LECTURE 2

1. Universal enveloping algebra

Construction 1.1. *Recall we have a forgetful functor* $\mathsf{oblv} : \mathsf{Alg}_k \to \mathsf{Lie}_k$ *from the category of associative algebras to that of Lie algebras. This functor admits a left adjoint*

$$U : \mathsf{Lie}_k \to \mathsf{Alg}_k$$

that sends a Lie algebra $\mathfrak{g}$ *to the associative algebra*

$$U(\mathfrak{g}) = T(\mathfrak{g})/\langle xy - yx - [x,y],\ x,y \in \mathfrak{g}\rangle.$$

Here

$$T(\mathfrak{g}) \coloneqq \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$$

is the tensor algebra of the underlying vector space of $\mathfrak{g}$*, and* $\langle xy - yx - [x,y],\ x,y \in \mathfrak{g}\rangle$ *is the two-sided ideal generated by elements of the form* $xy - yx - [x,y]$.

The associative algebra $U(\mathfrak{g})$ *is called the* ***universal enveloping algebra*** *of* $\mathfrak{g}$.

Let $U(\mathfrak{g})\text{–}\mathsf{mod}$ be the abelian category of left modules for $U(\mathfrak{g})$.

Lemma 1.2. *There is an equivalence*

$$\mathfrak{g}\text{–}\mathsf{mod} \simeq U(\mathfrak{g})\text{–}\mathsf{mod}$$

that commutes with forgetful functors to Vect_k.

Proof. For a given vector space V, a $\mathfrak{g}$-module structure on V is a Lie algebra homomorphism $\mathfrak{g} \to \mathsf{oblv}(\mathfrak{gl}(V))$. By adjunction, this is the same as a homomorphism $U(\mathfrak{g}) \to \mathfrak{gl}(V)$, i.e., a left $U(\mathfrak{g})$-module structure on V.

□

Construction 1.3. *The tensor algebra* $T(\mathfrak{g})$ *is naturally* graded. *But this grading does* not *descent to* $U(\mathfrak{g})$ *because* $xy - yx - [x,y]$ *is not a homogenous element. Instead,* $U(\mathfrak{g})$ *has an exhausted filtration*

$$\mathsf{F}^{\leq n} U(\mathfrak{g}) \coloneqq \mathsf{im}(\mathsf{F}^{\leq n} T(\mathfrak{g}) \to U(\mathfrak{g}))$$

that is compatible with the algebra structure, i.e.,

$$\mathsf{F}^{\leq m} U(\mathfrak{g}) \underset{k}{\otimes} \mathsf{F}^{\leq n} U(\mathfrak{g}) \xrightarrow{\mathsf{mult}} \mathsf{F}^{\leq m+n} U(\mathfrak{g}).$$

Taking associated graded pieces, we obtain a graded algebra

$$\mathsf{gr}^\bullet U(\mathfrak{g}) \coloneqq \bigoplus_{n\geq 0} \mathsf{F}^{\leq n} U(\mathfrak{g}) / \mathsf{F}^{< n} U(\mathfrak{g}).$$

By the universal property of the tensor algebra, we have a unique homomorphism $T(\mathfrak{g}) \to \mathsf{gr}^\bullet U(\mathfrak{g})$ *whose restriction on* $\mathfrak{g} \subset T(\mathfrak{g})$ *is the composition* $\mathfrak{g} \to \mathsf{F}^{\leq 1} U(\mathfrak{g}) \to \mathsf{gr}^1 U(\mathfrak{g}) \subset \mathsf{gr}^\bullet U(\mathfrak{g})$. *Denote this composition by* $x \mapsto \bar{x}$. *Note that we have* $\bar{x}\bar{y} = \bar{y}\bar{x}$ *as elements in* $\mathsf{gr}^2 U(\mathfrak{g})$ *because the*

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term $[x,y]$ is killed by the surjection $\mathsf{F}^{\leq 2}U(\mathfrak{g}) \to \mathsf{gr}^2 U(\mathfrak{g})$. It follows that we have a commutative diagram of surjective maps:

$$\begin{array}{ccc} T(\mathfrak{g}) & \xrightarrow{x \mapsto \bar{x}} & \mathsf{gr}^\bullet U(\mathfrak{g}) \\ \downarrow & \nearrow_{\phi} & \\ \mathsf{Sym}(\mathfrak{g}), & & \end{array}$$

where $\mathsf{Sym}(\mathfrak{g}) := T(\mathfrak{g})/\langle xy - yx \rangle$ is the symmetric algebra *of $\mathfrak{g}$. In particular, $\mathsf{gr}^\bullet U(\mathfrak{g})$ is a commutative algebra.*

Remark 1.4. Note that $\mathsf{gr}^\bullet U(\mathfrak{g})$ being commutative is equivalent to $[\mathsf{F}^i U(\mathfrak{g}), \mathsf{F}^j U(\mathfrak{g})] \subset \mathsf{F}^{i+j-1} U(\mathfrak{g})$, where we write $\mathsf{F}^{-n} U(\mathfrak{g}) = 0$ for $n > 0$.

Theorem 1.5 (Poincaré–Birkhoff–Witt, a.k.a. PBW)**.** *For any Lie algebra $\mathfrak{g}$, the above homomorphism $\phi : \mathsf{Sym}(\mathfrak{g}) \to \mathsf{gr}^\bullet U(\mathfrak{g})$ is an isomorphism.*

Corollary 1.6. *Let $\{x_i\}_{i \in I}$ be a basis of $\mathfrak{g}$ as a vector space. Choose a total order on the set I. Then the set $\{x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_n}^{m_n} \mid n \geq 0, i_1 < i_2 < \cdots < i_n, m_1, m_2, \ldots, m_n \in \mathbb{Z}^{>0}\}$ is a basis of the vector space $U(\mathfrak{g})$.*

Corollary 1.7. *If $\mathfrak{g}$ is a finite-dimensional algebra, then $U(\mathfrak{g})$ is left and right Noetherian.*

Proof. A filtered ring A is left (resp. right) Noetherian if its assoicated graded ring $\mathsf{gr}^\bullet A$ is so. See [MR, Chapter 1, Theorem 6.9][1]. □

2. Verma modules

From now on, we fix a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ and choose $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, i.e., a Cartain subalgebra and a Borel subalgebra of it. Recall we have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}$, $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$, $\mathfrak{t} \simeq \mathfrak{b}/\mathfrak{n}$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$[2].

Construction 2.1. *The projection $\mathfrak{b} \to \mathfrak{t}$ induces a restriction functor $\mathfrak{t}\text{–}\mathsf{mod} \to \mathfrak{b}\text{–}\mathsf{mod}$. Note that we have*

$$\mathfrak{t}\text{–}\mathsf{mod} \simeq U(\mathfrak{t})\text{–}\mathsf{mod} \simeq \mathsf{Sym}(\mathfrak{t})\text{–}\mathsf{mod} \simeq \mathsf{QCoh}(\mathfrak{t}^*). \tag{2.1}$$

Hence for any $\lambda \in \mathfrak{t}^*$, the skyscrapter sheaf at λ gives a 1-dimensional representation

$$k_\lambda \in \mathfrak{t}\text{–}\mathsf{mod}.$$

In other words, for $x \in \mathfrak{t}$, its action on k_λ is given by the scaler $\lambda(x)$.

We abuse notation and write k_λ for the corresponding object in $\mathfrak{b}\text{–}\mathsf{mod}$.

Remark 2.2. Note that any 1-dimensional $\mathfrak{b}$-module V is of the form k_λ. Indeed, the Lie homomorphism $\mathfrak{b} \to \mathfrak{gl}(V)$ must kill $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ because $\mathfrak{gl}(V)$ is abelian.

Definition 2.3. Consider the restriction functor $\mathfrak{g}\text{–}\mathsf{mod} \to \mathfrak{b}\text{–}\mathsf{mod}$ and its left adjoint

$$\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}} : \mathfrak{b}\text{–}\mathsf{mod} \to \mathfrak{g}\text{–}\mathsf{mod}.$$

For any weight $\lambda \in \mathfrak{t}^*$, we define the **Verma module** to be

$$M_\lambda := \mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(k_\lambda) \in \mathfrak{g}\text{–}\mathsf{mod}.$$

[1] Sketch: a left ideal $I \subset A$ defines a left ideal $\mathsf{gr}^\bullet I \subset \mathsf{gr}^\bullet A$ with $\mathsf{gr}^n I = ((I + \mathsf{F}^{n-1}A) \cap \mathsf{F}^n A)/\mathsf{F}^{n-1}A$. This assignment is injective.

[2] We didn't mention the last one in the last lecture, but it follows easily from the root decomposition.

Remark 2.4. Explicitly, we have

$$M_\lambda \simeq U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} k_\lambda.$$

In particular, M_λ is infinite-dimensional.

Definition 2.5. By adjunction, there is a $\mathfrak{b}$-linear map $k_\lambda \to M_\lambda$ corresponding to the identity morphism $M_\lambda \to M_\lambda$ in $\mathfrak{g}-\mathsf{mod}$. After fixing a nonzero vector 1_λ of k_λ, we obtain a vector $v_\lambda \in M_\lambda$. We call it a **highest weight vector** of M_λ.

The meaning of this name will be explained shortly. Note that by definition, $\mathfrak{n} \cdot v_\lambda = 0$ and v_λ is a λ-eigenvector for the $\mathfrak{t}$-action.

Exercise 2.6. This is Homework 1, Problem 1. Prove:

(1) The map

$$U(\mathfrak{n}^-) \underset{k}{\otimes} U(\mathfrak{b}) \xrightarrow{\mathsf{mult}} U(\mathfrak{g})$$

is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodules.

(2) As an $\mathfrak{n}^-$-module, M_λ is freely generated by v_λ, i.e.,

$$U(\mathfrak{n}^-) \to M_\lambda,\ x \mapsto x \cdot v_\lambda$$

is an isomorphism.

As a contrary, we have:

Lemma 2.7. *The $\mathfrak{n}$-action on M_λ is locally finite.*

Proof. By the above exercise, we have $M_\lambda = \bigcup_i \mathsf{F}^i U(\mathfrak{g}) \cdot v_\lambda$, where $F^\bullet U(\mathfrak{g})$ is the PBW filtration on $U(\mathfrak{g})$. Each $\mathsf{F}^i U(\mathfrak{g}) \cdot v_\lambda$ is finite dimensional. Hence we only need to show these subspaces are $\mathfrak{n}$-stable. For $u \in \mathsf{F}^i U(\mathfrak{g})$ and $x \in \mathfrak{n}$ we have

$$x \cdot (u \cdot v_\lambda) = u \cdot (x \cdot v_\lambda) + [x, u] \cdot v_\lambda.$$

By definition $x \cdot v_\lambda = 0$. Then we win because $[x, u] \in [\mathfrak{g}, \mathsf{F}^i U(\mathfrak{g})] \subset \mathsf{F}^{i-1} U(\mathfrak{g})$. □

We are going to describe the $\mathfrak{t}$-action on M_λ. We need some definitions.

Definition 2.8. Let $V \in \mathfrak{t}-\mathsf{mod}$. We say V is a **weight module** if $V = \oplus_{\lambda \in \mathfrak{t}^*} V_\lambda$, where $V_\lambda \subset V$ is the λ-eigenspace. We say λ is a **weight** of V if $V_\lambda \neq 0$. Vectors in V_λ are called **λ-weight vectors**.

Remark 2.9. A $\mathfrak{t}$-module V is a weight module iff the action is locally finite and semisimple. This means for any $v \in V$, the subspace $\mathfrak{t} \cdot v$ is finite-dimensional and any $x \in \mathfrak{t}$ is sent to a diagonalizable endomorphism in $\mathfrak{gl}(\mathfrak{t} \cdot v)$.

Remark 2.10. A $\mathfrak{t}$-module is a weight module iff the corresponding quasi-coherent sheaf on $\mathfrak{t}^*$ is a direct sum of 1-dimensional skyscrapters at closed points.

Example 2.11. By the root decomposition, $\mathfrak{g}$ is a weight module when viewed as a $\mathfrak{t}$-module via the adjoint action. Nonzero weights are roots.

Example 2.12. The object $U(\mathfrak{t}) \in \mathfrak{t}-\mathsf{mod}$ is not a weight module. Indeed, it corresponds to the structure sheaf of $\mathfrak{t}^*$.

Remark 2.13. Weight modules in $\mathfrak{t}-\mathsf{mod}$ are closed under taking subquotients (e.g. by Remark 2.10), but not closed under extensions.

Proposition 2.14. *The Verma module M_λ is a weight module, and the weights are given exactly by*
$$\lambda - \sum_{\alpha\in\Phi^+} n_\alpha \alpha,\ n_\alpha \in \mathbb{Z}^{\geq 0}.$$
Moreover, each weight space is finite-dimensional.

Proof. First, note that $v_\lambda \in M_\lambda$ is a λ-weight vector because it is the image of $1_\lambda \in k_\lambda$.

By the PBW theorem (Corollary 1.6), $U(\mathfrak{n}^-)$ has a basis consists of weight vectors whose weights are $-\sum_{\alpha\in\Phi^+} n_\alpha\alpha,\ n_\alpha \in \mathbb{Z}^{\geq 0}$. Also, each weight space is finite dimensional.

Let $x \in U(\mathfrak{n}^-)$ be such a weight vector and μ be its weight. By the following equation, $x \cdot v_\lambda \in M_\lambda$ is a $(\lambda+\mu)$-weight vector:
$$t\cdot(x\cdot v_\lambda) = x\cdot(t\cdot v_\lambda) + [t,x]\cdot v_\lambda,\ t\in\mathfrak{t}.$$
Then we win by Exercise 2.6.

□

Definition 2.15. We define a partial order $\leq$ on $\mathfrak{t}^*$ such that $\mu_1 \leq \mu_2$ iff $\mu_2 - \mu_1 \in \mathbb{Z}^{\geq 0}\Phi^+$.

Note that under the above partial order, the weight of $v_\lambda \in M_\lambda$ is indeed the highest one.

Example 2.16. Consider the case $\mathfrak{g} = \mathfrak{sl}_2$ equipped with its standard Cartan and Borel subalgebras. A weight $\lambda \in \mathfrak{t}^*$ is the same as a scaler $l := \langle \lambda, \check{\alpha}\rangle$, where $\check{\alpha} := h := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right) \in \mathfrak{t}$ is the coroot.

Since $\langle \alpha, \check{\alpha}\rangle = 2$, the weights of the Verma module M_l are of the form $l-2n,\ n\geq 0$. For each such $l' := l-2n$, since $\mathfrak{n}^-$ is 1-dimensional, the l'-weight space of M_l is also 1-dimensional. Namely, it is spaned by $f^n \cdot v_l \in M_l$, where $f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0\end{smallmatrix}\right)$ generates $\mathfrak{n}^-$.

Exercise 2.17. This is Homework 1, Problem 2[3]. In the case $\mathfrak{g} = \mathfrak{sl}_2$, show the Verma module M_l is irreducible unless $l \in \mathbb{Z}^{\geq 0}$. In the latter case, show there is a non-split short exact sequence
$$0 \to M_{-l-2} \to M_l \to L_l \to 0 \qquad (2.2)$$
such that L_l is a finite-dimensional irreducible $\mathfrak{sl}_2$-module with highest weight l.

We return to the study of general semisimple Lie algebra $\mathfrak{g}$.

Theorem 2.18. *The Verma module M_λ admits a unique irreducible quotient module L_λ, and the highest weight of L_λ is λ. In particular, L_λ and L'_λ are non-isomorphic for $\lambda \neq \lambda'$.*

Proof. Any proper submodule $N \subset M_\lambda$ is a weight module whose weights do not contain λ. It follows that the union of all the proper submodules satisfies the same property. By construction, this is the maximal proper submodule of M_λ. Then L_λ is the corresponding quotient.

□

3. Category $\mathcal{O}$

Roughly speaking, the Bernstein–Gelfand–Gelfand (a.k.a. BGG) category $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$−mod consisting of objects similar to Verma modules. Let us first give the traditional definition:

Definition 3.1. We define the **category** $\mathcal{O}$ to be the full subcategory of $\mathfrak{g}$−mod consisting of objects M satisfying the following properties:

(O1) M is finitely generated as a $\mathfrak{g}$-module;

[3]Warning: the solution in Gaitsgory's notes contains a critical typo and the last paragraph there should be justified. Also, don't forget to show L_l is irreducible.

(O2) M is a weight module;
(O3) The action of $\mathfrak{n}$ on M is locally finite.

Example 3.2. We have already seen that the Verma modules $M_\lambda \in \mathcal{O}$.

Lemma 3.3. *The subcategory $\mathcal{O}$ of $\mathfrak{g}{-}\mathsf{mod}$ is closed under taking sub-quotients and finite direct sums. In particular, $\mathcal{O}$ is an abelian category.*

Proof. For (O1), $U(\mathfrak{g})$ is Noetherian. For (O2), Remark 2.13. The claim for (O3) is obvious. □

Warning 3.4. *The subcategory $\mathcal{O}$ is* not *closed under extensions. This can be seen by considering $\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(N)$ where N is a finite dimensional $\mathfrak{t}$-module that does not have a weight decomposition.*

Lemma 3.5. *Any object $M \in \mathcal{O}$ is Noetherian, i.e., satisfies the ascending chain condition for subobjects.*

Proof. Follows from the fact that $U(\mathfrak{g})$ is Noetherian. □

Proposition 3.6. *Any object $M \in \mathcal{O}$ is a quotient of a finite successive extension of Verma modules. In particular, M is finitely generated as an $\mathfrak{n}^-$-module.*

Proof. The last claim follows from the first one because of Exercise 2.6.

By (O1), M is generated by a finite-dimensional subspace M_0 as a $\mathfrak{g}$-module. By (O2), we can enlarge M_0 and assumme it is a finite direct sum of weight spaces. By (O3), $U(\mathfrak{b})\cdot M_0 = U(\mathfrak{n})\cdot M_0$ is finite-dimensional. Hence we may assume M_0 is stable under the $\mathfrak{b}$-action. By adjunction, we have a $\mathfrak{g}$-linear map

$$\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(M_0) \to M,$$

which is surjective because M_0 generates M as a $\mathfrak{g}$-module. It remains to show M_0 is a successive extension of 1-dimensional $\mathfrak{b}$-modules. We state this as the following lemma.

Lemma 3.7. *Let $M \in \mathcal{O}$ and $M_0 \subset M$ be a finite-dimensional subspace stable under the $\mathfrak{b}$-action. Then the $\mathfrak{n}$-action on M_0 is nilpotent and M_0 is a successive extension of 1-dimensional $\mathfrak{b}$-modules.*

Proof. Note that the second claim follows from the first one. Namely, let $N_0 \subset M_0$ be the subspace annihilated by $\mathfrak{n}$. This is a sub-$\mathfrak{b}$-representation because $\mathfrak{n}$ is an ideal of $\mathfrak{b}$. The first claim implies $N_0 \neq 0$. Since N_0 is annihilated by $\mathfrak{n}$, it is in the image of the restriction functor $\mathfrak{t}{-}\mathsf{mod} \to \mathfrak{b}{-}\mathsf{mod}$. It follows that N_0 is a direct sum of 1-dimensional $\mathfrak{b}$-representations because it is a weight module. Replacing M_0 by M_0/N_0, we win by induction.

It remains to prove the first claim. We only need to show $\mathfrak{n}$ acts nilpotently on any weight vector $v \in M_0$. Let $x \in \mathfrak{n}$ be a weight vector. A direct calculation shows $x \cdot v$ is a weight vector whose weight is the sum of those of v and x. In particular, the weight of $x \cdot v$ is strictly greater than that of v with respect to the partial order $\prec$. Since the set of weights of M_0 is finite, we see $\mathfrak{n}$ acts nilpotently on v.

□

□[Proposition 3.6]

Corollary 3.8. *Let $M \in \mathcal{O}$. Then each weight space of M is finite-dimensional.*

Proof. Follows from Proposition 2.14 and Proposition 3.6.

□

Exercise 3.9. This is Homework 1, Problem 3. Recall for any $V_1, V_2 \in \mathfrak{g}{-}\mathsf{mod}$, the tensor product $V_1 \otimes V_2$ of the underlying vector spaces has a natural $\mathfrak{g}$-module structure defined by $x \cdot (v_1 \otimes v_2) := (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$.

(1) Prove: if V_1 and V_2 are weight modules, so is $V_1 \otimes V_2$. Determine the weights and weight spaces of $V_1 \otimes V_2$ in term of those for V_1 and V_2.
(2) Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Prove: the tensor product of two Verma modules is not contained in $\mathcal{O}$.

References

[MR] McConnell, John C., James Christopher Robson, and Lance W. Small. Noncommutative noetherian rings. Vol. 30. American Mathematical Soc., 2001.