LECTURE 3

In this lecture, we give a quick review of the theory of algebraic groups. This theory is analogous to that of complex Lie groups, but the techniques are more algebraic and some proofs are subtler[1]. Standard textbooks include: [B], [H] and [Sp]. See [M] for a modern treatment of this theory, which is also my favourite.

1. Algebraic Groups

Definition 1.1. An **algebraic group** over k is a finite type k-scheme G equipped with a group structure, i.e., a multiplication map $m : G \times G \to G$ subject to axioms similar to those for an abstract group.

Homomorphisms between algebraic groups are defined in the obvious way. Let Grp_k be the category of algebraic groups.

Remark 1.2. As in the study of abstract groups, the unit and the inversion maps are determined by the multiplication map. We denote them respectively by:

$$e : \mathsf{pt} \to G,\ \sigma : G \to G,$$

where $\mathsf{pt} \coloneqq \mathsf{Spec}(k)$.

Construction 1.3. *Let G be an algebraic group. For any commutative k-algebra A, write*

$$G(A) \coloneqq \mathsf{Hom}(\mathsf{Spec}(A), G)$$

be the set of maps $\mathsf{Spec}(A) \to G$ between k-schemes. The group structure on G induces a group structure on $G(A)$.

Note that for $A \to B$, we have a homomorphism $G(A) \to G(B)$. Hence we obtain a functor

$$G(-) : \mathsf{CAlg}_k \to \mathsf{Grp}$$

from the category of commutative k-algebras to the category of (abstract) groups. By the Yoneda lemma, the algebraic group G is determined by this functor.

Example 1.4. The **additive group** $\mathbb{G}_{\mathsf{a}}$ is defined such that $\mathbb{G}_{\mathsf{a}}(A) = A$, viewed as a commutative group under addition. The underlying k-scheme is the affine line $\mathbb{A}^1$.

Example 1.5. The **multplicative group** $\mathsf{GL}_1 = \mathbb{G}_{\mathsf{m}}$ is defined such that $\mathbb{G}_{\mathsf{m}}(A) = A^\times$, i.e. the subset of unit elements in A, viewed as a commutative group under multiplication. The underlying k-scheme is $\mathbb{A}^1 \smallsetminus 0$, i.e., the affine line with the origin removed.

Example 1.6. One can define algebraic groups $G \coloneqq \mathsf{GL}_n$, SL_n, SO_n, etc. such that $G(A)$ is the group of matrices of the corresponding type with coefficients in A.

Example 1.7. One can define the algebraic group PGL_n such that $\mathcal{O}_{\mathsf{PGL}_n}$ is a subring of $\mathcal{O}_{\mathsf{GL}_n}$[2].

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[1]Especially if one allows positive-characteristic or non-algebraic-closed base field k.

[2]Warning: $\mathsf{PGL}_n(A) \neq \mathsf{GL}_n(A)/\mathsf{GL}_1(A)$ for general A. In fact, viewed as functors in A, the LHS is the sheafification of the RHS in the fpqc topology.

Remark 1.8. Not every functor $\mathsf{CAlg}_k \to \mathsf{Grp}$ comes from an algebraic group. For example, any k-vector space V defines a functor

$$\mathsf{GL}_V(-) : \mathsf{CAlg}_k \to \mathsf{Grp}$$

that sends A to the group of A-linear automorphisms of $A \otimes V$. This functor is not represented by an algebraic group unless V is finite-dimensional.

2. Hopf algebras

From now on, we assume G is affine[3][4].

Construction 2.1. *An affine algebraic group G is determined by its ring of functions $\mathcal{O}_G$. The maps $m : G \times G \to G$ and $e : \mathsf{pt} \to G$ correspond to homomorphisms between algebras*

$$\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G,\ \epsilon : \mathcal{O}_G \to k,$$

which are called the ***comultiplication*** *and* ***counit*** *maps of $\mathcal{O}_G$. Together with the usual multiplication and unit maps of $\mathcal{O}_G$, we obtain a* ***bialgebra*** *$(\mathcal{O}_G, \cdot, \Delta)$.*

Note that this bialgebra is commutative but not cocommutative unless G is so.

The inverse map $\sigma : G \to G$ corresponds to a homomorphism $S : \mathcal{O}_G \to \mathcal{O}_G$, which is called the ***antipode*** *of $\mathcal{O}_G$. This makes $\mathcal{O}_G$ into a commutative* ***Hopf algebra****.*

Remark 2.2. A Hopf algebra is a bialgebra A equipped with an antipode map $S : A \to A$ subject to a certain axiom. For our purposes, it is less useful to memorize this axiom than to imagine it amounts to say "$\mathsf{Spec}(A)$"[5] has an inversion map.

Example 2.3. For $G = \mathbb{G}_{\mathsf{a}}$, we have $\mathcal{O}_G = k[t]$ and

$$\Delta(t) = t \otimes 1 + 1 \otimes t,\ \epsilon(f) = f(0),\ S(f)(t) = f(-t).$$

Example 2.4. For $G = \mathbb{G}_{\mathsf{m}}$, we have $\mathcal{O}_G = k[t, t^{-1}]$ and

$$\Delta(t) = t \otimes t,\ \epsilon(f) = f(1),\ S(f)(t) = f(t^{-1}).$$

Example 2.5. The universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra is a Hopf algebra. The comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g} \subset U(\mathfrak{g})$ and its compatibility with the multiplication. Similarly, the antipode is determined by $S(x) = -x$, $x \in \mathfrak{g}$.

The Hopf algebra $U(\mathfrak{g})$ is cocommutative but not commutative unless $\mathfrak{g}$ is abelian.

Remark 2.6. Using the Hopf algebra structure on $U(\mathfrak{g})$, the tensor product structure in $\mathfrak{g}{-}\mathsf{mod}$ can be defined as follows. Let V_1 and V_2 be left $U(\mathfrak{g})$-modules. Their tensor product $V_1 \otimes V_2$ is naturally a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$-module. Restricting along Δ, we can view $V_1 \otimes V_2$ as a $U(\mathfrak{g})$-module.

Remark 2.7. There are also interesting Hopf algebras that are neither commutative nor cocommutative. For example, *quantum algebras* are such gadgets. See [L] for a standard textbook.

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[3] Any affine algebraic group over field of characteristic 0 is smooth.

[4] Projective algebraic groups, a.k.a., abelian varieties, are also important and play a central role in modern mathematics.

[5] Note however that this does not make sense if A is not commutative.

3. Tangent spaces

As in the theory of Lie groups, the tangent space of an algebraic group at its unit is a Lie algebra. To desribe this, let us review the definition of tangent spaces in algebraic geometry.

Definition 3.1. Let X be any k-scheme and $x \in X$ be a k-point, i.e., a map $x : \mathsf{Spec}(k) \to X$. The **tangent space** of X at x, denoted by $T_x X$, is the *set* of dotted arrows making the following diagram commute:

$$\begin{array}{ccc} \mathsf{Spec}(k) & \xrightarrow{\;x\;} & X \\ \big\downarrow & \nearrow & \\ \mathsf{Spec}(k[\epsilon]/\epsilon^2). & & \end{array}$$

Here the vertical map is given by the homomorphism $k[\epsilon]/\epsilon^2 \to k$, $\epsilon \mapsto 0$.

Elements in $T_x X$ are called **tangent vectors** of X at x.

Tangent vectors are related to *derivations*. Let us review its definition in the algebraic setting.

Definition 3.2. Let A be a k-algebra and M be an A-module. A **k-derivation** of A into M is a k-linear map $D : A \to M$ satisfying the **Lebniz rule**

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

Let $\mathsf{Der}(A, M)$ be the set of such k-derivations. This is naturally an A-module.

Definition 3.3. Let X be a k-scheme and $\mathcal{M}$ be an $\mathcal{O}_X$-module. A **k-derivation** of $\mathcal{O}_X$ into $\mathcal{M}$ is a k-linear morphism $D : \mathcal{O}_X \to \mathcal{M}$ such that for any open subscheme $U \subset X$, $D(U) : \mathcal{O}_X(U) \to \mathcal{M}(U)$ is a k-derivation. Let $\mathsf{Der}(\mathcal{O}_X, \mathcal{M})$ be the space of k-derivations.

Construction 3.4. *Let $X = \mathsf{Spec}(A)$ be an affine scheme and $x : \mathsf{Spec}(k) \to X$ be given by a homomorphism $\phi : A \to k$. View k as an A-module via this homomorphism and denote it by k_x. For any $D \in \mathsf{Der}(A, k_x)$, the map*

$$A \to k[\epsilon]/\epsilon^2,\ f \mapsto \phi(f) + D(f)\epsilon$$

is a homomorphism and thereby gives a map $\mathsf{Spec}(k[\epsilon]/\epsilon^2) \to X$, which is an element in $T_x X$. It is easy to see this gives a bijection

$$\mathsf{Der}(A, k_x) \simeq T_x X.$$

In particular, we obtain an A-module structure on $T_x X$. Note that the action of A factors through $A \twoheadrightarrow k_x$.

For general k-scheme, the above construction gives a bijection

$$\mathsf{Der}(\mathcal{O}_X, k_x) \simeq T_x X,$$

where k_x is viewed as the skyscrapter sheaf at x. Indeed, this follows from the obvious fact that $T_x X$ only depends on a Zariski neighborhood of x in X.

Construction 3.5. *Let $f : X \to Y$ be a morphism between k-schemes. Let $x \in X$ and $y := f(x) \in Y$ be k-points. There is an obvious k-linear map*

$$df : T_x X \to T_y Y$$

*given by composing with f. We call it the **differential** of f.*

We have the following obvious result:

Lemma 3.6. *Let X and Y be k-schemes. Let $x \in X$ and $y \in Y$ be k-points. For $\partial_1 \in T_x X$ and $\partial_2 \in T_y Y$, write $\partial_1 \oplus \partial_2 \in T_{(x,y)}(X \times Y)$ for the map*

$$\mathsf{Spec}(k[\epsilon]/\epsilon^2) \xrightarrow{(\partial_1,\partial_2)} X \times Y.$$

Then the map

$$T_x X \times T_y Y \to T_{(x,y)}(X \times Y),\ (\partial_1, \partial_2) \mapsto \partial_1 \oplus \partial_2 \tag{3.1}$$

induces an isomorphism $T_x X \oplus T_y Y = T_{(x,y)}(X \times Y)$.

4. Lie Algebras and Algebraic Groups

Notation 4.1. *Let G be an algebraic group. Define*

$$\mathsf{Lie}(G) \coloneqq T_e G.$$

Example 4.2. For $G = \mathsf{GL}_n$, we have $\mathsf{Lie}(\mathsf{GL}_n) \simeq \mathfrak{gl}_n$. In particular, $\mathsf{Lie}(\mathsf{GL}_n)$ is naturally a Lie algebra. We have similar results for other classical subgroups of GL_n.

Remark 4.3. Consider the multiplication map $m : G \times G \to G$ and its differential $dm : \mathsf{Lie}(G \times G) \to \mathsf{Lie}(G)$. Via the identification $\mathsf{Lie}(G) \oplus \mathsf{Lie}(G) \simeq \mathsf{Lie}(G \times G)$, we obtain a binary operation on $\mathsf{Lie}(G)$:

$$\mathsf{Lie}(G) \oplus \mathsf{Lie}(G) \simeq \mathsf{Lie}(G \times G) \xrightarrow{dm} \mathsf{Lie}(G).$$

By definition, this binary operation is associative and $0 \in \mathsf{Lie}(G)$ is the unit of it. Hence it sends (∂_1, ∂_2) to $\partial_1 + \partial_2$. This is the algebraic analogue of the formula $\exp(tu) \cdot \exp(tv) = \exp(t(u+v))(1 + O(t^2))$, $u, v \in \mathsf{Lie}(G)$ that appears in the study of Lie groups.

In particular, we have a short exact sequence of groups:

$$0 \to \mathsf{Lie}(G) \to G(k[\epsilon]) \to G(k) \to 0, \tag{4.1}$$

where the group structure on $\mathsf{Lie}(G)$ is given by $(\partial_1, \partial_2) \mapsto \partial_1 + \partial_2$.

Theorem 4.4. *There is a canonical functor $\mathsf{Grp}_k \to \mathsf{Lie}_k$ sending G to $\mathsf{Lie}(G)$ equipped with a natural Lie bracket, such that for $G = \mathsf{GL}_n$, the Lie bracket on $\mathsf{Lie}(\mathsf{GL}_n)$ is given by that on $\mathfrak{gl}_n$.*

Remark 4.5. There are several equivalent ways to construct the Lie bracket on $\mathsf{Lie}(G)$. Below are two of them:

- The Hopf algebra approach: given $\partial_1, \partial_2 \in \mathsf{Lie}(G)$, viewed as k-derivations $\mathcal{O}_G \to k_e$, consider the composition[6]

  $$\mathcal{O}_G \xrightarrow{\Delta} \mathcal{O}_G \otimes \mathcal{O}_G \xrightarrow{\partial_1\otimes\partial_2 - \partial_2\otimes\partial_1} k_e.$$

  One can show this is also a k-derivation. Then we define $[\partial_1, \partial_2] \in \mathsf{Lie}(G)$ to be the element corresponding to this k-derivation. The axioms of Hopf algebras imply this is indeed a Lie bracket. This is the algebraic analogue of the formula $\exp(tu) \cdot \exp(tv) \cdot \exp(-tu) \cdot \exp(-tv) = \exp(t^2[u,v])(1 + O(t^3))$, $u, v \in \mathsf{Lie}(G)$ that appears in the study of Lie groups.
- The adjoint representation approach: the isomorphism $\mathsf{Lie}(G) \simeq \mathsf{Der}(\mathcal{O}_G, k_e)$ can be generalized to $\mathsf{Lie}(G) \otimes R \simeq \mathsf{Der}(\mathcal{O}_G, R_e)$ that are functorial in $R \in \mathsf{CAlg}_k$. It follows that the short exact sequence (4.1) can be generalized to short exact sequences

  $$0 \to \mathsf{Lie}(G) \otimes R \to G(R[\epsilon]) \to G(R) \to 0$$

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[6]For non-affine G, we use the morphisms between $\mathcal{O}_X$-modules $\mathcal{O}_G \xrightarrow{\Delta} \mathsf{mult}_*(\mathcal{O}_G \otimes \mathcal{O}_G) \xrightarrow{\partial_1\otimes\partial_2 - \partial_2\otimes\partial_1} \mathsf{mult}_*(k_e \otimes k_e) \simeq k_e$.

that are functorial in R. Note that we have a canonical splitting $G(R) \to G(R[\epsilon])$ induced by the *embedding* $R \to R[\epsilon]$. These splittings provide adjoint actions of $G(R)$ on $\mathsf{Lie}(G) \otimes R$ that are functorial in R. In other words, we have a homomorphism between algebraic groups:

$$\mathsf{Ad} : G \to \mathsf{GL}_{\mathsf{Lie}(G)}. \tag{4.2}$$

The differential of this homomorphism at the unit element e gives a k-linear map

$$\mathsf{ad} : \mathsf{Lie}(G) \to \mathfrak{gl}_{\mathsf{Lie}(G)}.$$

Then we define $[\alpha_1, \alpha_2] := \mathsf{ad}(\alpha_1)(\alpha_2)$. One can show this is indeed a Lie bracket. Note that by construction ad is indeed the adjoint presentation of $\mathsf{Lie}(G)$.

Remark 4.6. The functor $\mathsf{Grp}_k \to \mathsf{Lie}_k$ is unique if stated properly. See [M, Theorem 10.23].

Warning 4.7. *It is not true that every Lie algebra can be obtained from algebraic groups. See [Bou, I, §5, Exercise 6] for a counterexample.*

Remark 4.8. Let G be an affine algebraic group and $\mathfrak{g} := \mathsf{Lie}(G)$ be its Lie algebra. The Hopf algebras $\mathcal{O}_G$ and $U(\mathfrak{g})$ are related as follows.

Consider the dual $U(\mathfrak{g})^* := \mathsf{Hom}_k(U(\mathfrak{g}), k)$ as a topological vector space[7]. The cocommutative Hopf algebra structure on $U(\mathfrak{g})$ induces a commutative topological Hopf algebra structure on it[8].

On the other hand, consider the maximal ideal $\mathfrak{m} \subset \mathcal{O}_G$ corresponding to the unit point $e : \mathsf{pt} \to G$. Let $\hat{\mathcal{O}}_G$ be the $\mathfrak{m}$-adic completion of $\mathcal{O}_G$. The commutative Hopf algebra structure on $\mathcal{O}_G$ induces a commutative topological Hopf algebra structure on it.

We have

$$U(\mathfrak{g})^* \simeq \hat{\mathcal{O}}_G$$

as commutative topological Hopf algebras such that $(\mathsf{F}^{\leq k} U(\mathfrak{g}))^* \simeq \hat{\mathcal{O}}_G / \mathfrak{m}^k \hat{\mathcal{O}}_G$. Note that this gives another way to define the Lie bracket on $\mathsf{Lie}(G)$.

Example 4.9. Note that the Lie algebras of $\mathbb{G}_{\mathsf{a}}$ and $\mathbb{G}_{\mathsf{m}}$ are isomorphic, it follows that $\hat{\mathcal{O}}_{\mathbb{G}_{\mathsf{a}}} \simeq k[[t]]$ and $\hat{\mathcal{O}}_{\mathbb{G}_{\mathsf{m}}} = k[[t-1]]$ are isomorphic. Up to a scaler, this isomorphism is given by

$$k[[t-1]] \to k[[t]], \ f \mapsto f(\exp(t)).$$

Note that given a power series $a_0 + a_1(t-1) + a_2(t-1)^2 + \cdots$, the series $a_0 + a_1(\exp(t) - 1) + a_2(\exp(t) - 1)^2 + \cdots$ indeed converges in the t-adic topology.

5. Representations of Algebraic Groups

In this section, G is an affine algebraic group.

Definition 5.1. A **representation of** G, or equivalently a G**-module** is a k-vector space V equipped with a natural transformation $G(-) \to \mathsf{GL}_V(-)$ as functors $\mathsf{CAlg}_k \to \mathsf{Grp}$.

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[7]For a vector space V equipped with the discrete topology, the dual V^* is equipped with the weakest topology such that for any finite-dimensional subspace $V_0 \subset V$, the map $V^* \to V_0^*$ is continuous. Here V_0^* is equipped with the discrete topology. Equivalently, we can define V^* as an object in the pro-category $\mathsf{Pro}(\mathsf{Vect}_{k,\mathsf{fd}})$ of finite dimensional vector spaces.

[8]Here we must use the *complete tensor product* $U(\mathfrak{g})^* \hat{\otimes} U(\mathfrak{g})^*$ instead of the usual tensor product. By design, this is the dual of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Let V and W be G-modules, a G**-linear map** $\phi : V \to W$ is a k-linear map such that for any $A \in \mathsf{CAlg}_k$, the following diagram commutes

$$\begin{array}{ccc} G(A) & \longrightarrow & \mathsf{GL}_V(A) \\ \downarrow & & \downarrow \phi\circ- \\ \mathsf{GL}_W(A) & \xrightarrow{-\circ\phi} & \mathsf{Hom}_A(A\otimes V, A\otimes W). \end{array}$$

Let $\mathsf{Rep}(G)$ be the category of G-modules.

Example 5.2. Any V can be equipped with a trivial G-module structure such that the homomorphisms $G(A) \to \mathsf{GL}_V(A)$ are trivial.

Example 5.3. The homomorphism (4.2) defines a G-module structure on its Lie algebra $\mathsf{Lie}(G)$. We call it the **adjoint representation** of G.

Proposition 5.4. *The category* $\mathsf{Rep}(G)$ *is an abelian category and the forgetful functor* $\mathsf{Rep}(G) \to \mathsf{Vect}_k$ *is exact.*

Remark 5.5. If V is finite-dimensional, then GL_V is represented by an algebraic group. A G-module structure on V is just a homomorphism $G \to \mathsf{GL}_V$ between algebraic groups.

Warning 5.6. *Evaluate at* $k \in \mathsf{CAlg}_k$*, we obtain a group homomorphism* $G(k) \to \mathsf{GL}_V(k) = \mathsf{GL}(V)$*. But a* G*-module structure contains more information than such a homomorphism. This can be seen from the following exercise. (Note that for* $k = \mathbb{C}$*, the* abstract *groups* $\mathbb{G}_\mathsf{a}(\mathbb{C}) = \mathbb{C}$ *and* $\mathbb{G}_\mathsf{m}(\mathbb{C}) = \mathbb{C}^\times$ *are isomorphic via the exponential map.)*

Exercise 5.7. This is Homework 1, Problem 4.

(1) Find all maps between k-schemes $\mathbb{A}^1 \to \mathbb{A}^1 \smallsetminus 0$.
(2) Find all 1-dimensional representations of the additive group $\mathbb{G}_\mathsf{a}$.
(3) Find all maps between k-schemes $\mathbb{A}^1 \smallsetminus 0 \to \mathbb{A}^1 \smallsetminus 0$.
(4) Find all 1-dimensional representations of the multiplicative group $\mathbb{G}_\mathsf{m}$.

Proposition 5.8. *Any* G*-action on a vector space* V *is locally finite, i.e.,* V *is the union of its finite-dimensional subrepresentations.*

Proposition 5.9. *There is a canonical equivalence*

$$\mathsf{Rep}(G) \simeq \mathcal{O}_G\text{-}\mathsf{comod}$$

from the category of G*-modules to the category of* $\mathcal{O}_G$*-comodules. This equivalence is compatible with the forgetful functors to* Vect_k*.*

Remark 5.10. The functor $\mathsf{Rep}(G) \simeq \mathcal{O}_G\text{-}\mathsf{comod}$ is constructed as follows. Let $V \in \mathsf{Rep}(G)$. Consider $A := \mathcal{O}_G$ and the homomorphism $G(A) \to \mathsf{GL}_V(A)$. The identity map $G \to G$ can be written as $\mathsf{Spec}(A) \to G$ which corresponds to an element in $G(A)$[9]. Consider the image of this element in $\mathsf{GL}_V(A)$, which is a A-linear map $A \otimes V \to A \otimes V$. This is the same as a k-linear map $V \to A \otimes V$, i.e., a k-linear map

$$V \to \mathcal{O}_G \otimes V.$$

One can verify this defines a $\mathcal{O}_G$-comodule structure on V.

[9] Warning: this is *not* the unit element of this group.

Construction 5.11. *There is a forgetful functor*

$$\mathsf{Rep}(G) \to \mathsf{Lie}(G)\text{–}\mathsf{mod}$$

that can be constructed in the following equivalent ways:

- *Let we first suppose $V \in \mathsf{Rep}(G)$ is finite-dimensional. Then we have a homomorphism between algebraic groups $G \to \mathsf{GL}_V$ which induces a homomorphism between their Lie algebras $\mathsf{Lie}(G) \to \mathsf{Lie}(\mathsf{GL}_V) = \mathfrak{gl}(V)$, i.e., a $\mathsf{Lie}(G)$-module structure on V.*

  When V is infinite-dimensional, the functor $\mathsf{GL}_V(-)$ is no longer represented by an algebraic group, but there is a formal method to define its Lie algebra and make the above construction work.
- *Any $V \in \mathsf{Rep}(G)$ has an $\mathcal{O}_G$-comodule structure. By co-restricting along the map $\mathcal{O}_G \to \hat{\mathcal{O}}_G$, we obtain a (continuous) comodule structure for $\hat{\mathcal{O}}_G \simeq U(\mathfrak{g})^*$. Passing to duality, we obtain a module structure for $U(\mathfrak{g})$.*

Remark 5.12. On the level of k-points, the conjugate action of $G(k)$ on G induces an (abstract) action of $G(k)$ on $\mathsf{Lie}(G)$. This can be generalized to an action of $G(A)$ on $A \otimes \mathsf{Lie}(G)$ for any $A \in \mathsf{CAlg}_k$ and thereby obtain the desired G-module structure on $\mathsf{Lie}(G)$.

Proposition 5.13. *If G is connected, then the functor $\mathsf{Rep}(G) \to \mathsf{Lie}(G)\text{–}\mathsf{mod}$ is fully faithful. In particular, the G-invariance and $\mathsf{Lie}(G)$-invariance for a G-module are the same.*

Definition 5.14. If G is connected, we say an object $V \in \mathsf{Lie}(G)\text{–}\mathsf{mod}$ is **G-integrable** if it is contained in the essential image of the above functor.

Warning 5.15. *The similar claim for derived categories is false. In other words, extensions of G-integrable modules are not necessarily G-integrable. The multiplicative group $\mathbb{G}_{\mathsf{m}}$ is a counterexample.*

6. Semisimple Algebraic Groups

Theorem 6.1. *Any semisimple Lie algebra $\mathfrak{g}$ can be realized as the Lie algebra of an algebraic group. In the category of connected algebraic groups G with $\mathsf{Lie}(G) \simeq \mathfrak{g}$, there is a final object G_{ad} and an initial object G_{sc}. Moreover, the homomorphisms*

$$G_{\mathsf{sc}} \to G \to G_{\mathsf{ad}}$$

are isogenies*, i.e., are surjective and have finite kernels.*

Example 6.2. For $\mathfrak{g} = \mathfrak{sl}_n$, we have $G_{\mathsf{sc}} = \mathsf{SL}_n$ and $G_{\mathsf{ad}} = \mathsf{PGL}_n$.

Definition 6.3. We say G is **semisimple**[10] if it is connected and its Lie algebra is semisimple. For a semisimple algebraic group G, we say it is **of adjoint type** (resp. **simply connected**) if it is of the form G_{ad} (resp. G_{sc}).

Theorem 6.4. *If G_{sc} is a simply-connected semisimple algebraic group, then any finite-dimensional $\mathfrak{g}$-module is G_{sc}-integrable, i.e.,*

$$\mathsf{Rep}(G_{\mathsf{sc}})_{\mathsf{fd}} = \mathfrak{g}\text{–}\mathsf{mod}_{\mathsf{fd}}.$$

Warning 6.5. *The similar claim for infinite-dimensional representation is false. This can be seen from the following exercise.*

Exercise 6.6. This is Homework 1, Problem 5. Let G be any semisimple algebraic group with Lie algbra $\mathfrak{g}$. Prove: any Verma module of $\mathfrak{g}$ is not G-integrable.

[10]This is an ad hoc definition that only is only correct under our assumptions on k.

Theorem 6.7. *Let G be any semisimple algebraic group with Lie algebra $\mathfrak{g}$. Then the abelian categories $\mathsf{Rep}(G)$ and $\mathfrak{g}\text{–}\mathsf{mod}_{\mathsf{fd}}$ are semisimple*[11]*. Simple objects in $\mathsf{Rep}(G)$ are finite-dimensional, and an object $V \in \mathsf{Rep}(G)$ is simple iff it is an simple object in $\mathfrak{g}\text{–}\mathsf{mod}$.*

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[11] I.e., any object can be written as a direct sum of simple objects.