $\mathcal{O}_Y$-module** is an object $\mathcal{F} \in \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{qc}}$ euipped with isomorphisms $\mathsf{act}^*\mathcal{F} \to \mathsf{pr}^*\mathcal{F}$ that satisfies the cocycle condition over $K \times K \times Y$.

Let $\mathcal{O}_Y\text{–}\mathsf{mod}^K_{\mathsf{qc}}$ be the category of K-equivariant quasi-coherent $\mathcal{O}_Y$-modules.

Remark 2.2. The category $\mathcal{O}_Y\text{–}\mathsf{mod}^K_{\mathsf{qc}}$ is an abelian category and the forgetful functor $\mathcal{O}_Y\text{–}\mathsf{mod}^K_{\mathsf{qc}} \to \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{qc}}$ is exact. However, this functor is *not* fully faithful. *Being equivariant is a structure rather than a property.*

[4]Any action of a smooth connected solvable algebraic group B on a finite type separated k-scheme X has a fixed k-point.

Example 2.3. The structure sheaf $\mathcal{O}_Y$ has an obvious K-equivariant structure.

Construction 2.4. *Suppose the quotient Y/K exsits*[5]*. Let $\pi : Y \to Y/K$ be the projection map. Note that $\pi \circ \mathsf{act} = \pi \circ \mathsf{pr}$. Hence we have a functor*

$$\mathcal{O}_{Y/K}\text{-}\mathsf{mod}_{\mathsf{qc}} \to \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}^K,\ \mathcal{M} \mapsto \pi^*\mathcal{M} \tag{2.1}$$

that sends $\mathcal{M}$ to $\pi^*\mathcal{M}$ equipped with the obvious equivariant structure.

By the flat descent of quasi-coherent sheaves, we have:

Proposition 2.5. *The functor* (2.1) *is an equivalence.*

At least when K is affine, the inverse of (2.1) can be constructed as follows.

Construction 2.6. *Let $\mathcal{F} \in \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}^K$. The K-equivariant structure induces a morphism $\mathcal{F} \to \mathsf{act}_* \circ \mathsf{pr}^*\mathcal{F}$. Taking global sections, we obtain a map*

$$\mathcal{F}(Y) \to \mathsf{pr}^*\mathcal{F}(K \times Y) \simeq \mathcal{O}(K) \otimes \mathcal{F}(Y).$$

One can show the cocycle condition for the K-equivariant structure on $\mathcal{F}$ is translated to the associative law for a right *K-action on $\mathcal{F}(Y)$. By taking inverse, we obtain a functor*

$$\Gamma(Y, -) : \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}^K \to \mathsf{Rep}(K).$$

One can check the multplication map $\mathcal{O}(Y)\otimes\mathcal{O}(Y) \to \mathcal{O}(Y)$ and the action map $\mathcal{O}(Y)\otimes\mathcal{F}(Y) \to \mathcal{F}(Y)$ are K-linear.

Remark 2.7. Let $g \in K$ be a closed point and consider the map $g : Y \to Y$ given by its action. The K-equivariant structure on $\mathcal{F}$ provides an isomorphism $g^*\mathcal{F} \to \mathcal{F}$ and therefore $\mathcal{F} \to g_*\mathcal{F}$. Taking global sections, we obtain an automorphism $\Gamma(Y, \mathcal{F}) \to \Gamma(Y, \mathcal{F})$, which gives the right action of the point $g \in K$ on $\Gamma(Y, \mathcal{F})$. When $\mathcal{F} = \mathcal{O}$ is the structure sheaf, this is the usual formula $(\phi \cdot g)(y) = \phi(gy)$.

Construction 2.8. *The following construction is an infinitesimal variant of Construction 2.6. Consider $\mathfrak{k} := \mathsf{Lie}(K)$. Then for $\mathcal{F} \in \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}^K$, the underlying sheaf $\mathcal{F}$ has a natural action by $\mathfrak{k}$ such that the induced $\mathfrak{k}$-module structure on $\mathcal{F}(Y)$ coincides with that induced by the K-module structure in Construction 2.6.*

Namely, let $K^{(1)} := \mathsf{Spec}(\mathcal{O}_{K,e}/\mathfrak{m}_{K,e}^2)$ be the first neightborhood of the unit element e insider K. For any open subscheme, the maps $\mathsf{act}, \mathsf{pr}$ induce maps $\mathsf{act}_U^{(1)}, \mathsf{pr}_U^{(1)} : K^{(1)} \times U \to U$. The K-equivariant structure provides a morphism $\mathcal{F}|_U \to (\mathsf{act}_U^{(1)})_* \circ (\mathsf{pr}_U^{(1)})^*(\mathcal{F}|_U)$. Taking global sections, we obtain a map

$$\mathcal{F}(U) \to (\mathsf{pr}_U^{(1)})^*(\mathcal{F}|_U)(K^{(1)} \times U) \simeq \mathcal{O}_{K,e}/\mathfrak{m}_{K,e}^2 \otimes \mathcal{F}(U).$$

Note that the short exact sequence

$$0 \to \mathfrak{m}_{K,e}/\mathfrak{m}_{K,e}^2 \to \mathcal{O}_{K,e}/\mathfrak{m}_{K,e}^2 \to k \to 0$$

has a canonical spliting. Hence we obtain a map $\mathcal{F}(U) \to \mathfrak{m}_{K,e}/\mathfrak{m}_{K,e}^2 \otimes \mathcal{F}(U)$. By definition, we have $\mathfrak{k}^* \simeq \mathfrak{m}_{K,e}/\mathfrak{m}_{K,e}^2$. Hence we obtain a map

$$\mathfrak{k} \otimes \mathcal{F}(U) \to \mathcal{F}(U).$$

One can show[6] *this defines a $\mathfrak{k}$-module structure on $\mathcal{F}(U)$ for any U and therefore on $\mathcal{F}$.*

[5]More precisely, we mean the fppf sheafification of the functor $R \mapsto Y(R)/K(R)$ is represented by a k-scheme.

[6]Note: I do not have time to check if there should be a negative sign. I will return to this problem after the class.

Construction 2.9. *Now for any open subset $U \subset Y/K$, its inverse image $\pi^{-1}(U) \subset Y$ is preserved by the K-action. Hence we obtain a right K-representation structure on $\mathcal{F}(\pi^{-1}(U))$ compatible with the $\mathcal{O}(\pi^{-1}(U))$-module structure. One can show $\mathcal{O}(U) \simeq \mathcal{O}(\pi^{-1}(U))^K$ and $U \mapsto \mathcal{F}(\pi^{-1}(U))^K$ defines a quasi-coherent $\mathcal{O}_{Y/K}$-module, which we denote by $\mathcal{F}^K$. We have a functor*

$$\mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}^K \to \mathcal{O}_{Y/K}\text{-}\mathsf{mod}_{\mathsf{qc}},\ \mathcal{F} \mapsto \mathcal{F}^K$$

inverse to the functor (2.1).

Construction 2.10. *Let $V \in \mathsf{Rep}(K)$ be a K-representation. The quasi-coherent $\mathcal{O}_Y$-module $\mathcal{O}_Y \otimes V$ has a natural K-equivariant structure such that the K-representation structure on*

$$(\mathcal{O}_Y \otimes V)(\pi^{-1}(U)) \simeq \mathcal{O}_Y(\pi^{-1}(U)) \otimes V$$

is given by the (anti)diagonal action. Hence we obtain a funnctor

$$\mathsf{Rep}(K) \to \mathcal{O}_{Y/K}\text{-}\mathsf{mod}_{\mathsf{qc}},\ V \mapsto (\mathcal{O}_Y \otimes V)^K.$$

Remark 2.11. In fact, we have $\mathsf{Rep}(K) \simeq \mathsf{QCoh}(\mathsf{pt}/K)$ where pt/K is the classifying stack of K. Via this equivalence, the above functor corresponds to the pullback functor along the map $Y/K \to \mathsf{pt}/K$.

Construction 2.12. *For any character λ of T and the corresponding 1-dimensional B-representation $k_{-\lambda}$, we obtain a line bundle*[7]

$$\mathcal{L}^\lambda := (\mathcal{O}_G \otimes k_{-\lambda})^B \in \mathcal{O}_{G/B}\text{-}\mathsf{mod}_{\mathsf{qc}}$$

on the flag variety G/B. It is easy to see

$$\mathbb{X}(T) \simeq \mathbb{X}(B) \to \mathsf{Pic}(G/B),\ \lambda \mapsto \mathcal{L}^\lambda \tag{2.2}$$

is a homomorphism, which is called the ***characteristic map*** *for G.*

Remark 2.13. In fact, for any connected algebraic group G and its Borel subgroup B, we have an exact sequence

$$0 \to \mathbb{X}(G) \to \mathbb{X}(B) \to \mathsf{Pic}(G/B) \to \mathsf{Pic}(G) \to 0.$$

When G is semisimple, $\mathbb{X}(G) = 0$. When G is further simply connected, $\mathsf{Pic}(G) = 0$. See [M, Section 18].

Example 2.14. For $G = \mathsf{SL}_2$ equipped with its standard Borel and Cartan subgroups. Any character of B is of the form $\left(\begin{smallmatrix} t & * \\ 0 & t^{-1} \end{smallmatrix}\right) \mapsto t^n$ for some $n \in \mathbb{Z}$. Unwinding the definitions, the corresponding line bundle on $G/B \simeq \mathbb{P}^1$ is $\mathcal{O}(n)$. Indeed, its global section is the space of functions ϕ on SL_2 such that $\phi(g\left(\begin{smallmatrix} t & * \\ 0 & t^{-1} \end{smallmatrix}\right)) = t^n \phi(g)$.

Warning 2.15. *Other authors might choose different conventions and obtain the line bundle $\mathcal{O}(-n)$.*

Example 2.16. We have $\mathcal{L}^{-2\rho} \simeq \omega_{G/B}$.

Construction 2.17. *Note that $\mathcal{L}^\lambda$ is naturally equivariant with respect to the left multplication action of G on G/B. By Construction 2.6, we obtain a G-module structure on $\mathsf{H}^0(G/B, \mathcal{L}^\lambda)$. This representation is finite-dimensional because G/B is proper.*

Recall the following well-known result:

Theorem 2.18 (Borel-Weil-Bott). *When $\mathsf{char}(k) = 0$ and G is semisimple, if λ is dominant and integral, then $\mathsf{H}^0(G/B, \mathcal{L}^\lambda) \simeq L_\lambda$ and $\mathsf{H}^i(G/B, \mathcal{L}^\lambda) = 0$ for $i > 0$.*

[7] Note the negative sign!

Remark 2.19. The statement $\mathsf{H}^i(G/B,\mathcal{L}^\lambda)=0$ for $i>0$ remains true in positive characteristic, known as *Kempf's vanishing theorem.* See [J, Chapter 4] for a proof. However, $\mathsf{H}^0(G/B,\mathcal{L}^\lambda)$ is not always irreducible.

We also record the following result. For a proof, see [J, Chapter 4].

Theorem 2.20. *Let G be semisimple. The following conditions are equivalent:*

(i) *When viewed as an integral weight of $\mathfrak{t}$, λ is regular and dominant, i.e., $\langle\lambda,\check{\alpha}\rangle\in\mathbb{Z}^{>0}$ for any $\alpha\in\Phi^+$;*
(ii) *The line bundle $\mathcal{L}^\lambda$ is very ample;*
(iii) *The line bundle $\mathcal{L}^\lambda$ is ample.*

Remark 2.21. In fact, the characteristic map $\mathbb{X}(T)\to\mathsf{Pic}(\mathsf{Fl}_G)$ does not depend on the choice of B, as long as we replace T by the *abstract Cartan group* T_{abs} for G. More precisely, for any choice of B, there is a corresponding realization isomorphism $T_{\mathsf{abs}}\to T$, then the composition $\mathbb{X}(T_{\mathsf{abs}})\simeq\mathbb{X}(T)\to\mathsf{Pic}(\mathsf{Fl}_G)$ does not depend on B.

3. Bruhat decomposition

Recall G is assumed to be reductive. References for this section include [M, Section 21.h] and [J, Chapter 13].

Theorem 3.1 (Bruhat decomposition for $G(k)$)**.** *We have*

$$G(k)=\bigsqcup_{w\in W}B(k)wB(k),$$

where in the RHS we lift $w\in W$ to an element in $N_G(T)(k)$.

Example 3.2. For $G=\mathsf{GL}_n$, this decomposition follows from Gaussian elimination.

We also have the version for schemes.

Theorem 3.3 (Bruhat decomposition for G)**.** *For any $w\in W$, there is a unique smooth locally closed subscheme of G, denoted by BwB, such that $BwB(k)=B(k)wB(k)$. We have a disjoint union decomposition of underlying topological spaces:*

$$G=\bigsqcup_{w\in W}BwB.$$

Each subscheme BwB is stablized by the (B,B)-action on G and is equal to the (B,B)-orbit that contains w.

Remark 3.4. Using $B=NT$, we have $B(k)wB(k)=N(k)wB(k)=B(k)wN(k)$ and similarly $BwB=NwB=BwN$.

Taking quotient for the right B-action, we also have:

Theorem-Definition 3.5 (Bruhat decomposition for Fl_G)**.** *For any $w\in W$, there is a unique smooth locally closed subscheme of Fl_G, denoted by $\mathsf{Fl}_G^{=w}$, such that $\mathsf{Fl}_G^{=w}=BwB/B$. We have a disjoint union decomposition of underlying topological spaces:*

$$\mathsf{Fl}_G=\bigsqcup_{w\in W}\mathsf{Fl}_G^{=w}.$$

*Each subscheme $\mathsf{Fl}_G^{=w}$ is stablized by the B-action on Fl_G and is equal to the B-orbit/N-orbit that contains wB/B. The subschemes $\mathsf{Fl}_G^{=w}$ are called the **Bruhat cells** of the flag variety Fl_G.*

Example 3.6. For $G=\mathsf{SL}_2$ and the isomorphism $\mathsf{Fl}_G\simeq\mathbb{P}^1$ in Example 1.3. We hvae $\mathsf{Fl}_G^{=1}\simeq\{\infty\}$ and $\mathsf{Fl}_G^{=s}\simeq\mathbb{A}^1$.

The following lemma is easy on the level of k-points. A careful argument shows it is also true on the level of schemes.

Lemma 3.7. *We have* $\mathsf{Fl}_G^{=w} \simeq N/(\mathsf{Ad}_w(N) \cap N) \simeq \mathsf{Ad}_w(N) \cap N^-$.

Remark 3.8. When $\mathsf{char}(k) = 0$, for any unipotent algebraic group U, there is a well-defined isomorphism between *k-schemes* $\exp : \mathsf{Lie}(U) \to U$, where $\mathsf{Lie}(U)$ means the affine space scheme corresponding to the same-named vector space (see [M, Section 14.d]). It follows that $\mathsf{Fl}_G^{=w}$ is isomorphic to an affine space. This explains the word "cell".

Exercise 3.9. This is Homework 6, Problem 1. Prove: $\dim(\mathsf{Fl}_G^{=w}) \simeq \ell(w)$[8].

Definition 3.10. The (reduced) closures $\overline{\mathsf{Fl}_G^{=w}}$ of $\mathsf{Fl}_G^{=w}$ inside Fl_G are called the **Schubert varieties** for G.

Remark 3.11. Fortunately, the Schubert varieties are in general singular.

Theorem 3.12. *We have a disjoint union decomposition of underlying topological spaces*

$$\overline{\mathsf{Fl}_G^{=w}} = \bigsqcup_{w' \le w} \mathsf{Fl}_G^{=w'}.$$

Notation 3.13. *Because of the above theorem, we also write* $\mathsf{Fl}_G^{\le w} := \overline{\mathsf{Fl}_G^{=w}}$.

4. Statement of localization theorem

In this section, we state the main theorems of this course. Next time, we prove them.

Consider the G-action on the flag variety Fl_G. As in [Lecture 10, Construction 8.1], we have a homomorphism

$$a : U(\mathfrak{g}) \to \mathcal{D}(\mathsf{Fl}_G) \tag{4.1}$$

induced from the Lie algebra homomorphism $\mathfrak{g} \to \mathcal{T}(\mathsf{Fl}_G)$.

Construction 4.1. *We have adjoint functors*

$$\mathsf{Loc} : U(\mathfrak{g})\text{–}\mathsf{mod} \rightleftarrows \mathcal{D}_{\mathsf{Fl}_G}\text{–}\mathsf{mod}^l_{\mathsf{qc}} : \Gamma \tag{4.2}$$

constructed as follows:

- $\mathsf{Loc}(M) := \mathcal{D}_{\mathsf{Fl}_G} \otimes_{\underline{U(\mathfrak{g})}} \underline{M}$, *where* $\underline{M}$ *denotes the sheaf on* Fl_G *with constant values* M.
- $\Gamma(\mathcal{F})$ *is the space of global section of* $\mathcal{F}$, *equipped with the* $U(\mathfrak{g})$*-module structure obtained by restricting along* (4.1).

Remark 4.2. The above adjoint pair can not be equivalences because the homomorphism (4.1) is not an isormophism. Below is an imprecise but motivating explanation. The ring $U(\mathfrak{g})$ has the same size as $\mathsf{Sym}(\mathfrak{g})$, which is a commutative algebra of dimension $\dim(\mathfrak{g})$. On the other hand, the ring $\mathcal{D}(\mathsf{Fl}_G)$ has the same size as $\mathsf{Sym}_{\mathcal{O}(\mathsf{Fl}_G)}(\mathcal{T}(\mathsf{Fl}_G))$ (lying!), which is a commutative algebra of dimension $2\dim \mathsf{Fl}_G = \dim(\mathfrak{g}) - \dim(\mathfrak{t})$.

However, the differences are eliminated once we kill the kernel of the character of $Z(\mathfrak{g})$. Recall the latter is a commutative algebra of dimension $\dim(\mathfrak{t})$.

Construction 4.3. *Let* χ_0 *be the central character of the trivial* $\mathfrak{g}$*-module and write* $U(\mathfrak{g})_{\chi_0} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} k_{\chi_0}$. *In other words,* $U(\mathfrak{g})_{\chi_0}$ *is the quotient of* $U(\mathfrak{g})$ *by its double-sided ideal generated by* $\mathsf{ker}(\chi_0)$. *Note that*

$$U(\mathfrak{g})_{\chi_0}\text{–}\mathsf{mod} \subset U(\mathfrak{g})\text{–}\mathsf{mod}$$

[8] If you do not know the basics about reductive groups, prove this for semisimple G.

is the full subcategory of $U(\mathfrak{g})$-modules such that $Z(\mathfrak{g})$ acts via the character χ_0. Recall for any $w\in W$ we have

$$M_{w\bullet(-2\rho)},\, M^\vee_{w\bullet(-2\rho)},\, L_{w\bullet(-2\rho)} \in U(\mathfrak{g})_{\chi_0}\text{-}\mathsf{mod}.$$

Construction 4.4. *For any $w\in W$, let $i_w:\mathsf{Fl}_G^{=w}\to\mathsf{Fl}_G$ be the locally closed embedding. Consider the left $\mathcal{D}_{\mathsf{Fl}_G}$-modules:*

$$\begin{aligned}\Delta_w &:= i_{w,!}(\mathcal{O}_{\mathsf{Fl}_G^{=w}}),\\ \nabla_w &:= i_{w,*,\mathsf{dR}}(\mathcal{O}_{\mathsf{Fl}_G^{=w}}),\\ \mathsf{IC}_w &:= \mathsf{Im}(\Delta_w\to\nabla_w).\end{aligned}$$

Remark 4.5. Let us clarify the definitions of these objects in this remark.

We first translate $\mathcal{O}_{\mathsf{Fl}_G^{=w}}$ to the right $\mathcal{D}$-module

$$\omega_{\mathsf{Fl}_G^{=w}}\in\mathcal{D}_{\mathsf{Fl}_G^{=w}}\text{-}\mathsf{mod}^r_{\mathsf{qc}},$$

then apply the functors $i_{w,!}$ and $i_{w,*,\mathsf{dR}}$ defined in Lecture 11 to obtain complices of right $\mathcal{D}$-modules on Fl_G. However, by Kashiwara's lemma, $i_{w,*,\mathsf{dR}}(\omega_{\mathsf{Fl}_G^{=w}})$ is a genuine right $\mathcal{D}$-module, i.e., is contained in the abelian category. Moreover, it is *holonomic* by [Lecture 11, Fact 10.5]. Hence by [Lecture 11, Fact 10.1 and Example 9.3],

$$i_{w,!}(\omega_{\mathsf{Fl}_G^{=w}})\simeq\mathbb{D}\circ i_{w,*,\mathsf{dR}}\circ\mathbb{D}(\omega_{\mathsf{Fl}_G^{=w}})\simeq\mathbb{D}(i_{w,*,\mathsf{dR}}(\omega_{\mathsf{Fl}_G^{=w}}))$$

is also a holonomic right $\mathcal{D}$-module. Now Δ_w and ∇_w are defined to be the holonomic left $\mathcal{D}$-modules corresponding to them.

As for IC_w, by the base-change isomorphism ([Lecture 11, Fact 8.1]), we have an equivalence $\mathsf{Id}\simeq i_w^!\circ i_{w,*,\mathsf{dR}}$. Via adjunction, it induces a natural transformation $i_{w,!}\to i_{w,*,\mathsf{dR}}$, which then induces a morphism $\Delta_w\to\nabla_w$.

Remark 4.6. The symbol IC stands for *intersection cohomology* because IC_w corresponds to the intersection cohomology sheaf on the Schubert variety $\mathsf{Fl}_G^{\le w}$ via the Riemann–Hilbert correspondence.

Theorem 4.7 (Localization theorem)**.** *We have*

(1) *The homomorphism* (4.1) *induces an isomorphism*

$$U(\mathfrak{g})_{\chi_0}\simeq\mathcal{D}(\mathsf{Fl}_G).$$

(2) *The isomorphism in (1) induces functors inverse to each other:*

$$\mathsf{Loc}:U(\mathfrak{g})_{\chi_0}\text{-}\mathsf{mod}\rightleftarrows\mathcal{D}_{\mathsf{Fl}_G}\text{-}\mathsf{mod}^l_{\mathsf{qc}}:\Gamma.$$

(3) *Via the equivalences in (2), we have*

$$M_{w\bullet(-2\rho)}\longleftrightarrow\Delta_w,\; M^\vee_{w\bullet(-2\rho)}\longleftrightarrow\nabla_w,\; L_{w\bullet(-2\rho)}\longleftrightarrow\mathsf{IC}_w.$$

Remark 4.8. The adjoint functors in (2) are compatible with (4.2) in the obvious way.

Exercise 4.9. This is Homework 6, Problem 2. Deduce the BGG theorem from the localization theorem[9].

Exercise 4.10. This is Homework 6, Problem 3. Prove the isomorphism in (1) for the special case $G=\mathsf{SL}_2$.

[9]HintL prove any subquotient of Δ_w is set-theoretically support on the Schubert variety $\mathsf{Fl}_G^{\le w}$.

5. Strongly equivariant $\mathcal{D}$-modules

Note that the block $\mathcal{O}_{\chi_0}$ is *not* contained in $U(\mathfrak{g})_{\chi_0}\text{-mod}$ because each module in $\mathcal{O}_{\chi_0}$ is annihilated by $\ker(\chi_0)^N$, $N >> 0$ rather than just by $\ker(\chi_0)$. In this section, we give a description of the category

$$\mathcal{O}'_{\chi_0} := \mathcal{O}_{\chi_0} \cap U(\mathfrak{g})_{\chi_0}\text{-mod}$$

via localization theory.

Definition 5.1. Let Y be a smooth k-scheme acted by an algebraic group K. We say a left $\mathcal{D}$-module $\mathcal{F} \in \mathcal{O}_Y\text{-mod}^l_{\mathsf{qc}}$ is equipped with a **weakly K-equivariant** structure if

(i) It is equipped with a K-equivariant structure as a quasi-coherent $\mathcal{O}_Y$-module;
(ii) The action morphism $\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F}$ is compatible with the K-equivariant structures.

We say $\mathcal{F} \in \mathcal{O}_Y\text{-mod}^l_{\mathsf{qc}}$ is equipped with a **strongly K-equivariant** structure if we further have:

(iii) The $\mathfrak{k}$-module structure on $\mathcal{F}$ obtained in Construction 2.8 is the same as the $\mathfrak{k}$-module structure obtained by restricting along the map $\underline{U(\mathfrak{k})} \to \mathcal{D}_Y$.

We also translate the above definitions to right $\mathcal{D}$-modules.

Example 5.2. The structure sheaf $\mathcal{O}_Y$, viewed as a left $\mathcal{D}$-module, has an obvious weakly K-equivariant structure, and this structure is strong.

Remark 5.3. Being weakly equivariant is a structure rather than a property. However, one can show being strongly equivariant is actually a property when K is connected.

One the other hand, in the derived category of $\mathcal{D}$-modules, being strongly equivariant is also a structure rather than a property. Note that strongly equivariant complices of $\mathcal{D}$-modules are not the same as complices of strongly equivariant $\mathcal{D}$-modules.

I do not recommend to learn the traditional proof of the following result. Modern techniques about $\mathcal{D}$-modules would provide a more illuminating proof.

Proposition 5.4. *If the quotient Y/K exists (which is automatically smooth), then the functor π^* induces an equivalence*

$$\pi^* : \mathcal{D}_{Y/K}\text{-mod}_{\mathsf{qc}} \to \mathcal{D}_Y\text{-mod}_{\mathsf{qc}}^{K\text{-strong}}.$$

Example 5.5. The "six functors" introduced last time can be updated to functors between strongly K-equivariant $\mathcal{D}$-modules, at least after taking cohomologies.

Theorem 5.6 (Localization thoerem, continued)**.** *We have*

(4) *The equivalences in Theorem 4.7 restricts to equivalences*

$$\mathsf{Loc} : \mathcal{O}'_{\chi_0} \rightleftarrows \mathcal{D}_{\mathsf{Fl}_G}\text{-mod}_{\mathsf{c}}^{l,B\text{-strong}} : \Gamma.$$

Remark 5.7. Note that on the geometric side, we use *coherent* $\mathcal{D}$-modules. This is because objects in $\mathcal{O}'_{\chi_0}$ are finitely generated. Equivalently, we can use $\mathcal{D}_{\mathsf{Fl}_G}\text{-mod}_{\mathsf{hol}}^{l,B\text{-strong}}$ becuse any strongly B-equivariant coherent $\mathcal{D}$-module on Fl_G is holonomic. This is essentially because there are finitely many B-orbits on Fl_G.

Warning 5.8. *Although the B-orbits and N-orbits on Fl_G are the same, we must use strongly equivariant condition for B rather than for N. Otherwise the global section would not be weight modules. Namely, being strongly K-equivariant implies the $\mathfrak{g}$-module structure on the global section is K-integrable, and any object in $\mathcal{O}$ is required to be B-integrable (rather than just N-integrable).*

Remark 5.9. If we want an equivalence about the actual block $\mathcal{O}_{\chi_0}$, we need:

- Replace Fl_G by the *basic affine space*[10] G/U;
- Impose weakly equivariant structure with respect to the *right* T-action on G/U;
- Restrict to the full subcategory generated by subquotients and extensions of strongly T-equivariant objects. These objects are called $(T,0)$-monodromic objects.

This is beyond the scope of our course.

6. Twisted $\mathcal{D}$-modules

In this section, we sketch the story for other blocks $\mathcal{O}_{\varpi(\lambda)}$.

For any weight λ, there is a variant $\mathcal{D}^\lambda_{\mathsf{Fl}_G}$ of $\mathcal{D}_{\mathsf{Fl}_G}$, called the sheaf of **$\lambda$-twisted differential operators**. Recall $\mathcal{D}_{\mathsf{Fl}_G}$ is the universal enveloping algebra of the split Picard algebroid $\widetilde{\mathcal{T}}_{\mathsf{Fl}_G} := \mathcal{O}_{\mathsf{Fl}_G} \oplus \mathcal{T}_{\mathsf{Fl}_G}$ (see [Lecture 10, Remark 5.4]). For any λ, one can construct a Picard algebroid $\widetilde{\mathcal{T}}^\lambda_{\mathsf{Fl}_G}$, and $\mathcal{D}^\lambda_{\mathsf{Fl}_G}$ is defined to be its universal enveloping algebra[11]. When λ is integral, $\mathcal{D}^\lambda_{\mathsf{Fl}_G}$ is just the sheaf of differential operators on the line bundle $\mathcal{L}^\lambda$. For more details, see [BB] and [G, Section 9]. Now:

- For any λ and $\chi := \varpi(\lambda)$, we have $U(\mathfrak{g})_\chi \simeq \mathcal{D}^\lambda(\mathsf{Fl}_G)$.
- When λ is dot-dominant, the functor Γ is exact.
- When λ is dominant, the functor Γ is an equivalence. Also, Verma, dual Verma and irreducible modules correspond respectively to $!/*/\mathsf{IC}$ objects.

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[10]This is a bad name: it is not affine!

[11]In particular, we have a PBW filtration such that $\mathsf{gr}^\bullet \mathcal{D}^\lambda_{\mathsf{Fl}_G} \simeq \mathsf{Sym}^\bullet_{\mathcal{O}_{\mathsf{Fl}_G}} \mathcal{T}_{\mathsf{Fl}_G}$. Moreover, $\mathsf{F}^{\leq 1}\mathcal{D}^\lambda_{\mathsf{Fl}_G} \simeq \widetilde{\mathcal{T}}^\lambda_{\mathsf{Fl}_G}$