LECTURE 8

1. Hecke algebras and Kazhdan–Lusztig conjecture

Last time we introduced the Verma–BGG theorem, which gives a complete answer to when $[M_\lambda : L_\mu] \neq 0$ and $M_\mu \subset M_\lambda$. One could be more ambitious and ask the following question:

Question 1. *Can we find a formula or an algorithm that calculates the multiplicities $[M_\lambda : L_\mu]$ for amy λ, μ?*

The climax of this study was when Kazhdan–Lusztig proposed their famous conjecture in 1979, soon followed by independent proofs given by Beilinson–Bernstein and Brylinski–Kashiwara using a same *geometric* method. This method received the name *localization theory*, and marked the birth of the subject called *geometric representation theory*[1]. The ultimate goal of this course is to introduce this localization theory.

Roughly speaking, the KL conjecture says:

Conjecture 1.1. *In the principle block $\mathcal{O}_0$, the multiplicities $[M_\lambda : L_\mu]$ can be calculated using combinatorial data associated to the* Hecke algebra *of W.*

Let us first define the Hecke algebra of W, which plays a significant role in modern representation theory. We will only introduce the basics. There are many good references for this subject, and I would recommend [EMTW, Sect. 3].

Recall the Weyl group W can be generated by simple reflections $s \in S$ subject to the following relations

- (Order 2) For any $s \in S$, $s^2 = 1$;
- (Braid relation) For any $s \neq t \in S$,
$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}},$$
where $m_{st} \in \{2, 3, 4, 6\}$.

It follows that the group algebra $\mathbb{Z}W$ is generated by similar generators and relations over $\mathbb{Z}$. Rougly speaking, the Hecke algebra $\mathcal{H}$ of W is a deformation of the group algebra $\mathbb{Z}W$ using an indeterminate q as parameter, where we keep the braid relation but change the order 2 requirement. For reasons I cannot fully explain, $q = 0$ is not allowed. For reasons I do not want to explain *now*, it is more convenient to use $v := q^{-1/2}$ as the indeterminate. In other words, the base of this deformation is $\mathsf{Spec}(A)$ for $A = \mathbb{Z}[v^\pm] = \mathbb{Z}[q^{\pm 1/2}]$.

Definition 1.2. The **Hecke algebra** $\mathcal{H} := \mathcal{H}(W)$ is the (unital) associative algebra over $\mathbb{Z}[v^\pm]$ generated by the symbols $\{\delta_s \,|\, s \in S\}$ subject to the following relations:

- (Quadratic relation) For any $s \in S$, $(\delta_s - v^{-1})(\delta_s + v) = 0$.

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[1] When KL made their conjecture, they were inspired by Springer's geometric theory on representations of the Weyl group, published a few years ago. Kostant also made many pioneer works on the geometry of adjoint orbits.

- (Braid relation) For any $s \neq t \in S$,

$$\underbrace{\delta_s \delta_t \delta_s \cdots}_{m_{st}} = \underbrace{\delta_t \delta_s \delta_t \cdots}_{m_{st}}.$$

Warning 1.3. *There is another set of conventions, where the quadratic relation is $(\delta_s - q)(\delta_s + 1) = 0$. These two conventions define equivalent algebras via the change of variables $\delta_s \mapsto q^{-1/2}\delta_s$. Beware of this issue when comparing the literatures.*

Proposition-Definition 1.4. *For any $w \in W$, choose a reduced expression $w = s_1 s_2 \cdots s_{\ell(w)}$ and define*

$$\delta_w := \delta_{s_1} \delta_{s_2} \cdots \delta_{s_{\ell(w)}}.$$

*Then $\delta_w \in \mathcal{H}$ does not depend on the choice of the reduced expression, and $\{\delta_w\}_{w \in W}$ is a free basis of $\mathcal{H}$ as a $\mathbb{Z}[v^{\pm}]$-module. We call it the **standard basis** of the Hecke algebra $\mathcal{H}$.*

Remark 1.5. Taking $v = 1$, i.e., taking the tensor product $\mathcal{H} \otimes_{\mathbb{Z}[v^{\pm}]} (\mathbb{Z}[v^{\pm}]/(v-1))$, we recover the group algebra $\mathbb{Z}W$, and the image of the standard basis is the obvious basis of $\mathbb{Z}W$.

Exercise 1.6. This is Homework 4, Problem 1. Prove:

(1) For $w, w' \in W$ such that $\ell(w) + \ell(w') = \ell(ww')$, we have

$$\delta_w \delta_{w'} = \delta_{ww'}.$$

(2) For $w \in W$ and $s \in S$, we have

$$\delta_w \delta_s = \begin{cases} \delta_{ws} & \text{if } w < ws, \\ (v^{-1} - v)\delta_w + \delta_{ws} & \text{if } w > ws, \end{cases}$$

and

$$\delta_s \delta_w = \begin{cases} \delta_{sw} & \text{if } w < sw, \\ (v^{-1} - v)\delta_w + \delta_{sw} & \text{if } w > ws. \end{cases}$$

Remark 1.7. Note that the above exercise provides an algorithm to calculate the multiplication of $\mathcal{H}$ in terms of the standard basis. In particular, $\mathcal{H}$ can be *defined* via the standard basis and these relations.

Note that each δ_s is invertible with inverse given by

$$\delta_s^{-1} = \delta_s + (v - v^{-1}).$$

Hence by the above exercise, we obtain:

Lemma 1.8. *Each standard basis element δ_w is invertible, and we have*

$$\delta_{w^{-1}}^{-1} = \delta_w \mod \langle \delta_{w'} \rangle_{w' < w}.$$

Definition 1.9. The **Kazhdan–Lusztig involution**, or **bar involution**

$$\mathcal{H} \to \mathcal{H},\ h \mapsto \overline{h}$$

is the $\mathbb{Z}$-linear homomorphism determined by

$$\overline{\delta_s} = \delta_s^{-1},\ \overline{v} = v^{-1}.$$

Theorem-Definition 1.10. *There exist a unique subset $\{b_w\}_{w \in W} \subset \mathcal{H}$ such that for any $w \in W$,*

- *(Self-duality) $\overline{b_w} = b_w$;*
- *(Degree bound)*

$$b_w = \delta_w + \sum_{w' < w} h_{w',w} \delta_{w'}$$

for some polynomials $h_{w',w} \in v\mathbb{Z}[v]$ with vanishing constant term.

*This subset is called the **Kazhdan–Lusztig basis** of $\mathcal{H}$. The coefficients $h_{w',w}$ are called the **Kazhdan–Lusztig polynomials**.*

Convention 1.11. *We also set $h_{w,w} = 1$ and $h_{w',w} = 0$ if $w' \nleq w$.*

Example 1.12. We have $b_{\mathsf{Id}} = \delta_{\mathsf{Id}} = 1$.

Example 1.13. For $s \in S$, an immediate calculation shows $b_s = \delta_s + v$ and therefore $h_{1,s} = v$.

Remark 1.14. You are strongly encouraged to look at [EMTW, Sect. 3.3.1], where the KL polynomials for the Weyl group of $\mathfrak{sl}_3$ are calculated.

Now comes the main course.

Conjecture 1.15 (Kazhdan–Lusztig)**.** *For any $w, w' \in W$, we have*

$$[M_{w'\cdot 0} : L_{w\cdot 0}] = h_{w',w}(1).$$

Remark 1.16. According to the Verma–BGG theorem, $[M_{w'\cdot 0} : L_{w\cdot 0}] \neq 0$ iff $w \cdot 0 \leq_{\subset} w' \cdot 0$. By (the dominant version of) [Proposition 5.8, Lecture 7], this condition is equivalent to $w \geq w'$. Hence the conjecture would imply $h_{w',w}(1) \neq 0$ iff $w' \leq w$. Note that by definition, the "only if" part is true.

Remark 1.17. Although the conjecture only uses the value of $h_{w',w}$ at $v = 1$, it is not possible to define this value without knowing the deformation $\mathcal{H}$.

Remark 1.18. You are encouraged to view the Verma module $M_{w\cdot 0}$ as the incarnation of the standard basis element δ_w in $\mathcal{O}$, and view the irreducible module $L_{w\cdot 0}$ as the incarnation of the KL basis element b_w. In future lectures, we will introduce their incarnations in the geometry of G/B.

KL also made the following conjecture, which was latter proved by them using geometric methods:

Conjecture 1.19 (KL Positivity)**.** *The coefficients of $h_{w',w}(v)$ are non-negative integers.*

Remark 1.20. The pair (W, S) satisfies the axioms of a **Coxeter system**, and Hecke algebra, as well as the positivity conjecture, make sense for any Coxeter group. However, we no longer have geometric tools (like the flag variety G/B) to tackle this conjecture, and the first proof, by Elias–Williamson, only came in 2010's. For more details, see [EMTW].

2. More on $\mathcal{O}$

We still need a lot of preparations to present the geometric proofs of these conjectures. But before that, there are some remaining representation-theoretic topics that we need to address. Let me motivate them via the KL theory[2]:

- We will study the *contragradient duality* in $\mathcal{O}$, which is the incarnation of the KL involution on $\mathcal{H}$.
- We will study the derived category of $\mathcal{O}$, or equivalently, the *Ext-groups*. It turns out the coefficients of the KL polynomials are related to the dimensions of these Ext-groups. But before that, we need to study the *projective objects* in $\mathcal{O}$.

[2]We are not following the standard or historical order of presenting the theory of $\mathcal{O}$. Usually people would first introduce topics listed below, and state the Verma–BGG theorem and KL conjecture much later. I choose this order for two reasons: (i) I think the multiplicities $[M_\lambda : L_\mu]$ should be put on the central stage of this story; (ii) I want to highlight the KL conjecture and its geometric proof.

- We will introduce the *translation functors*, which allow us to calculate the multiplicities $[M_\lambda : L_\mu]$ in any integral block[3] using the information in the principle block $\mathcal{O}_0$.

There are other important topics that we do not have time to cover:

- Soergel's theory and his proof of the KL conjecture.
- Jantzen's filtrations and its geometric incarnation.
- Koszul duality and Langlands duality for $\mathcal{O}$.
- More...

3. Contragradient duality

We start with the following obvious construction.

Construction 3.1. *Let M be a weight (=semisimple) $\mathfrak{t}$-module. We define its **dual weight module** as*
$$M^{*,\mathsf{wt}} := \bigoplus_{\lambda\in\mathfrak{t}^*} (M^{\mathsf{wt}=\lambda})^*$$
where $(M^{\mathsf{wt}=\lambda})^*$ is a $\mathfrak{t}$-module of weight $-\lambda$.

The following lemma is obvious.

Lemma 3.2. *Let M be a weight $\mathfrak{t}$-module. The embedding*
$$M^{*,\mathsf{wt}} \simeq \bigoplus_{\lambda\in\mathfrak{t}^*} (M^{\mathsf{wt}=\lambda})^* \hookrightarrow \prod_{\lambda\in\mathfrak{t}^*} (M^{\mathsf{wt}=\lambda})^* \simeq M^*$$
identifies $M^{*,\mathsf{wt}}$ as the subspace of vectors v such that $U(\mathfrak{t})\cdot v$ is finite-dimensional.

Using the root decomposition of $\mathfrak{g}$, it is easy to deduce the following.

Corollary 3.3. *Let M be a weight $\mathfrak{g}$-module, then $M^{*,\mathsf{wt}}$ is a sub-$\mathfrak{g}$-module of M^*.*

For a Verma module M_λ, the above construction would define a *lowest* weight module, which can no longer belong to $\mathcal{O}$. Hence we need to find a way to correct the signs of the weights.

Construction 3.4. *Consider the automorphism of the root system (E,Φ) given by multiplication by -1. By the classification of semisimple Lie algebras, we obtain an automorphism $\tau : \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$, which is called the **Cartan involution** on $\mathfrak{g}$. Note that $\tau\circ\tau = \mathsf{Id}$.*

We abuse notation and let $\tau : \mathfrak{g}-\mathsf{mod} \to \mathfrak{g}-\mathsf{mod}$ be the automorphism induced by τ. Note that the functor τ is compatible with the forgetful functors to Vect, and sends weight modules to weight modules. Also, $\tau(M)^{\mathsf{wt}=\lambda} \simeq M^{\mathsf{wt}=-\lambda}$.

Remark 3.5. By construction, the restriction $\tau|_{\mathfrak{t}}$ is multiplication by -1. Note that $\tau(\mathfrak{b}) = \mathfrak{b}^-$ because Φ^+ is sent to $\Phi^- = -\Phi^+$.

Example 3.6. For $\mathfrak{g} = \mathfrak{sl}_n$, the Cartan involution is given by $\tau(A) = -A^T$.

Construction 3.7. *Let $\mathcal{C} \subset \mathfrak{g}-\mathsf{mod}$ be the full subcategory of weight $\mathfrak{g}$-modules such that each weight subspace is finite-dimensional. For $M \in \mathcal{C}$, define*
$$M^\vee := \tau(M^{*,\mathsf{wt}}).$$
Note that
$$(M^\vee)^{\mathsf{wt}=\lambda} \simeq (M^{\mathsf{wt}=\lambda})^*.$$
In particular, the λ-weight subspaces of $M^\vee$ and M have equal dimensions.

[3]For non-integral blocks, inspired by early works of Jantzen, Soergel ([S]) reduced the problem to the study of an integral block for another semisimple Lie algebra whose Weyl group is $W_{[\lambda]}$. See [H, Sect. 13.13] and the references there for more information.

The following lemma is obvious:

Lemma 3.8. *The functor*

$$\mathcal{C}^{\mathsf{op}} \to \mathcal{C},\ M \mapsto M^\vee.$$

is a contravariant involution, i.e., $(M^\vee)^\vee \simeq M$.

Theorem 3.9. *If M belongs to $\mathcal{O}$, so does $M^\vee$. In particular, the functor*

$$\mathcal{O}^{\mathsf{op}} \to \mathcal{O},\ M \mapsto M^\vee.$$

is a contravariant involution. We call it the ***contragradient duality*** *on $\mathcal{O}$.*

Proof. It is easy to see $\mathcal{O} \subset \mathcal{C}$ is closed under extensions. Hence the theorem follows from the following proposition.

□

Proposition 3.10. *For any $\lambda \in \mathfrak{t}^*$, we have $L_\lambda^\vee \simeq L_\lambda$.*

Proof. Since L_λ is an irreducible $\mathfrak{g}$-module, so is $L_\lambda^\vee$. By contruction, $L_\lambda^\vee$ has highest weight λ, and $(L_\lambda^\vee)^{\mathsf{wt}=\lambda}$ is 1-dimensional. For any λ-weight vector $v \in L_\lambda^\vee$, we have $\mathfrak{n} \cdot v = 0$ by considering the weights. This induces a nonzero $\mathfrak{g}$-linear map $M_\lambda \to L_\lambda^\vee$ sending v_λ to v. Since $L_\lambda^\vee$ is irreducible, this map is surjective. Therefore it must identify $L_\lambda^\vee$ with the unique irreducible quotient of M_λ.

□

Corollary 3.11. *Each block of $\mathcal{O}$ is stable under the contragradient duality.*

Definition 3.12. For any $\lambda \in \mathfrak{t}^*$, we call $M_\lambda^\vee$ the **dual Verma module** corresponding to M_λ.

Corollary 3.13. *For any $\lambda \in \mathfrak{t}^*$, the dual Verma module $M_\lambda^\vee$ has a unique irreducible submodule isomorphic to $L_\lambda^\vee \simeq L_\lambda$.*

Lemma 3.14. *For $\lambda, \mu \in \mathfrak{t}^*$, we have* $\dim \mathsf{Hom}_{\mathcal{O}}(M_\lambda, M_\mu^\vee) = \delta_{\lambda,\mu}$. *In particular, any composition $M_\lambda \twoheadrightarrow L_\lambda \hookrightarrow M_\lambda^\vee$ is a generator of the 1-dimensional vector space $\mathsf{Hom}_{\mathcal{O}}(M_\lambda, M_\lambda^\vee)$.*

Proof. Knowing a $\mathfrak{g}$-linear map $M_\lambda \to M_\mu^\vee$ is equivalent to knowing a λ-weight vector v in $M_\mu^\vee$ such that $\mathfrak{n} \cdot v = 0$. By definition, this is equivalent to knowing a functional $f : M_\mu \to k$ such that

- It factors as $M_\mu \twoheadrightarrow M_\mu^{\mathsf{wt}=\lambda} \to k$;
- It annilates $\mathfrak{n}^- \cdot M_\mu$.

Here the second condition is due to $\tau(\mathfrak{n}) = \mathfrak{n}^-$. Since M_μ is free over $U(\mathfrak{n}^-)$, we have $M_\mu \simeq (\mathfrak{n}^- \cdot M_\mu) \oplus M_\mu^{\mathsf{wt}=\mu}$. Hence $\lambda \neq \mu$ implies $f = 0$. In the case $\lambda = \mu$, f is determined by a functional $M_\lambda^{\mathsf{wt}=\lambda} \to k$, and the space of it is 1-dimensional.

□

Remark 3.15. The content of the lemma can be summarized as: the objects $\{M_\lambda^\vee\}_{\lambda \in \mathfrak{t}^*}$ is right orthogonal to the objects $\{M_\lambda\}_{\lambda \in \mathfrak{t}^*}$ in $\mathcal{O}$. Next time, we will prove this claim remains true even in the derived category. In other words, $\mathsf{Ext}^i_{\mathcal{O}}(M_\lambda, M_\mu^\vee) = 0$ for $i > 0$. For now, let us prove the case $i = 1$.

Lemma 3.16. *For $\lambda, \mu \in \mathfrak{t}^*$,* $\mathsf{Ext}^1_{\mathcal{O}}(M_\lambda, M_\mu^\vee) = 0$.

Proof. We need to show any following short exact sequence in $\mathcal{O}$ splits:

$$0 \to M_\mu^\vee \to N \to M_\lambda \to 0. \tag{3.1}$$

Consider the $\mathfrak{b}$-linear map $k_\lambda \to M_\lambda$. By pullback along this map, we obtain a short exact sequence of $\mathfrak{b}$-modules:

$$0 \to M_\mu^\vee \to N' \to k_\lambda \to 0,$$

where $N' = N \times_{M_\lambda} k_\lambda$. By the universal property of M_λ, we only need to show this sequence splits.

Note that $\mathsf{wt}(N') = \mathsf{wt}(M_\mu^\vee) \cup \{\lambda\} = \{\mu' \,|\, \mu' \leq \mu\} \cup \{\lambda\}$. If $\lambda \nless \mu$, then $\mathsf{wt}(N')$ contains no weight strictly greater than λ. Hence any λ-weight vector would give a desired splitting. If $\lambda < \mu$, we can pass to duality of (3.1) and obtain a short exact sequence

$$0 \to M_\lambda^\vee \to N^\vee \to M_\mu \to 0.$$

By the previous case, this sequence splits. Hence so is the original one. □

Example 3.17. For $\mathfrak{g} = \mathfrak{sl}_2$ and the coordinate $l = \langle \lambda, \check{\alpha} \rangle$, we have:

- If $l \notin \mathbb{Z}^{\geq 0}$, then $M_l \simeq L_l \simeq M_l^\vee$.
- If $l \in \mathbb{Z}^{\geq 0}$, then we have a nonsplit short exact sequence $0 \to L_l \to M_l^\vee \to L_{-l-2} \to 0$.

Remark 3.18. In future lectures, we will see the contragradient duality corresponds to the Verdier duality in geometry, and the latter can be related to the KL involution.

4. Projective modules

Recall the following definitions.

Definition 4.1. Let $\mathcal{A}$ be an abelian category. We say an object $P \in \mathcal{A}$ is **projective** if the functor $\mathsf{Hom}_{\mathcal{A}}(P, -)$ is exact.

We say $\mathcal{A}$ **has enough projectives** if every object $M \in \mathcal{A}$ admits a surjection $P \twoheadrightarrow M$ such that P is projective.

For an object $M \in \mathcal{A}$, we say a surjection $P \twoheadrightarrow M$ exhibits P as a **projective cover** of M if P is projective and the map $P \twoheadrightarrow M$ is an **essential surjection**, i.e., for any proper subobject $Q \subset P$, the composition $Q \to P \to M$ is not surjective.

Dually, we have:

Definition 4.2. Let $\mathcal{A}$ be an abelian category. We say an object $I \in \mathcal{A}$ is **injective** if the corresponding object in $\mathcal{A}^{\mathsf{op}}$ is projective.

We say $\mathcal{A}$ **has enough injectives** if $\mathcal{A}^{\mathsf{op}}$ has enough projectives.

For an object $M \in \mathcal{A}$, we say an injection $M \hookrightarrow I$ exhibits I as an **injective hull** of M if the corresponding morphism in $\mathcal{A}^{\mathsf{op}}$ gives a projective cover.

We will prove the following two theorems. The second one can be viewed as a blackbox[4].

Theorem 4.3. *The category $\mathcal{O}$ has enough projectives and injectives.*

Theorem 4.4. *Let $\mathcal{A}$ be an abelian category that has enough projectives and every object of $\mathcal{A}$ has finite length. Then:*

1. *Any object $M \in \mathcal{A}$ admits a projective cover, and any two projective covers $P_1 \twoheadrightarrow M$ and $P_2 \twoheadrightarrow M$ are isomorphic[5].*
2. *Any indecomposable projective object $P \in \mathcal{A}$ admits a unique irreducible quotient.*

[4] I fail to find a good reference for this well-known result, hence I provide a proof in the appendix.

[5] However, the isomorphism $P_1 \to P_2$ is *not* unique. Therefore, we can not say *the* projective cover.

(3) *There is a bijection:*

$$\{\textit{Isomorphism classes of irreducible objects in } \mathcal{A}\} \simeq$$
$$\{\textit{Isomorphism classes of indecomposable projective objects in } \mathcal{A}\}$$
$$L \longleftrightarrow P$$

such that (i) P is isomorphic to a projective cover of L; (ii) L is isomorphic to the irreducible quotient of P.

(4) *If P and L correspond to each other in (3), then for any $M \in \mathcal{A}$, we have*

$$\dim \mathsf{Hom}_{\mathcal{A}}(P, M) = [M : L].$$

Remark 4.5. We leave the dual version of the above theorem (for injective objects) to the readers.

Remark 4.6. In general, if an abelian category $\mathcal{A}$ has enough projectives (resp. injectives), then we can define the bounded above[6] (resp. bounded below) derived category $\mathcal{D}^-(\mathcal{A})$ (resp. $\mathcal{D}^+(\mathcal{A})$). However, it is much subtler to define the *unbounded* derived category.

But this subtlety does not occur for $\mathcal{O}$. We will see $\mathcal{O}$ has finite projective (resp. injective) dimension and thereby only the *bounded* derived category $\mathcal{D}^b(\mathcal{A})$ is needed.

Let us first assume the above theorems and study the projective objects in $\mathcal{O}$. The story for injective objects can be obtained using the contragradient duality.

Notation 4.7. *For any $\lambda \in \mathfrak{t}^*$, we denote a projective cover of L_λ by P_λ (which is well-defined up to non-unique isomorphisms).*

Similarly, we denote an injective hull of L_λ by I_λ.

Exercise 4.8. This is Homework 4, Problem 2. For any $\lambda \in \mathfrak{t}^*$, prove:

(1) The surjection $P_\lambda \twoheadrightarrow L_\lambda$ factors as $P_\lambda \to M_\lambda \to L_\lambda$.
(2) The obtained map $P_\lambda \to M_\lambda$ is surjective and exhibits P_λ as a projective cover of M_λ.

Corollary 4.9. *For any $M \in \mathcal{O}$, we have*

$$\dim \mathsf{Hom}_{\mathcal{O}}(P_\lambda, M) = [M : L_\lambda].$$

To prove Thorem 4.3, we need the following lemma

Lemma 4.10. *For $\chi = \varpi(\lambda)$, the functor*

$$\mathcal{O}_\chi \to \mathsf{Vect},\ M \mapsto M^{\mathsf{wt}=\lambda}.$$

is representable.

Remark 4.11. If λ is dot-dominant, then the above functor is represented by the Verma module M_λ. See [Lemma 3.7, Lecture 7]. The proof below is a slight modification of that proof.

Proof. For any $n \in \mathbb{Z}^{>0}$, let $I_{\lambda,n} \subset U(\mathfrak{g})$ be the left ideal genetated by the following elements:

- The element $t - \lambda(t)$ for any $t \in \mathfrak{t}$;
- The element $x_1 x_2 \cdots x_n$ for $x_i \in \mathfrak{n}^+$.

Let $M_{\lambda,n} \coloneqq U(\mathfrak{g})/I_{\lambda,n}$ be the quotient $U(\mathfrak{g})$-module. Note that $M_{\lambda,1}$ is just the Verma module. It is easy to see $M_{\lambda,n} \in \mathcal{O}$. Let $M_{\lambda,n,\chi} \in \mathcal{O}_\chi$ be the corresponding direct summand in the block $\mathcal{O}_\chi$.

[6] We always use cohomological convention when talking about chain complices.

We claim for n large enough, the module $M_{\lambda,n,\chi}$ represents the desired functor[7].

Indeed, for any $N \in \mathcal{O}_\chi$, $\mathsf{Hom}_{\mathcal{O}}(M_{\lambda,n}, N) \simeq \mathsf{Hom}_{\mathcal{O}_\chi}(M_{\lambda,n,\chi}, N)$ is the set of λ-weight vector v in N such that $(\mathfrak{n}^+)^n \cdot v = 0$. Then we win because weights occuring in $\mathcal{O}_\chi$ have upper bounded with respect to the partial order $\leq$.

□

Proof of Thoerem 4.3. By the contragradient duality, we only need to prove $\mathcal{O}$ has enough projectives. By the block decomposition, we only need to prove $\mathcal{O}_\chi$ has enough projectives. Using dévissage, we only need to show any irreducbile $L_\lambda \in \mathcal{O}_\chi$ admits a surjection $P \twoheadrightarrow L_\lambda$ with P being projective.

Let $P \in \mathcal{O}_\chi$ represents the functor in Lemma 4.10, i.e., $\mathsf{Hom}_{\mathcal{O}_\chi}(P, M) \simeq M^{\mathsf{wt}=\lambda}$. This functor is exact and therefore P is projective. Taking $M := L_\lambda$, any nonzero highest weight vector of L_λ gives a nonzero morphism $P \to L_\lambda$. Since L_λ is irreducible, this morphism is surjective as desired.

□[Thoerem 4.3]

Exercise 4.12. This is Homework 4, Problem 3. For $\lambda \in \mathfrak{t}^*$, let $P \twoheadrightarrow L_\lambda$ be the surjection constructed in the above proof, i.e., P represents the functor

$$\mathcal{O}_\chi \to \mathsf{Vect},\ M \mapsto M^{\mathsf{wt}=\lambda}.$$

Prove:

(1) This map factors as $P \to P_\lambda \twoheadrightarrow L_\lambda$. Moreover, $P \to P_\lambda$ is surjective.
(2) For $\mathfrak{g} = \mathfrak{sl}_2$, the obtained map $P \to P_\lambda$ happens to be an isomorphism[8].
(3) In general, $P \to P_\lambda$ is not an isomorphism[9].

Appendix A. Projective covers in Artinian and Noetherian category

Proof of Theorem 4.4(1). We first prove projective covers are isomorphic if they exist. This is true for any abelian category. Let $P_1 \xrightarrow{p_1} M$ and $P_2 \xrightarrow{p_2} M$ be projective covers of $M \in \mathcal{A}$. By the lifting property of P_1, the morphism $P_1 \xrightarrow{p_1} M$ factors as $P_1 \xrightarrow{\phi} P_2 \xrightarrow{p_2} M$. The morphism $P_1 \xrightarrow{\phi} P_2$ must be surjective because otherwise $P_2 \xrightarrow{p_2} M$ is not essential. By the lifting property of P_2, the identity morphism $P_2 = P_2$ factors as $P_2 \xrightarrow{\varphi} P_1 \xrightarrow{\phi} P_2$. The morphism $P_2 \xrightarrow{\varphi} P_1$ must be surjective because otherwise $P_1 \xrightarrow{p_1} M$ is not essential. But $P_2 \xrightarrow{\varphi} P_1$ is also injective because ϕ is a left inverse of it. It follows that both φ and ϕ are isomorphisms.

Now we prove any $M \in \mathcal{A}$ admits a projective cover. Let $P \xrightarrow{p} M$ be a surjection such that P is projective and $\mathsf{length}(P)$ is minimal among all such surjections. We claim this is a projective cover. We only need to show p is an essential surjection. Suppose $Q \xrightarrow{i} P$ is a subobject such that $q := p \circ i$ is surjective. We only need to show i is an isomorphism. We can assume $\mathsf{length}(Q)$

[7]By Yoneda lemma, this claim implies $M_{\lambda,n,\chi}$ and $M_{\lambda,n+1,\chi}$ are isomorphic for $n >> 0$. In fact, one can directly prove the obvious map $M_{\lambda,n+1,\chi} \to M_{\lambda,n,\chi}$ is an isomorphism for $n >> 0$. Sketch: let $I'_{\lambda,n} \subset U(\mathfrak{b})$ be the left ideal generated by the same set of elements. Then $M_\lambda \simeq \mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(U(\mathfrak{b})/I'_{\lambda,n})$. We have

$$I'_{\lambda,n}/I'_{\lambda,n+1} \simeq \bigoplus_{\alpha_1,\cdots,\alpha_n \in \Phi^+} k_{\lambda+\sum_{i=1}^n \alpha_i}$$

as $U(\mathfrak{b})$-modules. Hence we have a short exact sequence in $\mathcal{O}$:

$$0 \to \bigoplus_{\alpha_1,\cdots,\alpha_n \in \Phi^+} M_{\lambda+\sum_{i=1}^n \alpha_i} \to M_{\lambda,n+1} \to M_{\lambda,n} \to 0.$$

Now for $n >> 0$, the kernel will not be contained in the block $\mathcal{O}_\chi$ by considering the weights.

[8]Hint: using Corollary 4.9.

[9]Hint: what we have learned so far can (at least) prove this for $\mathfrak{sl}_3$ and $\mathfrak{sl}_4$.

is minimal among all such subobjects. By the lifting property of P, the morphism $P \xrightarrow{p} M$ factors as $P \xrightarrow{r} Q \xrightarrow{q} M$. Consider the composition $Q \xrightarrow{i} P \xrightarrow{r} Q$. It must be surjective because otherwise $\mathsf{Im}(r \circ i) \subset Q$ would contradict the minimal assumption about $\mathsf{length}(Q)$. But then $r \circ i$ must be an isomorphism because $\mathsf{length}(\ker(r \circ i)) = \mathsf{length}(Q) - \mathsf{length}(Q) = 0$. It follows that Q is a direct summand of P and therefore is also projective. Then we must have $Q \simeq P$ because of the minimal assumption about $\mathsf{length}(P)$.

□[Theorem 4.4(1)]

To prove (2), recall the following well-known result:

Lemma A.1. *Let $\mathcal{A}$ be any abelian category and $M \in \mathcal{A}$ be an indecomposable object of finite length. Then:*

(i) *Any $\phi : M \to M$ is either an isomorphism or nilpotent.*
(ii) *If $\phi, \varphi : M \to M$ is such that $\phi + \varphi$ is an isomorphism, then one of them is an isomorphism.*

Proof. It is obvious that (ii) follows from (i). To prove (i), suppose ϕ is not nilpotent. The descending chain $M \supset \mathsf{Im}(\phi) \supset \mathsf{Im}(\phi^2) \supset \cdots$ must stablize at a nonzero subobject $N \subset M$. Then ϕ stablizes N and $\phi|_N : N \to N$ is an isomorphism. Also, for $n >> 0$, ϕ^n induces a surjection $\varphi_n : M \to N$. By definition $\varphi_n|_N = (\phi|_N)^n$. Hence $\phi_n|_N$ is also an isomorphism. Then N is a direct summand of M. Since M is indecomposable, we must have $N \simeq M$ and therefore ϕ is surjective. By considering lengths, it is an isomorphism.

□

Proof of Theorem 4.4(2). Suppose $P \twoheadrightarrow L_1$ and $P \twoheadrightarrow L_2$ are two non-isomorphic irreducible quotients. Let $K_1 \subset P$ and $K_2 \subset P$ be the kernels. Then the morphism $K_1 \oplus K_2 \to P$ must be surjective. By the lifting property of P, the identity morphism $P = P$ factors as $P \to K_1 \oplus K_2 \to P$. Let ϕ_i be the composition $P \to K_i \to P$. Then $\phi_1 + \phi_2 = \mathsf{Id}$. By the above lemma, one of ϕ_1, ϕ_2 is an isomorphism. But this is absurd.

□[Theorem 4.4(2)]

Proof of Theorem 4.4(3). Let L be an irreducible object and P be a projective cover of it. Then P is indecomposble because otherwise $P \twoheadrightarrow L$ is not essential. Also, by (2), L is the unique irreducible quotient of P.

On the other hand, let P be an indecomposable object and L be the unique irreducible quotient of it. We only need to show $P \twoheadrightarrow L$ is essential. But the same proof as in (1) suffices for this purpose.

□[Theorem 4.4(3)]

Proof of Theorem 4.4(4). Using dévissage, we can reduce to the case when M is irreducible. In this case, both sides are either 0 or 1, depending on whether M and L are isomorphic.

□[Theorem 4.4(4)]

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